DSSR

Discussion Paper No. 106

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January, 2020

Data Science and Service Research
Discussion Paper

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Extreme values, means, and inequality measurement*

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This version: January 15, 2020

Abstract. We examine some ordinal measures of inequality that are familiar from the literature. These measures have a quite simple structure in that their values are determined by combinations of specific summary statistics such as the extreme values and the arithmetic mean of a distribution. In spite of their common appearance, there seem to be no axiomatizations available so far, and this paper is intended to fill that gap. In particular, we consider the absolute and relative variants of the range; the max-mean and the mean-min orderings; and quantile-based measures. In addition, we provide some empirical observations that are intended to illustrate that, although these orderings are straightforward to define, some of them display a surprisingly high correlation with alternative (more complex) measures. *Journal of Economic Literature* Classification Nos.: H24, I31.

Keywords: Economic Index Numbers; Mean Values; Luxembourg Income Studies.

* We are grateful to Giorgia Menta for excellent research assistance. We thank Takuya Hasebe and seminar participants at Fukuoka University and Sophia University for comments and suggestions. Financial support from the Fonds de Recherche sur la Société et la Culture of Québec and the Fonds National de la Recherche Luxembourg (Grant C18/SC/12677653) is gratefully acknowledged.

† Kamaga is appointed as a visiting associate professor at Graduate School of Economics and Management at Tohoku University.

1 Introduction

The measurement of income inequality has been an active field of investigation for over a century, and early classical contributions include those of Lorenz (1905), Gini (1912), Pigou (1912), and Dalton (1920). While much of the literature focuses on a relative notion of inequality (that is, on scale-invariant measures), absolute indices (which are translation-invariant) are examined as well. Centrist or intermediate measures that represent compromises between the relative and the absolute approach are discussed in Kolm (1976a,b), Pfingsten (1986), and Bossert and Pfingsten (1990). The normative approach connects inequality to welfare and can be traced back to Kolm (1969), Atkinson (1970), and Sen (1973) in the case of relative measures, and to Kolm (1969) and Blackorby and Donaldson (1980) if an absolute notion of inequality is adopted. Ethical measures of inequality in an ordinal setting are analyzed by Blackorby and Donaldson (1984), Ebert (1987), and Dutta and Esteban (1992).

In this paper, we follow an ordinal approach to inequality measurement and, therefore, focus on inequality orderings. Our main results provide characterizations of some simple measures of inequality that are familiar from the literature. The first of these are range-based measures which perform inequality comparisons by means of the difference between maximal and minimal income in the absolute case, and the ratio of the maximum and the minimum in a relative setting. The max-mean orderings use the difference and the ratio of the maximum and the arithmetic mean and the mean-min measures employ the arithmetic mean and the minimal income. In addition, we examine inequality orderings that focus on the income gaps (in the absolute case) or the income shares (for relative measures) of the top or bottom quantile of an income distribution. All of these inequality orderings satisfy three standard axioms, namely, S-convexity, continuity, and replication invariance. However, as far we are aware, they have not been axiomatized yet.

Clearly, these measures are rather coarse because of their limited use of income distribution statistics, so that we do not advocate their use over all competing suggestions. Nevertheless, as discussed by Leigh (2009, p. 162) in the context of justifying the use of the top income shares, when some data is absent or reliable estimates of the entire income distribution are not available, they can serve as a useful proxy for measuring inequality. In particular, in light of the interdependence between different parts of the income distribution resulting from economic activities, they could be a useful and easy-to-use tool for drawing inferences about overall inequality from limited data; see Atkinson (2007, pp. 19–25) and Atkinson, Piketty, and Saez (2011, pp. 7–12) for discussions regarding top income shares. Therefore, we think that it is worthwhile to provide axiomatic characterizations of those inequality orderings.

Among the orderings we consider, the range-based inequality orderings that compare the distance between (or the ratio of) the maximal and the minimal income do not utilize the average income. In this sense, these inequality orderings are coarser than the others. To present axiomatic characterizations of these inequality orderings, we employ some suitably adapted axioms that appeared in the literature on ranking sets of outcomes under complete uncertainty. These properties, reformulated in the context of income inequality measurement, are concerned with how we should rank income distributions when the information on the realized income levels in the distributions is reliable but that on their frequency distribution is not. Our characterizations of the other inequality orderings, on the other hand, rely on properties regarding the composition of progressive and regressive transfers in addition to standard axioms.

In addition to presenting their axiomatic characterizations, it is important to empirically examine the usefulness of these inequality orderings. In analogy with Leigh's (2007) study of the relative performance of top income shares in comparison with other inequality measures, we provide an empirical analysis of the correlation between the range-based and quantile-based orderings and some classical indices including the Gini coefficient. We find that there is some surprisingly significant agreement when considering the movements of the measures and more commonly-employed inequality orderings.

In the following section, we introduce our basic notation and definitions. The range-based measures, the max-mean orderings, the mean-min ordinal indices, and the quantile shares and gaps are characterized in Section 3. In each case, axiomatizations of both the requisite absolute ordering and its relative counterpart are provided. Section 4 contains our empirical study and Section 5 concludes. The independence of the axioms used in our characterizations is established in an appendix.

2 Notation and definitions

2.1 Range-based and related inequality orderings

Let $\mathbb{N}$ be the set of positive integers. The sets of all real numbers, all non-negative real numbers, and all positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$. For $n \in \mathbb{N}$, let $\mathbf{1}^n$ denote the n-dimensional vector consisting of n ones and, for all $i \in \{1, \ldots, n\}$, e^i is the i^{th} unit vector in $\mathbb{R}^n$. For simplicity, we suppress the dependence of this unit vector on n; the dimension of e^i will always be apparent from the context. For all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^n$, the arithmetic mean of x is denoted by $\mu(x)$; that is, $\mu(x) = \sum_{i=1}^n x_i/n$.

We distinguish two domains that are relevant in this paper. In the context of absolute inequality orderings, incomes may take on any real value and, analogously, relative inequality orderings are restricted to positive incomes. Thus, we define the (variable-population) domains $D = \cup_{n \in \mathbb{N}} \Omega^n$, where $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. A vector $x \in D$ is interpreted as an income distribution.

An *inequality ordering* is an ordering $R \subseteq D^2$ and we write xRy for $(x, y) \in R$. Thus, the expression xRy means that the income inequality in x is at least as high as the inequality in y. The asymmetric part of R is P and the symmetric part of R is I.

An absolute inequality ordering is invariant to equal absolute changes of all incomes. That is, it is required to satisfy the axiom of translation invariance.

Translation invariance. For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}^n$, and for all $\delta \in \mathbb{R}$,

$$(x + \delta \mathbf{1}^n) I x.$$

Analogously, a relative inequality ordering is invariant to changes in the scaling of all incomes by a common positive factor.

Scale invariance. For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $\lambda \in \mathbb{R}_{++}$,

$$\lambda x I x.$$

The first two orderings that we consider in this paper are the *absolute range* R^a_{xn} associated with $\Omega = \mathbb{R}$ and the *relative range* R^r_{xn} with the domain generated by $\Omega = \mathbb{R}_{++}$, defined as follows. For all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n$, and for all $y \in \mathbb{R}^m$, we let

$$xR^a_{xn}y \;\Leftrightarrow\; \max\{x_1,\ldots,x_n\} - \min\{x_1,\ldots,x_n\} \geq \max\{y_1,\ldots,y_m\} - \min\{y_1,\ldots,y_m\}.$$

Cowell (2011, p. 155) refers to a representation of this ordering as the *range.* The measure that is obtained by dividing R^a_{xn} by the mean income $\mu(x)$ (which requires the domain to be restricted to $\mathbb{R}_{++}$) is what he labels the *standardized range.* The latter also appears in Sen (1973, p. 24).

The relative counterpart of the absolute range is the relative range R^r_{xn}, defined by

$$xR^r_{xn}y \;\Leftrightarrow\; \frac{\max\{x_1,\ldots,x_n\}}{\min\{x_1,\ldots,x_n\}} \geq \frac{\max\{y_1,\ldots,y_m\}}{\min\{y_1,\ldots,y_m\}}$$

for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$.

The *absolute max-mean inequality ordering* $R^a_{x\mu}$ is defined by letting, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n$, and for all $y \in \mathbb{R}^m$,

$$xR^a_{x\mu}y \;\Leftrightarrow\; \max\{x_1,\ldots,x_n\} - \mu(x) \geq \max\{y_1,\ldots,y_m\} - \mu(y).$$

The scale-invariant counterpart of $R^a_{x\mu}$ is the *relative max-mean inequality ordering* $R^r_{x\mu}$, defined as

$$xR^r_{x\mu}y \;\Leftrightarrow\; \frac{\max\{x_1,\ldots,x_n\}}{\mu(x)} \geq \frac{\max\{y_1,\ldots,y_m\}}{\mu(y)}$$

for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$.

The *absolute mean-min inequality ordering* $R^a_{\mu n}$ is given by

$$xR^a_{\mu n}y \;\Leftrightarrow\; \mu(x) - \min\{x_1,\ldots,x_n\} \geq \mu(y) - \min\{y_1,\ldots,y_m\}$$

for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n$, and for all $y \in \mathbb{R}^m$. Chakravarty (2010, p. 34) refers to a representation of this ordering as the *absolute maximin index* because of its link to the maximin social welfare function.

Finally, the *relative mean-min inequality ordering* $R^r_{\mu n}$ is obtained by defining, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^n_{++}$, and for all $y \in \mathbb{R}^m_{++}$,

$$xR^r_{\mu n}y \;\Leftrightarrow\; \frac{\mu(x)}{\min\{x_1,\ldots,x_n\}} \geq \frac{\mu(y)}{\min\{y_1,\ldots,y_m\}}$$

or, equivalently,

$$xR^r_{\mu n}y \;\Leftrightarrow\; \frac{\min\{x_1,\ldots,x_n\}}{\mu(x)} \leq \frac{\min\{y_1,\ldots,y_m\}}{\mu(y)}.$$

Hence, according to $R^r_{\mu n}$, inequality increases if and only if the ratio of the minimum income to the mean income decreases. In analogy to the absolute case, Chakravarty (2010, p. 24) uses the term *relative maximin index* for a representation of $R^r_{\mu n}$.

