修士学位論文要約（平成29年３月）

一般化対称関数を計算するエネルギー効率の良いしきい値回路に関する研究

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Energy-efficient Threshold Circuits Computing Generalized Symmetric Functions

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In this thesis, we study a threshold circuit C computing generalized symmetric functions $d_{f,k}$, and investigate the relationship among the size and the energy of C. Then, the size of C is defined to be the number of gates in C and the energy of C is defined to be the maximum number of gates outputting "1" over all inputs to C. We first prove that $d_{f,k}$ of nk variables can be computable by a threshold circuit C with size $s = O(e \cdot 2^{n/(e-2)} + n)$ for any energy $e \geq 3$, and size $s = O(2^n)$ for energy $e = 2$. In contrast of this upper bound, we also prove that a threshold circuit C computing $d_{f,k}$ of nk variables requires size $s = \Omega(2^{n/6e^2})$ for any energy $e \geq 2$. In addition we give a lower and an upper bounds of a threshold circuit C computing $d_{f,k}$ of nk variables in the case where energy is extreme small, that is $e = 1$. We prove that a threshold circuit C computing $d_{f,k}$ of nk variables requires size $s = \{(k-1)^n + (k+1)^n\}/2^k$ for energy $e = 1$. We also prove that $d_{f,k}$ of nk variables can be computable by a threshold circuit C with size $s = \Omega((k+1)^n)$ and energy $e = 1$. Thus these results imply that threshold circuits computing generalized symmetric functions $d_{f,k}$ have tradeoff between size and energy.

1. Introduction

A logic circuit is a computational model of a Boolean function. In precise, a logic circuit is a directed acyclic graph which has input nodes of in-degree 0 and an output node of out-degree 0 corresponding to input and output of a Boolean function respectively, and other nodes called gates which are elements computing basic Boolean functions. A computational power of a logic circuit depends on computational power of gates contained in the circuit, and there are much research to analyze computational power of a variety of logic circuits composed of gates having different power.

In this thesis, we investigate the computational power of threshold circuits computing generalized symmetric functions in terms of size energy complexity. A threshold circuit is a logic circuit composed of threshold gates which compute linear threshold functions. This circuit is generalized computational model of logic circuits consisting of AND, OR, NOT gates because threshold gates have more computational power than those gates. Threshold circuits have attracted considerable attention in circuit complexity, and much research has been devoted to understand their computation for a few decades [2]. However, the computational power of threshold circuits is still in the dark [1].

An energy complexity is one of the complexity measures of threshold circuits. As a neural network in the brain carries out information processing by conveying electrical signals (i.e., "firing") among neurons, we can view a threshold circuit as a network computing a Boolean function by conveying Boolean values (i.e., "1") among threshold gates. The energy of a threshold circuit is then defined as the maximum number of gates outputting "1" in the circuit, where the maximum is taken over all the input assignments to the circuit [3].

2. Definitions

A *threshold gate* g is a logic gate computing a linear threshold function of an arbitrary integer z of inputs, which is identified by weight $\boldsymbol{w}(g) \in \mathbb{R}^z$ for the z inputs and an threshold $t(g) \in \mathbb{R}$, where the ith component of $\boldsymbol{w}(g)$, denoted by $\boldsymbol{w}(g)[i]$, is a weight for ith input. We define the output $g(\boldsymbol{x})$ of g as follows: For every $\boldsymbol{x} \in \{0,1\}^z$,

$$g(\boldsymbol{x}) = \mathrm{sign}\left(\sum_{i=1}^{z} \boldsymbol{w}(g)[i]\boldsymbol{x}[i] - t(g)\right)$$

where $\mathrm{sign}(z) = 1$ if $z \geq 0$ and $\mathrm{sign}(z) = 0$ if $z < 0$.

A *threshold circuit* C is a feedforward circuit consisting of threshold gates, and is expressed by a directed acyclic graph. Let n be the number of inputs to C, then C has n input nodes of in-degree 0, each of which corresponds to one of the n input variables $\boldsymbol{x} = (\boldsymbol{x}[1], \boldsymbol{x}[2], \ldots, \boldsymbol{x}[n])$, while the other nodes correspond to threshold gates. The inputs to a gate g in C consist of the inputs $\boldsymbol{x}$ and the outputs of some gates directed to g. Let g_s be one of the gates of out-degree 0, and we regard the output $g_s(\boldsymbol{x})$ of g_s as the *output* $C(\boldsymbol{x})$ *of* C: $C(\boldsymbol{x}) = g_s(\boldsymbol{x})$ for every input $\boldsymbol{x} \in \{0,1\}^n$. We call g_s the *top gate* of C. A threshold circuit C *computes* a Boolean function $f : \{0,1\}^n \to \{0,1\}$ if $C(\boldsymbol{x}) = f(\boldsymbol{x})$ for every $\boldsymbol{x} \in \{0,1\}^n$. We define *size* s of C as the number of gates in C, and define the *energy* e of C as

$$e = \max_{\boldsymbol{x} \in \{0,1\}^n} \sum_{i=1}^{s} g_i(\boldsymbol{x}),$$

where $g_i(\boldsymbol{x})$ is the output of g_i when input of C is $\boldsymbol{x}$.