2.2 Quantile-based inequality orderings

In order to discuss the inequality orderings that are based on top and bottom income shares and gaps, we need to employ a slightly modified framework. Let $q \in \mathbb{N}$ with $q \geq 3$. The set D of income distributions considered now is defined by $D = \cup_{n \in \mathbb{N}} \Omega^{nq}$, where $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. This modification guarantees that q equal-sized groups of individuals in an income distribution are well-defined. Note that, for any $n \in \mathbb{N}$ and for any $x \in \Omega^{nq}$, there exists a unique permutation π_x of $\{1, \ldots, nq\}$ such that $x_{(\,)} = (x_{\pi_x(1)}, \ldots, x_{\pi_x(nq)})$ is a non-decreasing rearrangement of x and, for all $i, j \in \{1, \ldots, nq\}$ with $i < j$, if $x_{\pi_x(i)} = x_{\pi_x(j)}$ then $\pi_x(i) < \pi_x(j)$. That is, $\pi_x^{-1}(i)$ is interpreted as the income rank of individual i from the bottom in x, where ties of income levels are broken with respect to individual names represented by numbers. For any $n \in \mathbb{N}$, for any $x \in \Omega^{nq}$, and for any $\ell \in \{1, \ldots, q\}$, we define $G_\ell(x)$ by

$$G_\ell(x) = \{i \in \{1, \ldots, nq\} \mid (\ell - 1)n + 1 \leq \pi_x^{-1}(i) \leq \ell n\},$$

that is, $G_\ell(x)$ is the group of individuals in the ℓ^{th} q-quantile in x. In this paper, the ℓ^{th} q-quantile of income distribution x represents the ℓ^{th} worse-off group of individuals according to the income ranking π_x^{-1}, rather than the ℓ^{th} cut-off point. Therefore, if $q = 10$, $G_1(x)$ is the group of individuals in the bottom decile and $G_{10}(x)$ is that in the top decile. For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}_{++}$, and for all $\ell \in \{1, \ldots, q\}$, we write $\mu_\ell(x)$ as the mean income of the ℓ^{th} q-quantile of x, that is, $\mu_\ell(x) = \sum_{i \in G_\ell(x)} x_i / n$.

According to the modification of the domain of an inequality ordering, we say that an inequality ordering R on D is absolute if it satisfies the translation invariance axiom reformulated as follows.

Translation invariance*. For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}$, and for all $\delta \in \mathbb{R}$,

$$(x + \delta \mathbf{1}^{nq}) I x.$$

Analogously, an inequality ordering R is said to be relative if it satisfies the following reformulation of the scale-invariance property.

Scale invariance*. For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}_{++}$, and for all $\lambda \in \mathbb{R}_{++}$,

$$\lambda x I x.$$

We define the *top income gap inequality ordering* R_t^a by letting, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}$, and for all $y \in \mathbb{R}^{mq}$,

$$x R_t^a y \;\Leftrightarrow\; \mu_q(x) - \mu(x) \geq \mu_q(y) - \mu(y).$$

The scale-invariant analogue of R_t^a is the (relative) *top income share inequality ordering* R_t^r, defined as follows. For all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}_{++}$, and for all $y \in \mathbb{R}^{mq}_{++}$,

$$x R_t^r y \;\Leftrightarrow\; \frac{\sum_{i \in G_q(x)} x_i}{\sum_{i=1}^{nq} x_i} \geq \frac{\sum_{i \in G_q(y)} y_i}{\sum_{i=1}^{mq} y_i}.$$

Since the pioneering work by Piketty (2001), top income shares have been widely employed in the literature on the empirical analysis of inequality in the long run; see, for instance, Atkinson, Piketty, and Saez (2011) and Leigh (2009). Note that, since $\sum_{i \in G_q(x)} x_i / \sum_{i=1}^{nq} x_i = \mu_q(x)/(q\mu(x))$, an ordinally equivalent representation of R_t^r is given by

$$xR_t^r y \;\Leftrightarrow\; \frac{\mu_q(x)}{\mu(x)} \geq \frac{\mu_q(y)}{\mu(y)}.$$

The *bottom income gap inequality ordering* R_b^a is given by letting, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}^{nq}$, and for all $y \in \mathbb{R}^{mq}$,

$$xR_b^a y \;\Leftrightarrow\; \mu(x) - \mu_1(x) \geq \mu(y) - \mu_1(y).$$

Finally, we define a relative analogue of the bottom income gap inequality ordering. The *bottom income share inequality ordering* is the inequality ordering R_b^r defined as follows. For all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^{nq}$, and for all $y \in \mathbb{R}_{++}^{mq}$,

$$xR_b^r y \;\Leftrightarrow\; \frac{\sum_{i \in G_1(x)} x_i}{\sum_{i=1}^{nq} x_i} \leq \frac{\sum_{i \in G_1(y)} y_i}{\sum_{i=1}^{mq} y_i}.$$

Analogously to the top income share inequality ordering, an ordinally equivalent representation of R_b^r is given by

$$xR_b^r y \;\Leftrightarrow\; \frac{\mu(x)}{\mu_1(x)} \geq \frac{\mu(y)}{\mu_1(y)}.$$

3 Characterizations

The use of translation invariance is restricted to absolute inequality orderings, whereas scale invariance is employed in the relative case. All other axioms can be defined for both options, that is, for $\Omega = \mathbb{R}$ and for $\Omega = \mathbb{R}_{++}$. Each of the following subsections addresses one type of ordering considered in this paper.

3.1 Range inequality orderings

Our first axiom in this subsection requires that the inequality ordering R is *anonymous,* paying no attention to the names of the individuals. Clearly, this is a fundamental property that requires no further discussion.

Anonymity. For all $n \in \mathbb{N}$ and for all $x, y \in \Omega^n$, if x is a permutation of y, then xIy.

In addition to anonymity, the results of this subsection make use of properties that involve the comparison of income distributions of different dimensions. The first of these is straightforward. *Equality indifference* requires that all equal distributions are equally unequal, independent of the number of people involved. As is the case for anonymity, the intuitive appeal of this condition is immediate.

Equality indifference. For all $n, m \in \mathbb{N}$ and for all $\alpha, \beta \in \Omega^1$,

$$\alpha \mathbf{1}^n I \beta \mathbf{1}^m.$$

The first part of the following expansion-dominance axiom is borrowed from the literature on ranking sets of outcomes in the presence of complete uncertainty; see, for instance, Kannai and Peleg (1984) and Bossert and Slinko (2006). In contrast to that literature, we have to allow for incomes being equal within a distribution and, moreover, the role played by lowest incomes is different from that played by worst elements in a set of possible outcomes. Thus, our formulation differs from that in the literature on ranking sets. The second part of the property reflects the coarse nature of the inequality orderings discussed here by requiring that adding individuals with incomes between the extremes of a distribution does not increase inequality.

Expansion dominance. (i) For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, if $y_1 = \ldots = y_m > \max\{x_1, \ldots, x_n\}$, then

$$(y, x) P x.$$

(ii) For all $n \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $\alpha \in [\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}]$,

$$x R (x, \alpha).$$

Part (i) of the above expansion-dominance axiom is based on the observation that if an income distribution is expanded by adding any number of individuals with a common income level that is above the highest in the original distribution, the resulting larger distribution should display a higher level of inequality. Again, this is intuitively plausible because the new distribution increases maximal income without changing the distribution among those who are present prior to the expansion. Part (ii) clearly is more controversial because it reflects a feature of the range-based measures—namely, that they are insensitive with respect to expansions of a distribution that leave the extreme values unchanged.

Another modification of a requirement from the literature on choice under complete uncertainty is the following conditional version of an independence property. Again, the axiom differs from the corresponding condition for set rankings because of the different interpretation—primarily because equal income levels within a distribution have to be accommodated.

Conditional independence. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, for all $y \in \Omega^m$, and for all $\alpha \in \Omega^1$, if xPy, $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_m\}$, $\alpha \geq \max\{x_1, \ldots, x_n\}$, and $\alpha > \max\{y_1, \ldots, y_m\}$, then

$$(x, \alpha) R (y, \alpha).$$

Conditional independence is a robustness condition. Starting with two distributions x and y (not necessarily of the same population size), if x is considered more unequal than y, then the addition of an individual whose income exceeds the maximal income in y and is at least as high as the maximal income in x should not overturn this strict relation.

Our first observation shows that the conjunction of the four axioms of this subsection implies that an income distribution x of any dimension must be as unequal as the distribution that is composed of the maximal and the minimal values of x. See, for instance, Kannai and Peleg's (1984, p. 174) Lemma and Bossert and Slinko's (2006, pp. 108–109) Theorem 1 for analogous results in the context of set rankings.

Theorem 1. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. If R satisfies anonymity, equality indifference, expansion dominance, and conditional independence, then, for all $n \in \mathbb{N}$ and for all $x \in \Omega^n$,*

$$xI(\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\}).$$

Proof. Let $n \in \mathbb{N}$ and $x \in \Omega^n$. If $x_1 = \ldots = x_n$, the result follows from equality indifference. Now suppose that there exist $i, j \in \{1, \ldots, n\}$ such that $x_i \neq x_j$. Because of anonymity, without loss of generality, we can assume that $x_1 = \max\{x_1, \ldots, x_n\}$ and $x_n = \min\{x_1, \ldots, x_n\}$. If there exists $j \in \{1, \ldots, n-1\}$ such that $x_j = x_n$, let y be the vector consisting of all components x_j such that $x_j = x_n$. By equality indifference, it follows that

$$yI(x_n) = (\min\{x_1, \ldots, x_n\}).$$

If there are more than two different levels of income, successively augment y with the components of x that correspond to the next-highest income level, except those at the top level $x_1 = \max\{x_1, \ldots, x_n\}$. Let z be the vector of incomes that includes all levels strictly between x_1 and x_n. Repeated application of part (i) of expansion dominance and anonymity, along with transitivity, implies that we must have

$$(y, z)P(x_n).$$

If there exists $i \in \{2, \ldots, n\}$ such that $x_i = x_1 = \max\{x_1, \ldots, x_n\}$, let w be the vector consisting of those incomes except for x_1 itself. Augmenting the distribution (z, y) by w, it follows that, by definition, $(w, z, y) = (x_2, \ldots, x_n)$. Using part (i) of expansion dominance, anonymity, and transitivity again, we obtain

$$(x_2, \ldots, x_n)P(x_n).$$

By conditional independence, it follows that

$$x = (x_1, \ldots, x_n)R(x_1, x_n) = (\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\}). \tag{1}$$

Part (ii) of expansion dominance and anonymity (applied repeatedly if necessary) together imply

$$(\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\})Rx$$

and, combined with (1), it follows that

$$xI(\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\}),$$

as was to be established. ■

The following theorem characterizes all inequality orderings that satisfy the axioms defined in this subsection.

Theorem 2. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. R satisfies anonymity, equality indifference, expansion dominance, and conditional independence if and only if there exists an ordering $\succsim$ (with asymmetric and symmetric parts $\succ$ and $\sim$) on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; (\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\}) \succsim (\max\{y_1, \ldots, y_m\}, \min\{y_1, \ldots, y_m\});$$

(ii) $(\alpha, \alpha) \sim (\beta, \beta)$ for all $\alpha, \beta \in \Omega^1$;

(iii) $\succsim$ is increasing in its first argument.

Proof. 'If.' Anonymity follows from (i) in the theorem statement. Further, equality indifference follows from combining (i) and (ii).

To prove that part (i) of expansion dominance is satisfied, suppose that $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$ are such that $y_1 = \ldots = y_m > \max\{x_1, \ldots, x_n\}$. It follows that

$$\max\{x_1, \ldots, x_n, y_1, \ldots, y_m\} = y_1 > \max\{x_1, \ldots, x_n\}$$

and

$$\min\{x_1, \ldots, x_n, y_1, \ldots, y_m\} = \min\{x_1, \ldots, x_n\}$$

so that, by (i) in the theorem statement and the increasingness of $\succsim$ in its first argument (see (iii) in the theorem statement), it follows that $(x, y)Px$.

Next, we prove part (ii) of expansion dominance. Suppose that $n \in \mathbb{N}$, $x \in \Omega^n$, and $\alpha \in [\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}]$. This implies that

$$\max\{x_1, \ldots, x_n, \alpha\} = \max\{x_1, \ldots, x_n\} \;\text{ and }\; \min\{x_1, \ldots, x_n, \alpha\} = \min\{x_1, \ldots, x_n\}.$$

Thus, because $\succsim$ is reflexive, part (i) of the theorem statement implies that $xR(x, \alpha)$.

To conclude the proof of the 'if' part, we show that conditional independence is satisfied. To that end, suppose that $n, m \in \mathbb{N}$, $x \in \Omega^n$, $y \in \Omega^m$, and $\alpha \in \Omega^1$ are such that xPy, $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_m\}$, $\alpha \geq \max\{x_1, \ldots, x_n\}$, and $\alpha > \max\{y_1, \ldots, y_m\}$. It follows that

$$\max\{x_1, \ldots, x_n, \alpha\} = \max\{y_1, \ldots, y_m, \alpha\} = \alpha$$

and

$$\min\{x_1, \ldots, x_n, \alpha\} = \min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_m\} = \min\{y_1, \ldots, y_m, \alpha\}$$

so that

$$(\max\{x_1, \ldots, x_n, \alpha\}, \min\{x_1, \ldots, x_n, \alpha\}) \succsim (\max\{y_1, \ldots, y_m, \alpha\}, \min\{y_1, \ldots, y_m, \alpha\})$$

because $\succsim$ is reflexive. By part (i), it follows that $(x, \alpha)R(y, \alpha)$.

'Only if.' Suppose that R satisfies the axioms in the theorem statement. Define the relation $\succsim$ by letting, for all $(\alpha, \beta), (\alpha', \beta') \in S$,

$$(\alpha, \beta) \succsim (\alpha', \beta')$$

if and only if there exist $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$ such that xRy and

$$\alpha = \max\{x_1, \ldots, x_n\}, \beta = \min\{x_1, \ldots, x_n\}, \alpha' = \max\{y_1, \ldots, y_m\}, \beta' = \min\{y_1, \ldots, y_m\}.$$

By Theorem 1 and the transitivity of R, this relation is a well-defined ordering, and property (i) of the theorem statement follows by definition.

To establish that property (ii) is satisfied, suppose that $\alpha, \beta \in \Omega^1$. By equality indifference, it follows that $\alpha \mathbf{1}^n I \beta \mathbf{1}^m$ for all $n, m \in \mathbb{N}$ and, by property (i), it follows that

$$(\alpha, \alpha) \sim (\beta, \beta).$$

Finally, we prove property (iii). Suppose that $\alpha, \alpha', \beta \in \Omega^1$ are such that $\alpha > \alpha' \geq \beta$. Let $x = (\alpha, \alpha', \beta)$ and $y = (\alpha', \beta)$. Thus,

$$\max\{x_1, x_2, x_3\} = \alpha > \alpha' = \max\{y_1, y_2\} \text{ and } \min\{x_1, x_2, x_3\} = \beta = \min\{y_1, y_2\}.$$

By part (i) of expansion dominance, it follows that xPy and, by property (i), we obtain $(\alpha, \beta) \succ (\alpha', \beta)$ so that $\succsim$ is increasing in its first argument. ■

We now prove the two main results of this subsection. Adding translation invariance to the axioms of Theorem 2 characterizes the absolute range, whereas the relative range is obtained if scale invariance is used in the place of translation invariance. Because the 'if' parts of the proofs are straightforward, we only establish the reverse implications. The same remark applies to analogous results later in the paper.

Theorem 3. *Let $\Omega = \mathbb{R}$. R satisfies anonymity, equality indifference, expansion dominance, conditional independence, and translation invariance if and only if $R = R^a_{xn}$.*

Proof. Let $n \in \mathbb{N}$ and $x \in \Omega^n$. Translation invariance with $\delta = -\min\{x_1, \ldots, x_n\}$ requires that

$$(x_1 - \min\{x_1, \ldots, x_n\}, \ldots, x_n - \min\{x_1, \ldots, x_n\}) I x$$

and, by Theorem 2,

$$(\max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\}, 0) \sim (\max\{x_1, \ldots, x_n\}, \min\{x_1, \ldots, x_n\}).$$

Now let $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$. Using Theorem 2, it follows that

$$xRy \Leftrightarrow (\max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\}, 0) \succsim (\max\{y_1, \ldots, y_m\} - \min\{y_1, \ldots, y_m\}, 0)$$

and, because $\succsim$ is increasing in its first argument, this is equivalent to

$$\begin{aligned} xRy &\Leftrightarrow \max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\} \geq \max\{y_1, \ldots, y_m\} - \min\{y_1, \ldots, y_m\} \\ &\Leftrightarrow xR^a_{xn} y. \blacksquare \end{aligned}$$

As a remark aside, note that Theorems 1, 2, and 3 remain true if $\Omega = \mathbb{R}$ is replaced with $\Omega = \mathbb{R}_+$; this is apparent from inspecting their proofs.

Theorem 4. *Let $\Omega = \mathbb{R}_{++}$. R satisfies anonymity, equality indifference, expansion dominance, conditional independence, and scale invariance if and only if $R = R^r_{xn}$.*

Proof. Let $n \in \mathbb{N}$ and $x \in \Omega^n$. Scale invariance with $\lambda = 1/\min\{x_1, \ldots, x_n\}$ requires that

$$\left(\frac{x_1}{\min\{x_1,\ldots,x_n\}}, \ldots, \frac{x_n}{\min\{x_1,\ldots,x_n\}}\right) Ix$$

and, by Theorem 2,

$$\left(\frac{\max\{x_1,\ldots,x_n\}}{\min\{x_1,\ldots,x_n\}}, 1\right) \sim (\max\{x_1,\ldots,x_n\}, \min\{x_1,\ldots,x_n\})\,.$$

Now let $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$. Using Theorem 2, we obtain

$$xRy \;\Leftrightarrow\; \left(\frac{\max\{x_1,\ldots,x_n\}}{\min\{x_1,\ldots,x_n\}}, 1\right) \succsim \left(\frac{\max\{y_1,\ldots,y_m\}}{\min\{y_1,\ldots,y_m\}}, 1\right)$$

and, because $\succsim$ is increasing in its first argument, this is equivalent to

$$xRy \;\Leftrightarrow\; \frac{\max\{x_1,\ldots,x_n\}}{\min\{x_1,\ldots,x_n\}} \geq \frac{\max\{y_1,\ldots,y_m\}}{\min\{y_1,\ldots,y_m\}} \;\Leftrightarrow\; xR^r_{xm}y. \blacksquare$$

3.2 Max-mean inequality orderings

We characterize the absolute and relative max-mean inequality orderings using four axioms in addition to translation invariance and scale invariance, respectively.

For any $n \in \mathbb{N}$, an $n \times n$ matrix is *doubly stochastic* if all its elements are nonnegative and its rows and columns sum to one. Given $n \in \mathbb{N}$ and $x \in \Omega^n$, multiplying x by an $n \times n$ doubly stochastic matrix B yields an income distribution $Bx \in \Omega^n$ that has the same total income and is a smoothening of x in the sense that each component is a convex combination of x. Indeed, it is known that for any rank-ordered distribution $x \in \Omega^n$, Bx can be obtained by a finite sequence of progressive transfers (Hardy, Littlewood, and Pólya, 1934; Marshall and Olkin, 1979). The property of *Schur-convexity* (or *S-convexity,* for short) asserts that such a smoothening of an income distribution does not increase inequality.

S-convexity. For all $n \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $n \times n$ doubly stochastic matrices B, $xR(Bx)$.

Note that S-convexity is equivalent to the conjunction of anonymity and the well-known *Pigou-Dalton transfer principle* (Pigou, 1912; Dalton, 1920). Clearly, S-convexity is uncontroversial because the axiom captures the very notion of inequality measurement: if incomes move closer together, inequality cannot increase.

Continuity requires that small changes in incomes do not lead to large changes in inequality. This is another standard requirement commonly imposed on inequality orderings and other (ordinal) social indicators.

Continuity. For all $n \in \mathbb{N}$ and for all $x \in \Omega^n$, $\{y \in \Omega^n \mid yRx\}$ and $\{y \in \Omega^n \mid xRy\}$ are closed in Ω^n.

Replication invariance, which first appeared in Dalton (1920) under the name of the *principle of proportionate additions to persons,* requires that inequality be invariant under any k-fold replica of an income distribution.

Replication invariance. For all $n, k \in \mathbb{N}$ and for all $x \in \Omega^n$, $xI(\underbrace{x, \ldots, x}_{k \text{ times}})$.

Replication invariance in conjunction with translation invariance if $\Omega = \mathbb{R}$ or scale invariance if $\Omega = \mathbb{R}_{++}$ implies equality indifference. This can be verified as follows. Let $\Omega = \mathbb{R}$, $n, m \in \mathbb{N}$, and $\alpha, \beta \in \Omega^1$. Translation invariance implies $\alpha\mathbf{1}^n I \beta\mathbf{1}^n$. By replication invariance, we obtain $\beta\mathbf{1}^n I \beta\mathbf{1}^{nm}$ and $\beta\mathbf{1}^{nm} I \beta\mathbf{1}^m$. Since R is transitive, it follows that $\alpha\mathbf{1}^n I \beta\mathbf{1}^m$. Analogously, it can be verified that replication invariance and scale invariance together imply equality indifference if $\Omega = \mathbb{R}_{++}$.

The *composite transfer principle for top income* proposes specific consequences of a composition of rank-preserving progressive and regressive transfers involving three income recipients. Consider three individuals i, j, and n. Suppose that n is the best-off in the entire population and i is worse off than j. The axiom asserts that a composition of a progressive transfer from j to i and a regressive transfer from j to n increases inequality as long as the ranking of all individuals is preserved. This axiom strengthens an idea embodied in the *joint transfer axiom* in Sen (1974).