For a function $f : \{0,1,2,...,n\} \to \{0,1\}$ and a positive integer k, generalized symmetric functions $d_{f,k}$ are defined as follows: For every input $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k) \in (\{0,1\}^n)^k$,

$$d_{f,k}(\boldsymbol{x}) = f(\mathcal{H}(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_k)),$$

where we define $\mathcal{H}(\boldsymbol{x}) = \sum_{i=1}^{n} x[i]$ for $\boldsymbol{x} \in \{0,1\}^n$.

3. Our results

We first prove the relationship among the size s and the energy e if e is relatively large, that is $e \geq 2$. The following two theorems show the enough size to give a construction of a threshold circuit computing generalized symmetric functions for each energy e.

Theorem 1 *For any positive integers k and n, and any function $f : \{0,1,\ldots,n\} \to \{0,1\}$, $d_{f,k}$ is computable by a threshold circuit of size $O(e \cdot 2^{n/(e-2)} + n)$ and energy $e \geq 3$.*

Theorem 2 *For any positive integers k and n, and any function $f : \{0,1,\ldots,n\} \to \{0,1\}$, $d_{f,k}$ is computable by a threshold circuit of size 2^n and energy two.*

The following theorem shows the size required threshold circuits computing generalized symmetric functions with energy $e \geq 2$.

Theorem 3 *Let $e \geq 2$. For any positive integers k and n, and any function $f : \{0,1,\ldots,n\} \to \{0,1\}$, every threshold circuit of size s and energy e computing $d_{f,k}$ satisfies $2^{n/6e^2} \leq s$.*

Theorems 1, 2 and 3 imply that there exists tradeoff between size and energy of threshold circuit computing generalized symmetric functions. Compared Theorems 1 and 2 to Theorem 3, the upper bound of the size almost matches the lower bound. In precise, both of them are exponential of n if e is constant, and are polynomial of n if e is polynomial of n. So we can prove tradeoff between size and energy of energy-efficient threshold circuit computing generalized symmetric functions.

In contrast above case, we also prove the relationship among the size s and the energy e if e is extreme small, that is $e = 1$. Theorem 4 shows an upper bound of the size of a threshold circuit computing generalized symmetric functions and Theorem 5 shows a lower bound.

Theorem 4 *For positive integers k, n and any function $f : \{0,1,\ldots,n\} \to \{0,1\}$, $d_{f,k}$ is computable by a threshold circuit of size at most $(k+1)^n$ and energy one.*

Theorem 5 *For any two positive integers $n \geq 1$ and $k \geq 2$, and any function $f : \{0,1,\ldots,n\} \to \{0,1\}$, any threshold circuit of energy one that computes $d_{f,k}$ has size*

$$s \geq \frac{(k-1)^n + (k+1)^n}{2^k}.$$

Above two theorems imply optimal size of threshold circuit computing generalized symmetric functions with energy one. In Theorem 5, if integer k is constant,

$$\frac{(k-1)^n + (k+1)^n}{2^k} = \Omega((k+1)^n).$$

Therefore size of such the threshold gate is $\Theta((k+1)^n)$ if k is constant.

4. Conclusion

In this thesis, we investigate a threshold circuit C computing generalized symmetric functions $d_{f,k}$, and show that there is a tradeoff between the size s and the energy e. Furthermore, we give a construction of threshold circuit C computing generalized symmetric functions $d_{f,k}$ with energy one which has almost optimal size.

References

1) D. M. Kane and R. Williams. Super-linear gate and super-quadratic wire lower bounds for depth-two and depth-three threshold circuits. *Proc. STOC*, 633–643, 2016.
2) M. Minsky and S. Papert. *Perceptrons: An Introduction to Computational Geometry.* MIT Press, Cambridge, MA, 1988.
3) K. Uchizawa, R. Douglas and W. Maass. On the computational power of threshold circuits with sparse activity. *Neural Computation*, 18(12):2994–3008, 2008.