Composite transfer principle for top income. For all $n \in \mathbb{N}$ and for all $x, y \in \Omega^n$ with $x_k \le x_{k+1}$ and $y_k \le y_{k+1}$ for all $k \in \{1, \ldots, n-1\}$, if there exist $i, j \in \{1, \ldots, n-1\}$ with $i < j$ and $\delta, \varepsilon \in \mathbb{R}_{++}$ such that $x - y = \delta(e^i - e^j) + \varepsilon(e^n - e^j)$, then xPy.

The following theorem provides a preliminary result that is analogous to Theorem 1 of the previous section.

Theorem 5. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$ and suppose that R satisfies S-convexity, continuity, replication invariance, and the composite transfer principle for top income. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, if $\max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_m\}$ and $\mu(x) = \mu(y)$, then xIy.*

Proof. *Step 1.* Let $n \in \mathbb{N}$ with $n \ge 3$ and $x, y \in \Omega^n$ be such that $x_k \le x_{k+1}$ and $y_k \le y_{k+1}$ for all $k \in \{1, \ldots, n-1\}$, and suppose that there exist $\delta \in \mathbb{R}_{++}$ and $i, j \in \{1, \ldots, n-1\}$ with $i < j$ such that $x - y = \delta(e^i - e^j)$. We show that xRy.

Suppose, by way of contradiction, that xRy does not hold. Since R is complete, yPx holds. It follows from the completeness and continuity of R that $\{z \in \Omega^n \mid yPz\}$ is open and $x \in \{z \in \Omega^n \mid yPz\}$. Thus, there exists $\varepsilon \in \mathbb{R}_{++}$ such that $U_\varepsilon(x) \subseteq \{z \in \Omega^n \mid yPz\}$, where $U_\varepsilon(x)$ is the open ball with center at x and radius ε.

Let $\xi = \min\{\delta, \varepsilon\}/2$. Define $\bar{z} \in \Omega^n$ by $\bar{z}_i = x_i - \xi$, $\bar{z}_j = x_j + \xi/2$, $\bar{z}_n = x_n + \xi/2$, and $\bar{z}_k = x_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j, n\}$. Note that $\bar{z} - y = (\delta - \xi)(e^i - e^j) + (\xi/2)(e^n - e^j)$.

Furthermore, $\bar{z}_k \leq \bar{z}_{k+1}$ for all $k \in \{1, \ldots, n-1\}$. By the composite transfer principle for top income, we obtain $\bar{z}Py$. However, this is a contradiction since $\bar{z} \in U_\varepsilon(x) \subseteq \{z \in \Omega^n \mid yPz\}$.

Step 2. Let $n \in \mathbb{N}$ with $n \geq 2$ and $x, y \in \Omega^n$, and suppose that $\max\{x_1, \ldots, x_n\} > \max\{y_1, \ldots, y_n\}$ and $\mu(x) = \mu(y)$. We show that xRy.

Since S-convexity implies anonymity and R is transitive, we can without loss of generality assume that $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. We distinguish two cases.

(i) $n = 2$. Let $\delta = x_2 - y_2$. Since $y - x = \delta(e^1 - e^2)$, we obtain xRy by S-convexity.

(ii) $n \geq 3$. First, we define $\bar{x} \in \Omega^n$ by $\bar{x}_n = x_n$ and $\bar{x}_i = \sum_{i=1}^{n-1} x_i/(n-1)$ for all $i \in \{1, \ldots, n-1\}$. It follows from S-convexity that

$$x R \bar{x}.$$

We show that $\bar{x}Ry$, which proves that xRy because R is transitive. For any $z \in \Omega^n$, we define

$$B(z) = \{i \in \{1, \ldots, n-1\} \mid z_i > y_i\}$$

and

$$W(z) = \{i \in \{1, \ldots, n-1\} \mid z_i < y_i\}.$$

Note that $W(\bar{x}) \neq \varnothing$ since $\bar{x}_n = x_n > y_n$ and $\mu(\bar{x}) = \mu(x) = \mu(y)$. We further distinguish two cases.

(a) $B(\bar{x}) = \varnothing$. Since $\bar{x}_n > y_n$ and $\mu(\bar{x}) = \mu(y)$, $\bar{x}Ry$ follows from S-convexity.

(b) $B(\bar{x}) \neq \varnothing$. Note that there exist $\overline{m}, \underline{m} \in \{1, \ldots, n-1\}$ with $\overline{m} < \underline{m}$ such that

$$B(\bar{x}) = \{i \mid 1 \leq i \leq \overline{m}\} \text{ and } W(\bar{x}) = \{i \mid \underline{m} \leq i \leq n-1\}.$$

For all $i \in W(\bar{x})$, let

$$r_i = \frac{y_i - \bar{x}_i}{\sum_{j \in W(\bar{x})}(y_j - \bar{x}_j)}.$$

We define $\tilde{x} \in \Omega^n$ by $\tilde{x}_i = \bar{x}_i$ for all $i \in \{1, \ldots, n-1\} \backslash W(\bar{x})$, $\tilde{x}_i = \bar{x}_i + r_i(x_n - y_n)$ for all $i \in W(\bar{x})$, and $\tilde{x}_n = y_n$. It follows from S-convexity that

$$\bar{x} R \tilde{x}.$$

Note that $B(\tilde{x}) = B(\bar{x})$ and $W(\tilde{x}) = W(\bar{x})$ since $\mu(\bar{x}) = \mu(y)$ and $B(\bar{x}) \neq \varnothing$. Further, $\tilde{x}_i \leq \tilde{x}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. Since

$$\sum_{i \in B(\tilde{x}) \cup W(\tilde{x})} \tilde{x}_i = \sum_{i \in B(\tilde{x}) \cup W(\tilde{x})} y_i,$$

y is obtained from $\tilde{x}$ by a finite sequence of rank-preserving regressive transfers from individuals in $B(\tilde{x})$ to individuals $W(\tilde{x})$ choosing individuals in $B(\tilde{x})$ in ascending order and those in $W(\tilde{x})$ in descending order, respectively. Thus, it follows from Step 1 and the transitivity of R that

$$\tilde{x} R y.$$

Since R is transitive, we obtain $\bar{x}Ry$.

Step 3. Let $n \in \mathbb{N}$ and $x, y \in \Omega^n$, and suppose that $\max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_n\}$ and $\mu(x) = \mu(y)$. We show that xIy.

Again, from S-convexity and the transitivity of R, it follows that we can without loss of generality assume that $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for all $i \in \{1, \ldots, n-1\}$.

If $n = 1$, xIy follows from the reflexivity of R.

Now consider the case where $n \geq 2$. If $x_n = x_1$, then $x = y = (\mu(x), \ldots, \mu(x))$. Thus, it follows from the reflexivity of R that xIy.

In what follows, we assume that $x_n > x_1$, which implies $y_n > y_1$ as well. Suppose, by way of contradiction, that xIy does not hold. Without loss of generality, we assume yPx. Since R is complete and satisfies continuity, $\{z \in \Omega^n \mid yPz\}$ is open and $x \in \{z \in \Omega^n \mid yPz\}$. Thus, there exists $\varepsilon \in \mathbb{R}_{++}$ such that $U_\varepsilon(x) \subseteq \{z \in \Omega^n \mid yPz\}$. We define $\bar{z} \in \Omega^n$ by $\bar{z}_1 = x_1 - \varepsilon/2$, $\bar{z}_n = x_n + \varepsilon/2$, and $\bar{z}_i = x_i$ for all $i \in \{2, \ldots, n-1\}$. Note that $\bar{z}_i \leq \bar{z}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. Furthermore, $\bar{z}_n > x_n = y_n$ and $\mu(\bar{z}) = \mu(x) = \mu(y)$. Thus, it follows from Step 2 that $\bar{z}Ry$. However, this is a contradiction since $\bar{z} \in U_\varepsilon(x) \subseteq \{z \in \Omega^n \mid yPz\}$.

Step 4. We complete the proof. Let $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$. Suppose that $\max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_m\}$ and $\mu(x) = \mu(y)$. Let $\ell = nm$ and define $z, w \in \mathbb{R}^\ell$ by

$$z = (\underbrace{x, \ldots, x}_{m \text{ times}}) \text{ and } w = (\underbrace{y, \ldots, y}_{n \text{ times}}).$$

Note that

$$\max\{z_1, \ldots, z_\ell\} = \max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_m\} = \max\{w_1, \ldots, w_\ell\}$$

and

$$\mu(z) = \mu(x) = \mu(y) = \mu(w).$$

It follows from Step 3 that zIw. Since R satisfies replication invariance, we obtain xIz and yIw. Because R is transitive, xIy follows. ■

Parallel to Theorem 2, the following result characterizes all inequality orderings that satisfy the axioms introduced in this subsection. As the theorem shows, these orderings only utilize the maximum and average incomes and are increasing in the maximum income.

Theorem 6. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. R satisfies S-convexity, continuity, replication invariance, and the composite transfer principle for top income if and only if there exists a continuous ordering $\succsim$ on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \Leftrightarrow (\max\{x_1, \ldots, x_n\}, \mu(x)) \succsim (\max\{y_1, \ldots, y_m\}, \mu(y));$$

(ii) $\succsim$ is increasing in its first argument.

Proof. 'If.' Suppose that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in the theorem statement.

From property (i), R satisfies replication invariance.

Further, by properties (i) and (ii), R satisfies the composite transfer principle for top income.

To show that R satisfies S-convexity, let $n \in \mathbb{N}$, $x \in \Omega^n$, and B be an $n \times n$ doubly stochastic matrix. Since

$$\max\{(Bx)_1, \ldots, (Bx)_n\} \leq \max\{x_1, \ldots, x_n\} \text{ and } \mu(Bx) = \mu(x),$$

it follows from properties (i) and (ii) that $xR(Bx)$.

Next, to show that R satisfies continuity, let $n \in \mathbb{N}$ and $x \in \Omega^n$. We show that $\{y \in \Omega^n \mid yRx\}$ is closed in Ω^n. Let $\langle z^t \rangle_{t \in \mathbb{N}}$ be a sequence of vectors in $\{y \in \Omega^n \mid yRx\}$ and suppose that $\langle z^t \rangle_{t \in \mathbb{N}}$ converges to z. From property (i), it follows that, for all $t \in \mathbb{N}$,

$$(\max\{z_1^t, \ldots, z_n^t\}, \mu(z^t)) \succsim (\max\{x_1, \ldots, x_n\}, \mu(x)).$$

Since

$$\lim_{t \to \infty} \max\{z_1^t, \ldots, z_n^t\} = \max\{z_1, \ldots, z_n\} \text{ and } \lim_{t \to \infty} \mu(z^t) = \mu(z),$$

it follows from the continuity of $\succsim$ that

$$(\max\{z_1, \ldots, z_n\}, \mu(z)) \succsim (\max\{x_1, \ldots, x_n\}, \mu(x)).$$

From property (i), we obtain zRx. The proof that $\{y \in \Omega^n \mid xRy\}$ is closed in Ω^n is analogous.

'Only if.' Define the binary relation $\succsim$ on S by letting, for all $(\alpha, \beta), (\alpha', \beta') \in S$,

$$(\alpha, \beta) \succsim (\alpha', \beta')$$

if and only if there exist $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$ such that xRy and

$$\alpha = \max\{x_1, \ldots, x_n\}, \; \beta = \mu(x), \; \alpha' = \max\{y_1, \ldots, y_m\}, \; \beta' = \mu(y).$$

To show that property (i) is satisfied, let $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$. By the definition of $\succsim$,

$$xRy \;\Rightarrow\; (\max\{x_1, \ldots, x_n\}, \mu(x)) \succsim (\max\{y_1, \ldots, y_m\}, \mu(y)).$$

To show that the converse implication is true, suppose that

$$(\max\{x_1, \ldots, x_n\}, \mu(x)) \succsim (\max\{y_1, \ldots, y_m\}, \mu(y)).$$

By the definition of $\succsim$, there exist $\tilde{n}, \tilde{m} \in \mathbb{N}$, $\tilde{x} \in \Omega^{\tilde{n}}$, and $\tilde{y} \in \Omega^{\tilde{m}}$ such that $\tilde{x}R\tilde{y}$ and

$$\max\{x_1, \ldots, x_n\} = \max\{\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}\} \text{ and } \mu(x) = \mu(\tilde{x}),$$
$$\max\{y_1, \ldots, y_m\} = \max\{\tilde{y}_1, \ldots, \tilde{y}_{\tilde{m}}\} \text{ and } \mu(y) = \mu(\tilde{y}).$$

By Theorem 5, $xI\tilde{x}$ and $yI\tilde{y}$. Since R is transitive, we obtain xRy. Thus, $\succsim$ satisfies property (i).

Next, we show that $\succsim$ is an ordering on S. To this end, we show that, for any $(\alpha, \beta) \in S$, there exist $n \in \mathbb{N}$ and $x \in \Omega^n$ such that

$$\max\{x_1, \ldots, x_n\} = \alpha \text{ and } \mu(x) = \beta. \tag{2}$$

Let $(\alpha, \beta) \in S$ and $n \in \mathbb{N}$ with $n \geq 2$. We define $x \in \mathbb{R}^n$ by

$$x_n = \alpha \text{ and } x_i = \frac{n\beta - \alpha}{n-1} = \beta - \frac{\alpha - \beta}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}.$$

Note that x satisfies (2). It is straightforward that $x \in \Omega^n$ if $\Omega = \mathbb{R}$. We now suppose that $\Omega = \mathbb{R}_{++}$. Assuming that n is sufficiently large so that it satisfies

$$\beta > \frac{\alpha}{n},$$

it follows that, for all $i \in \{1, \ldots, n-1\}$,

$$x_i = \frac{n\beta - \alpha}{n-1} > \frac{\alpha - \alpha}{n-1} = 0.$$

Since $\alpha \geq \beta$, we obtain $x_n > 0$. Thus, $x \in \Omega^n$. Since R is an ordering and $\succsim$ satisfies property (i), $\succsim$ is an ordering on S.

Now we prove that $\succsim$ is continuous. Let $(\alpha, \beta) \in S$ and consider any sequence $\langle(\alpha^t, \beta^t)\rangle_{t\in\mathbb{N}}$ in $\{(\alpha', \beta') \in S \mid (\alpha', \beta') \succsim (\alpha, \beta)\}$ that converges to $(\alpha^*, \beta^*) \in S$. Let $n \in \mathbb{N}$ with $n \geq 2$. We define the sequence $\langle x^t\rangle_{t\in\mathbb{N}}$ in $\mathbb{R}^n$ by

$$x_n^t = \alpha^t \text{ and } x_i^t = \frac{n\beta^t - \alpha^t}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}.$$

Similarly, define $x, x^* \in \mathbb{R}^n$ by

$$x_n = \alpha \text{ and } x_i = \frac{n\beta - \alpha}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}$$

and

$$x_n^* = \alpha^* \text{ and } x_i^* = \frac{n\beta^* - \alpha^*}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}.$$

It follows that

$$\max\{x_1, \ldots, x_n\} = \alpha,\ \mu(x) = \beta,\ \max\{x_1^*, \ldots, x_n^*\} = \alpha^*,\ \mu(x^*) = \beta^*,$$

and, for all $t \in \mathbb{N}$,

$$\max\{x_1^t, \ldots, x_2^t\} = \alpha^t \text{ and } \mu(x^t) = \beta^t.$$

First, we suppose that $\Omega = \mathbb{R}$. Then, $\langle x^t\rangle_{t\in\mathbb{N}}$ is a sequence in Ω^n and $x, x^* \in \Omega^n$. Since $(\alpha^t, \beta^t) \succsim (\alpha, \beta)$ for all $t \in \mathbb{N}$, it follows from property (i) of $\succsim$ that x^tRx for all $t \in \mathbb{N}$. Since $\langle x^t\rangle_{t\in\mathbb{N}}$ converges to x^* and R satisfies continuity, we obtain x^*Rx. From property

(i) of $\succsim$, we obtain $(\alpha^*, \beta^*) \succsim (\alpha, \beta)$. Thus, $\{(\alpha', \beta') \in S \mid (\alpha', \beta') \succsim (\alpha, \beta)\}$ is closed. The proof that $\{(\alpha', \beta') \in S \mid (\alpha, \beta) \succsim (\alpha', \beta')\}$ is closed is analogous.

Now suppose that $\Omega = \mathbb{R}_{++}$. Since $\langle(\alpha^t, \beta^t)\rangle_{t\in\mathbb{N}}$ converges to (α^*, β^*), there exist $t^* \in \mathbb{N}$ and a sufficiently small $\varepsilon \in \mathbb{R}_{++}$ such that, for all $t \geq t^*$,

$$\alpha^* - \varepsilon < \alpha^t < \alpha^* + \varepsilon \text{ and } 0 < \beta^* - \varepsilon < \beta^t < \beta^* + \varepsilon.$$

Let

$$\lambda^* = \frac{\alpha^* + \varepsilon}{\beta^* - \varepsilon} \text{ and } \lambda = \frac{\alpha}{\beta}.$$

Further, let $\Lambda = \max\{\lambda^*, \lambda\}$. Note that

$$\frac{\alpha}{\beta} \leq \Lambda \text{ and } \frac{\alpha^*}{\beta^*} \leq \Lambda$$

and, for all $t \geq t^*$,

$$\frac{\alpha^t}{\beta^t} \leq \Lambda.$$

Thus, assuming that n is sufficiently large so that it satisfies $n > \Lambda$, it follows that $x_i^t, x_i, x_i^* \in \mathbb{R}_{++}$ for all $i \in \{1, \ldots, n-1\}$ and for all $t \geq t^*$. Therefore, $\langle x^{t^*+\ell}\rangle_{\ell\in\mathbb{N}}$ is a sequence in Ω^n and $x, x^* \in \Omega^n$. Since $(\alpha^t, \beta^t) \succsim (\alpha, \beta)$ for all $t \in \mathbb{N}$, it follows from property (i) of $\succsim$ that $x^{t^*+\ell}Rx$ for all $\ell \in \mathbb{N}$. Since $\langle x^{t^*+\ell}\rangle_{\ell\in\mathbb{N}}$ converges to x^* and R satisfies continuity, we obtain x^*Rx. From property (i) of $\succsim$, we obtain $(\alpha^*, \beta^*) \succsim (\alpha, \beta)$. Thus, $\{(\alpha', \beta') \in S \mid (\alpha', \beta') \succsim (\alpha, \beta)\}$ is closed. The proof that $\{(\alpha', \beta') \in S \mid (\alpha, \beta) \succsim (\alpha', \beta')\}$ is closed is analogous.

Finally, to show that $\succsim$ satisfies property (ii), let $(\alpha, \beta), (\alpha', \beta) \in S$ and suppose $\alpha > \alpha'$. Let $n \in \mathbb{N}$ with $n \geq 3$. We define $x, y \in \mathbb{R}^n$ by

$$x_n = \alpha \text{ and } x_i = \frac{n\beta - \alpha}{n-1} = \beta - \frac{\alpha - \beta}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}$$

and

$$y_n = \alpha' \text{ and } y_i = \frac{n\beta - \alpha'}{n-1} = \beta - \frac{\alpha' - \beta}{n-1} \text{ for all } i \in \{1, \ldots, n-1\}.$$

Note that $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ because $\alpha > \alpha' \geq \beta$. Let $\delta = \alpha - \alpha' > 0$. Then, for all $i \in \{1, \ldots, n-1\}$,

$$y_\ell - x_\ell = \frac{\delta}{n-1}.$$

Since $\max\{x_1, \ldots, x_n\} = \alpha$, $\max\{y_1, \ldots, y_n\} = \alpha'$, $\mu(x) = \mu(y) = \beta$, and $\succsim$ satisfies property (i), it suffices to show that $x, y \in \Omega^n$ and xPy.

First, we assume that $\Omega = \mathbb{R}$. To show that xPy, let $\varepsilon \in \mathbb{R}_{++}$ be such that

$$\varepsilon < \frac{\delta}{n-1}$$

and define $z \in \mathbb{R}^n$ by

$$\left.\begin{aligned} z_n &= y_n + \frac{1}{2}\left(\frac{n-2}{n-1}\delta + \varepsilon\right) = x_n - \frac{1}{2}\left(\frac{n}{n-1}\delta - \varepsilon\right) < x_n, \\ z_{n-1} &= y_{n-1} + \frac{1}{2}\left(\frac{n-2}{n-1}\delta + \varepsilon\right) = x_{n-1} + \frac{1}{2}\left(\frac{n}{n-1}\delta + \varepsilon\right) > x_{n-1}, \\ z_1 &= y_1 - \frac{\delta}{n-1} - \varepsilon = x_1 - \varepsilon < x_1, \\ z_\ell &= y_\ell - \frac{\delta}{n-1} = x_\ell \text{ for all } \ell \in \{2, \dots, n-2\}. \end{aligned}\right\} \tag{3}$$

Note that $z \in \Omega^n$ and $z_i \leq z_{i+1}$ for all $i \in \{1, \dots, n-1\}$. Furthermore, $\sum_{i=1}^n y_i = \sum_{i=1}^n z_i$. From S-convexity, it follows that zRy. Since

$$x - z = \varepsilon(e^1 - e^{n-1}) + \frac{1}{2}\left(\frac{n}{n-1}\delta - \varepsilon\right)(e^n - e^{n-1}),$$

it follows from the composite transfer principle for top income that xPz. Since R is transitive, we obtain xPy.

Next, we suppose that $\Omega = \mathbb{R}_{++}$. Assuming that n is sufficiently large so that it satisfies

$$\beta > \frac{\alpha}{n},$$

it follows that $x, y \in \Omega^n$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that

$$\varepsilon < \min\left\{\frac{\delta}{n-1}, x_1\right\}$$

and define $z \in \mathbb{R}^n$ by (3). By the same argument as in the case where $\Omega = \mathbb{R}$, we obtain xPy. ■

The subsection is concluded with characterizations of the absolute and relative max-mean inequality orderings.

Theorem 7. *Let $\Omega = \mathbb{R}$. R satisfies S-convexity, continuity, replication invariance, the composite transfer principle for top income, and translation invariance if and only if $R = R^a_{x\mu}$.*

Proof. From Theorem 6, it follows that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in Theorem 6. Thus, we can prove that $R = R^a_{x\mu}$ applying the same argument as in the proof of Theorem 3 using $\delta = -\mu(x)$ instead of $\delta = -\min\{x_1, \dots, x_n\}$. ■

Theorem 8. *Let $\Omega = \mathbb{R}_{++}$. R satisfies S-convexity, continuity, replication invariance, the composite transfer principle for top income, and scale invariance if and only if $R = R^r_{x\mu}$.*

Proof. The proof that $R = R^r_{x\mu}$ is analogous to the proof of Theorem 7 using the same argument as in the proof of Theorem 4. ■

3.3 Mean-min inequality orderings

We characterize the absolute and relative mean-min inequality orderings using an axiom dual to the composite transfer principle for top income, which we call the *composite transfer principle for bottom income.* Consider again three individuals i, j, and 1. Now suppose that j is better-off than i and 1 is the worst-off in the entire population. The composite transfer principle for bottom income asserts that a composition of a progressive transfer from i to 1 and a regressive transfer from i to j decreases inequality as long as the ranking of all individuals is preserved. This axiom is similar to the *transfer sensitivity* axiom in Shorrocks and Foster (1987). See also Kamaga (2018) and Bossert and Kamaga (2020). In the context of welfare measurement, the property employed by these authors is implied by the conjunction of the strong Pareto principle and the well-known axiom of *Hammond equity;* see Hammond (1979, p. 1132).

Composite transfer principle for bottom income. For all $n \in \mathbb{N}$ and for all $x, y \in \Omega^n$ with $x_k \leq x_{k+1}$ and $y_k \leq y_{k+1}$ for all $k \in \{1, \ldots, n-1\}$, if there exist $i, j \in \{2, \ldots, n\}$ with $i < j$ and $\delta, \varepsilon \in \mathbb{R}_{++}$ such that $x - y = \delta(e^1 - e^i) + \varepsilon(e^j - e^i)$, then yPx.

In analogy to the previous subsections, we begin with a preliminary result. This is followed by a characterization of all inequality orderings that satisfy the axioms of the previous subsection when the composite transfer principle for top income is replaced with the corresponding principle for bottom income.

Theorem 9. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$ and suppose that R satisfies S-convexity, continuity, replication invariance, and the composite transfer principle for bottom income. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, if $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_m\}$ and $\mu(x) = \mu(y)$, then xIy.*

Proof. *Step 1.* Let $n \in \mathbb{N}$ with $n \geq 3$ and $x, y \in \Omega^n$ be such that $x_k \leq x_{k+1}$ and $y_k \leq y_{k+1}$ for all $k \in \{1, \ldots, n-1\}$. Suppose there exist $i, j \in \{2, \ldots, n\}$ with $i < j$ and $\varepsilon \in \mathbb{R}_{++}$ such that $x - y = \varepsilon(e^j - e^i)$. We show that yRx.

Suppose, by way of contradiction, that yRx does not hold. Since R is complete, we obtain xPy. It follows from the completeness and continuity of R that $\{z \in \Omega^n \mid xPz\}$ is open and $y \in \{z \in \Omega^n \mid xPz\}$. Thus, there exists $\delta \in \mathbb{R}_{++}$ such that $U_\delta(y) \subseteq \{z \in \Omega^n \mid xPz\}$.

Let $\xi = \min\{\delta, \varepsilon\}/2$. Define $\bar{z} \in \Omega^n$ by $\bar{z}_1 = y_1 - \xi/2$, $\bar{z}_i = y_i - \xi/2$, $\bar{z}_j = y_j + \xi$, and $\bar{z}_k = y_k$ for all $k \in \{1, \ldots, n\} \setminus \{1, i, j\}$. Note that $x - \bar{z} = (\xi/2)(e^1 - e^i) + (\varepsilon - \xi)(e^j - e^i)$. Furthermore, $\bar{z}_k \leq \bar{z}_{k+1}$ for all $k \in \{1, \ldots, n-1\}$. By the composite transfer principle for bottom income, we obtain $\bar{z}Px$. However, this is a contradiction since $\bar{z} \in U_\delta(y) \subseteq \{z \in \Omega^n \mid xPz\}$.

Step 2. Let $n \in \mathbb{N}$ with $n \geq 2$ and $x, y \in \Omega^n$. We suppose that $\min\{x_1, \ldots, x_n\} > \min\{y_1, \ldots, y_n\}$ and $\mu(x) = \mu(y)$ and show that yRx.

Since S-convexity implies anonymity and R is transitive, we assume that x and y are arranged in ascending order, so that $\min\{x_1, \ldots, x_n\} = x_1$ and $\min\{y_1, \ldots, y_n\} = y_1$.

If $n = 2$, we immediately obtain yRx from S-convexity.

Now assume that $n \geq 3$. We define $\bar{y} \in \Omega^n$ by $\bar{y}_1 = y_1$ and $\bar{y}_i = \sum_{j=2}^{n} y_j/(n-1)$ for all $i \in \{2, \ldots, n\}$. From S-convexity, it follows that

$$y R \bar{y}.$$

For any $z \in \Omega^n$, we define $B(z)$ and $W(z)$ by

$$B(z) = \{i \in \{2, \ldots, n\} \mid z_i > x_i\}$$

and

$$W(z) = \{i \in \{2, \ldots, n\} \mid z_i < x_i\}.$$

Note that $B(\bar{y}) \neq \varnothing$ since $x_1 > \bar{y}_1$ and $\mu(x) = \mu(\bar{y})$. We distinguish two cases.

(a) $W(\bar{y}) = \varnothing$. It follows from S-convexity that $\bar{y} R x$. Since R is transitive, we obtain yRx.

(b) $W(\bar{y}) \neq \varnothing$. Then there exist $\overline{m}, \underline{m} \in \{2, \ldots, n\}$ with $\overline{m} < \underline{m}$ such that

$$B(\bar{y}) = \{i \mid 2 \leq i \leq \overline{m}\} \text{ and } W(\bar{y}) = \{i \mid \underline{m} \leq i \leq n\}.$$

For each $i \in B(\bar{y})$, define r_i by

$$r_i = \frac{\bar{y}_i - x_i}{\sum_{j \in B(\bar{y})} (\bar{y}_j - x_j)}.$$

We define $\tilde{y} \in \Omega^n$ by $\tilde{y}_i = \bar{y}_i$ for all $i \in \{2, \ldots, n\} \backslash B(\bar{y})$, $\tilde{y}_i = \bar{y}_i - r_i(x_1 - y_1)$ for all $i \in B(\bar{y})$, and $\tilde{y}_1 = x_1$. From S-convexity, we obtain

$$\bar{y} R \tilde{y}.$$

Note that $B(\tilde{y}) = B(\bar{y})$ and $W(\tilde{y}) = W(\bar{y})$. Further, $\tilde{y}_k \leq \tilde{y}_{k+1}$ for all $k \in \{1, \ldots, n-1\}$. By the construction of $\tilde{y}$, x is obtained from $\tilde{y}$ by a finite sequence of regressive transfers from individuals in $B(\tilde{y})$ to individuals in $W(\tilde{y})$ choosing individuals in $B(\tilde{y})$ in ascending order and those in $W(\tilde{y})$ in descending order, respectively. Thus, it follows from Step 1 and the transitivity of R that

$$\tilde{y} R x.$$

Since R is transitive, we obtain yRx.

Step 3. Let $n \in \mathbb{N}$ and $x, y \in \Omega^n$, and suppose that $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_n\}$ and $\mu(x) = \mu(y)$. We show that xIy.

We prove this claim by employing the same argument as in Step 3 of the proof of Theorem 5. Specifically, by the definition of $\bar{z} \in \Omega^n$ in Step 3 of the proof of Theorem 5, we obtain $\bar{z}_1 < x_1 = y_1$ and $\mu(\bar{z}) = \mu(x) = \mu(y)$. Thus, using Step 2, the proof is analogous to Step 3 of the proof of Theorem 5.

Step 4. Let $n, m \in \mathbb{N}$, $x \in \Omega^n$, $y \in \Omega^m$, and suppose that $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_m\}$ and $\mu(x) = \mu(y)$. Applying the same argument as in Step 4 of the proof of Theorem 5, it follows that xIy. ■

Note that, unlike Theorems 1, 2, and 3, the proof of Theorem 9 does not apply if $\Omega = \mathbb{R}$ is replaced with $\Omega = \mathbb{R}_+$; this is the case because Step 1 cannot be established on this alternative domain. For that reason, we allow for negative income values in the absolute case.

The following theorem axiomatizes the class of continuous inequality orderings that only utilize the mean and minimum incomes and are decreasing in the minimum income.

Theorem 10. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. R satisfies S-convexity, continuity, replication invariance, and the composite transfer principle for bottom income if and only if there exists a continuous ordering $\succsim$ on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \Leftrightarrow (\mu(x), \min\{x_1, \ldots, x_n\}) \succsim (\mu(y), \min\{y_1, \ldots, y_m\});$$

(ii) $\succsim$ is decreasing in its second argument.

Proof. 'If.' Suppose that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in the theorem statement. From properties (i) and (ii), R satisfies the composite transfer principle for bottom income. Further, R satisfies S-convexity since for any $n \in \mathbb{N}$, any $x \in \Omega^n$, and any $n \times n$ doubly stochastic matrix B,

$$\min\{(Bx)_1, \ldots, (Bx)_n\} \geq \min\{x_1, \ldots, x_n\} \text{ and } \mu(Bx) = \mu(x).$$

The proof that R satisfies continuity and replication invariance is analogous to the proof of Theorem 6.

'Only if.' The proof of the existence of the binary relation $\succsim$ on S satisfying property (i) is analogous to the proof of Theorem 6.

To show that $\succsim$ satisfies property (ii), let $(\alpha, \beta), (\alpha, \beta') \in S$ and suppose that $\beta > \beta'$. Let $n \in \mathbb{N}$ with $n \geq 3$. We define $x, y \in \mathbb{R}^n$ by

$$x_1 = \beta \text{ and } x_i = \frac{n\alpha - \beta}{n-1} = \alpha + \frac{\alpha - \beta}{n-1} \text{ for all } i \in \{2, \ldots, n\}$$

and

$$y_1 = \beta' \text{ and } y_i = \frac{n\alpha - \beta'}{n-1} = \alpha + \frac{\alpha - \beta'}{n-1} \text{ for all } i \in \{2, \ldots, n\}.$$

Note that $x, y \in \Omega^n$, $x_i \leq x_{i+1}$, and $y_i \leq y_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. Since $\min\{x_1, \ldots, x_n\} = \beta$, $\min\{y_1, \ldots, y_n\} = \beta'$, $\mu(x) = \mu(y) = \alpha$, and $\succsim$ satisfies property (i), it suffices to show that yPx.

Let $\delta = \beta - \beta' > 0$. Then, for all $i \in \{2, \ldots, n\}$,

$$y_i - x_i = \frac{\delta}{n-1}.$$

Let $\varepsilon \in \mathbb{R}_{++}$ be such that

$$\varepsilon < \frac{\delta}{n-1}.$$

We define $z \in \mathbb{R}^n$ by

$$\begin{aligned}
z_1 &= x_1 - \frac{1}{2}\left(\frac{n-2}{n-1}\delta + \varepsilon\right) = y_1 + \frac{1}{2}\left(\frac{n}{n-1}\delta - \varepsilon\right) > y_1, \\
z_2 &= x_2 - \frac{1}{2}\left(\frac{n-2}{n-1}\delta + \varepsilon\right) = y_2 - \frac{1}{2}\left(\frac{n}{n-1}\delta + \varepsilon\right) < y_2, \\
z_n &= x_n + \frac{\delta}{n-1} + \varepsilon = y_n + \varepsilon > y_n,
\end{aligned}$$

and

$$z_i = x_i + \frac{\delta}{n-1} = y_i$$

for all $i \in \{3, \ldots, n-1\}$. Note that $z \in \Omega^n$ and $z_i \leq z_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. It follows from S-convexity that

$$zRx.$$

Since

$$z - y = \frac{1}{2}\left(\frac{n}{n-1}\delta - \varepsilon\right)(e^1 - e^2) + \varepsilon(e^n - e^2),$$

it follows from the composite transfer principle for bottom income that yPz. Since R is transitive, we obtain yPx.

In either case (that is, $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}_{++}$), for any $(\alpha, \beta) \in S$ and for any $n \in \mathbb{N}$ with $n \geq 2$, the vector $x \in \mathbb{R}^n$ defined by $x_1 = \beta$ and $x_i = (n\alpha - \beta)/(n-1)$ for all $i \in \{2, \ldots, n\}$ satisfies $x \in \Omega^n$, $\mu(x) = \alpha$, and $\min\{x_1, \ldots, x_n\} = \beta$. Therefore, the proof that $\succsim$ is a continuous ordering on S is analogous to the corresponding proof in Theorem 6 presented for the case where $\Omega = \mathbb{R}$. ■

Finally, we characterize the absolute and relative mean-min inequality orderings.

Theorem 11. *Let $\Omega = \mathbb{R}$. R satisfies S-convexity, continuity, replication invariance, the composite transfer principle for bottom income, and translation invariance if and only if $R = R^a_{\mu n}$.*

Proof. From Theorem 10, it follows that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in Theorem 10. Applying the same argument as in the proof of Theorem 3 using $\delta = -\mu(x)$ instead of $\delta = -\min\{x_1, \ldots, x_n\}$, we obtain that, for any $n, m \in \mathbb{N}$, for any $x \in \Omega^n$, and for any $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; (0, \min\{x_1, \ldots, x_n\} - \mu(x)) \succsim (0, \min\{y_1, \ldots, y_m\} - \mu(y)).$$

Since $\succsim$ is decreasing in its second argument, this is equivalent to

$$\begin{aligned}
xRy &\Leftrightarrow \min\{x_1, \ldots, x_n\} - \mu(x) \leq \min\{y_1, \ldots, y_m\} - \mu(y) \\
&\Leftrightarrow \mu(x) - \min\{x_1, \ldots, x_n\} \geq \mu(y) - \min\{y_1, \ldots, y_m\} \\
&\Leftrightarrow xR^a_{\mu n}y. \;\blacksquare
\end{aligned}$$

Theorem 12. *Let $\Omega = \mathbb{R}_{++}$. R satisfies S-convexity, continuity, replication invariance, the composite transfer principle for bottom income, and scale invariance if and only if $R = R^r_{\mu n}$.*

Proof. From Theorem 10, it follows that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in Theorem 10. Applying the same argument as in the proof of Theorem 4 using $\lambda = 1/\mu(x)$ instead of $\lambda = 1/\min\{x_1, \ldots, x_n\}$, we obtain that, for any $n, m \in \mathbb{N}$, for any $x \in \Omega^n$, and for any $y \in \Omega^m$,

$$xRy \Leftrightarrow \left(1, \frac{\min\{x_1, \ldots, x_n\}}{\mu(x)}\right) \succsim \left(1, \frac{\min\{y_1, \ldots, y_m\}}{\mu(y)}\right).$$

Since $\succsim$ is decreasing in its second argument, this is equivalent to

$$\begin{aligned} xRy &\Leftrightarrow \frac{\min\{x_1, \ldots, x_n\}}{\mu(x)} \leq \frac{\min\{y_1, \ldots, y_m\}}{\mu(y)} \\ &\Leftrightarrow \frac{\mu(x)}{\min\{x_1, \ldots, x_n\}} \geq \frac{\mu(y)}{\min\{y_1, \ldots, y_m\}} \\ &\Leftrightarrow xR^r_{\mu n}y. \blacksquare \end{aligned}$$

3.4 Top income gaps and shares

We begin by presenting the restatements of S-convexity, continuity, and replication invariance defined on the requisite domain.

S-convexity*. For all $n \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $nq \times nq$ doubly stochastic matrices B, $xR(Bx)$.

Continuity*. For all $n \in \mathbb{N}$ and for all $x \in \Omega^{nq}$, $\{y \in \Omega^{nq} \mid yRx\}$ and $\{y \in \Omega^{nq} \mid xRy\}$ are closed in Ω^{nq}.

Replication invariance*. For all $n, k \in \mathbb{N}$ and for all $x \in \Omega^{nq}$, $xI(\underbrace{x, \ldots, x}_{k \text{ times}})$.

Transfer neutrality within quantiles postulates a consequence of a transfer between individuals in the same quantile. It requires that inequality be invariant with respect to a transfer within a quantile as long as the individuals involved remain in the same quantile. This axiom is an inequality-measurement analogue of the *incremental-equity* property introduced by Blackorby, Bossert, and Donaldson (2002) in the context of welfare measurement.

Transfer neutrality within quantiles. For all $n \in \mathbb{N}$ and for all $x, y \in \Omega^{nq}$, if $G_\ell(x) = G_\ell(y)$ for all $\ell \in \{1, \ldots, q\}$ and there exist $\ell' \in \{1, \ldots, q\}$ and $i, j \in G_{\ell'}(x)$ such that $x_i - y_i = y_j - x_j$ and $x_k = y_k$ for all $k \in \{1, \ldots, nq\} \setminus \{i, j\}$, then xIy.

The following theorem characterizes the class of inequality orderings that satisfy the four axioms presented above. As the theorem shows, this class consists of all continuous and S-convex orderings that only utilize the mean incomes of the quantiles.

Theorem 13. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. R satisfies S-convexity*, replication invariance*, continuity*, and transfer neutrality within quantiles if and only if there exists a continuous and S-convex ordering $\succsim^*$ on $S^* = \{z \in \Omega^q \mid z_\ell \leq z_{\ell+1}$ for all $\ell \in \{1, \ldots, q-1\}\}$ such that, for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,*

$$xRy \;\Leftrightarrow\; (\mu_1(x), \ldots, \mu_q(x)) \succsim^* (\mu_1(y), \ldots, \mu_q(y)). \tag{4}$$

Proof. 'If.' Suppose that there exists a continuous and S-convex ordering $\succsim^*$ on S^* satisfying (4). First, we show that R satisfies S-convexity*. Let $n \in \mathbb{N}$ and $x, y \in \Omega^{nq}$. Suppose that there exists an $nq \times nq$ doubly stochastic matrix B such that $y = Bx$. We show that xRy. Since $y = Bx$, it follows that, for all $k \in \{1, \ldots, nq\}$,

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \text{ and } \sum_{i=1}^{nq} x_{(i)} = \sum_{i=1}^{nq} y_{(i)};$$

see, for example, Hardy, Littlewood, and Pólya (1934), Marshall and Olkin (1979), and Dasgupta, Sen, and Starrett (1973). Thus, we obtain that, for all $k \in \{1, \ldots, q\}$,

$$\sum_{\ell=1}^{k} \mu_\ell(x) \leq \sum_{\ell=1}^{k} \mu_\ell(y) \text{ and } \sum_{\ell=1}^{q} \mu_\ell(x) = \sum_{\ell=1}^{q} \mu_\ell(y),$$

which implies that there exists a $q \times q$ doubly stochastic matrix B^* such that

$$B^*(\mu_1(x), \ldots, \mu_q(x)) = (\mu_1(y), \ldots, \mu_q(y)).$$

Since $\succsim^*$ is S-convex, we obtain

$$(\mu_1(x), \ldots, \mu_q(x)) \succsim^* (\mu_1(y), \ldots, \mu_q(y)).$$

From (4), xRy follows.

Next, we show that R satisfies continuity*. Let $n \in \mathbb{N}$ and $x \in \Omega^{nq}$. We show that $\{y \in \Omega^{nq} \mid yRx\}$ is closed in Ω^{nq}. Let $\langle z^t \rangle_{t \in \mathbb{N}}$ be a sequence of vectors in $\{y \in \Omega^{nq} \mid yRx\}$ and suppose that $\langle z^t \rangle_{t \in \mathbb{N}}$ converges to z. Since $z^t Rx$ for all $t \in \mathbb{N}$, it follows from (4) that, for all $t \in \mathbb{N}$,

$$(\mu_1(z^t), \ldots, \mu_q(z^t)) \succsim^* (\mu_1(x), \ldots, \mu_q(x)).$$

Since, for each $\ell \in \{1, \ldots, q\}$,

$$\lim_{t \to \infty} \mu_\ell(z^t) = \mu_\ell(z),$$

it follows from the continuity of $\succsim^*$ that

$$(\mu_1(z^*), \ldots, \mu_q(z^*)) \succsim^* (\mu_1(x), \ldots, \mu_q(x)).$$

From (4), we obtain zRx. The proof that $\{y \in \Omega^{nq} \mid xRy\}$ is closed in Ω^{nq} is analogous.

Next, to show that R satisfies replication invariance*, let $n, k \in \mathbb{N}$, $x \in \mathbb{R}^{nq}_{++}$, and $y = (\underbrace{x, \ldots, x}_{k \text{ times}}) \in \mathbb{R}^{knq}_{++}$. Note that for all $\ell \in \{1, \ldots, q\}$, $\mu_\ell(x) = \mu_\ell(y)$. Thus, we obtain $(\mu_1(x), \ldots, \mu_q(x)) I (\mu_1(y), \ldots, \mu_q(y))$, and xIy follows from (4).

Finally, we show that R satisfies transfer neutrality within quantiles. Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^{nq}_{++}$. Suppose that $G_\ell(x) = G_\ell(y)$ for all $\ell \in \{1, \ldots, q\}$ and there exist $\ell' \in \{1, \ldots, q\}$ and $i, j \in G_{\ell'}(x)$ such that $x_i - y_i = y_j - x_i$ and $x_k = y_k$ for all $k \in \{1, \ldots, nq\} \setminus \{i, j\}$. Then, again, $\mu_\ell(x) = \mu_\ell(y)$ for all $\ell \in \{1, \ldots, q\}$. Thus, from the same argument as above, xIy follows.

'Only if.' *Step 1.* We show that, for any $n \in \mathbb{N}$ and for any $x, y \in \mathbb{R}^{nq}_{++}$, xIy if $(y_{(\ell-1)n+1}, \ldots, y_{\ell n}) = (\mu_\ell(x), \ldots, \mu_\ell(x))$ for all $\ell \in \{1, \ldots, q\}$. This follows immediately if $n = 1$ because R satisfies anonymity. Now assume that $n \geq 2$, and let $x \in \mathbb{R}^{nq}_{++}$. Since R satisfies anonymity, without loss of generality, we assume $x = x_{(\,)}$. Hence, for all $\ell \in \{1, \ldots, q\}$, $G_\ell(x) = \{(\ell-1)n+1, \ldots, \ell n\}$. For all $\ell \in \{1, \ldots, q\}$, we define the subsets $B_\ell(x)$ and $W_\ell(x)$ of $G_\ell(x)$ by

$$B_\ell(x) = \{i \in G_\ell(x) \mid x_i > \mu_\ell(x)\}$$

and

$$W_\ell(x) = \{i \in G_\ell(x) \mid x_i < \mu_\ell(x)\}.$$

Further, define $y \in \mathbb{R}^{nq}_{++}$ by $(y_{(\ell-1)n+1}, \ldots, y_{\ell n}) = (\mu_\ell(x), \ldots, \mu_\ell(x))$ for all $\ell \in \{1, \ldots, q\}$. Note that $G_\ell(y) = G_\ell(x)$ for all $\ell \in \{1, \ldots, q\}$. If $B_\ell(x) = \varnothing$ for all $\ell \in \{1, \ldots, q\}$, then $x = y$. Thus, we obtain xIy. We now suppose that there exists $\ell \in \{1, \ldots, q\}$ such that $B_\ell(x) \neq \varnothing$. Note that for all $\ell \in \{1, \ldots, q\}$, $B_\ell(x) \neq \varnothing$ implies $W_\ell(x) \neq \varnothing$. Furthermore, $\sum_{i \in B_\ell(x)}(x_i - \mu_\ell(x)) = \sum_{i \in W_\ell(x)}(\mu_\ell(x) - x_i)$. Thus, y can be obtained from x by a finite sequence of progressive transfers from individuals in $B_\ell(x)$ to those in $W_\ell(x)$. Since none of these transfers change the quantile to which the donor and recipient belong, we obtain xIy by transfer neutrality within quantiles and the transitivity of R.

Step 2. To complete the proof, let $n, m \in \mathbb{N}$, $x \in \mathbb{R}^{nq}_{++}$, and $y \in \mathbb{R}^{mq}_{++}$. We define $\bar{x} \in \mathbb{R}^{nq}_{++}$ and $\bar{y} \in \mathbb{R}^{mq}_{++}$ by

$$(\bar{x}_{(\ell-1)n+1}, \ldots, \bar{x}_{\ell n}) = (\mu_\ell(x), \ldots, \mu_\ell(x))$$

and

$$(\bar{y}_{(\ell-1)m+1}, \ldots, \bar{y}_{\ell m}) = (\mu_\ell(y), \ldots, \mu_\ell(y))$$

for all $\ell \in \{1, \ldots, q\}$. Since R is transitive, it follows from Step 1 that

$$xRy \;\Leftrightarrow\; \bar{x}R\bar{y}.$$

Since R satisfies replication invariance, we obtain

$$\bar{x}I(\mu_1(x), \ldots, \mu_q(x))$$

and

$$\bar{y}I(\mu_1(y), \ldots, \mu_q(y)).$$

Therefore, by the transitivity of R, we obtain

$$xRy \;\Leftrightarrow\; (\mu_1(x), \ldots, \mu_q(x))R(\mu_1(y), \ldots, \mu_q(y)).$$

We define the ordering $\succsim^*$ on S^* by the restriction of R to $S^* \subset D$. Then, $\succsim^*$ satisfies (4). Further, since R satisfies S-convexity* and continuity*, $\succsim^*$ is continuous and S-convex on S^*. ■

The fifth axiom we use to characterize the top income gap inequality ordering and its relative counterpart is the *composite transfer principle for top quantile*. This axiom parallels the composite transfer principle for top income but the requirement is restricted to income distributions involving q individuals. Thus, it is logically weaker than the direct reformulation of the composite transfer principle for top income.

Composite transfer principle for top quantile. For all $x, y \in \Omega^q$ with $x_\ell \leq x_{\ell+1}$ and $y_\ell \leq y_{\ell+1}$ for all $\ell \in \{1, \ldots, q-1\}$, if there exist $\delta, \varepsilon \in \mathbb{R}_{++}$ and $i, j \in \{1, \ldots, q-1\}$ with $i < j$ such that $x - y = \delta(e^i - e^j) + \varepsilon(e^q - e^j)$, then xPy.

Adding the composite transfer principle for top quantile to the axioms of Theorem 13, we obtain the following preliminary result that is analogous to Theorem 5.

Theorem 14. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$ and suppose that R satisfies S-convexity**, *continuity**, *replication invariance**, *transfer neutrality within quantiles, and the composite transfer principle for top quantile. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$, if $\mu_q(x) = \mu_q(y)$ and $\mu(x) = \mu(y)$, then xIy.*

Proof. From Theorem 13, there exists a continuous and S-convex ordering $\succsim^*$ on S^* that satisfies (4). Note that for any $n \in \mathbb{N}$ and for any $x \in \Omega^{nq}$,

$$\mu(x) = \mu\left((\mu_1(x), \ldots, \mu_q(x))\right).$$

Thus, from Theorem 13, it suffices to show that, for all $x, y \in S^*$, if $x_q = y_q$ and $\mu(x) = \mu(y)$, then $x \sim^* y$. Note that this claim is analogous to the claim of Step 3 of the proof of Theorem 5. Further, Steps 1, 2, and 3 of the proof of Theorem 5 were established using vectors in $\{x \in \Omega^n \mid x_i \leq x_{i+1} \text{ for all } i \in \{1, \ldots, n-1\}\}$. Thus, letting $n = q$, we can prove the claim by employing the same argument as in the proof of Theorem 5. ■

The following theorem characterizes all inequality orderings that satisfy the axioms introduced in this subsection. These inequality orderings only utilize the mean incomes of the top quantile and the entire population and they are increasing in the mean income of the top quantile.

Theorem 15. *(a) Let $\Omega = \mathbb{R}$. R satisfies S-convexity**, *continuity**, *replication invariance**, *transfer neutrality within quantiles, and the composite transfer principle for top quantile if and only if there exists a continuous ordering $\succsim$ on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,

$$xRy \;\Leftrightarrow\; (\mu_q(x), \mu(x)) \succsim (\mu_q(y), \mu(y));$$

(ii) $\succsim$ is increasing in its first argument.

(b) Let $\Omega = \mathbb{R}_{++}$. R satisfies S-convexity, continuity*, replication invariance*, transfer neutrality within quantiles, and the composite transfer principle for top quantile if and only if there exists a continuous ordering $\succsim$ on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta > \alpha/q\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,

$$xRy \;\Leftrightarrow\; (\mu_q(x), \mu(x)) \succsim (\mu_q(y), \mu(y));$$

(ii) $\succsim$ is increasing in its first argument.

Proof. (a) 'If.' Suppose that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii). We define the ordering $\succsim^*$ on $S^* = \{z \in \Omega^q \mid z_\ell \leq z_{\ell+1} \text{ for all } \ell \in \{1, \dots, q-1\}\}$ as follows. For all $x, y \in S^*$,

$$x \succsim^* y \;\Leftrightarrow\; (x_q, \mu(x)) \succsim (y_q, \mu(y)).$$

Since $\succsim$ satisfies property (i) and $\mu(x) = \mu((\mu_1(x), \dots, \mu_q(x)))$ for all $n \in \mathbb{N}$ and for all $x \in \Omega^{nq}$, $\succsim^*$ satisfies (4) in Theorem 13. By the continuity of $\succsim$, $\succsim^*$ is continuous on S^*. Since $\succsim$ satisfies property (ii), $\succsim^*$ is S-convex. Thus, from Theorem 13, R satisfies S-convexity*, replication invariance*, continuity*, and transfer neutrality within quantiles. Furthermore, from properties (i) and (ii), R satisfies the composite transfer principle for top quantile.

'Only if.' We define the binary relation $\succsim$ on S by letting, for all $(\alpha, \beta), (\alpha', \beta') \in S$,

$$(\alpha, \beta) \succsim (\alpha', \beta')$$

if and only if there exist $n, m \in \mathbb{N}$, $x \in \Omega^{nq}$, and $y \in \Omega^{mq}$ such that xRy and

$$\alpha = \mu_q(x),\;\; \beta = \mu(x),\;\; \alpha' = \mu_q(y),\;\; \beta' = \mu(y).$$

We can prove that $\succsim$ satisfies property (i) by the same argument as in the corresponding proof of Theorem 6 using Theorem 14 instead of Theorem 5.

For any $(\alpha, \beta) \in S$ and for any $q \in \mathbb{N}$ with $q \geq 3$, the vector $x \in \mathbb{R}^q$ defined by

$$x_q = \alpha \;\text{ and }\; x_\ell = \frac{q\beta - \alpha}{q - 1} \;\text{ for all } \ell \in \{1, \dots, q-1\}$$

satisfies $\mu_q(x) = \alpha$ and $\mu(x) = \beta$. Further, $x \in \Omega^n$ follows; note that if $\Omega = \mathbb{R}_{++}$, $(\alpha, \beta) \in S$ satisfies $\beta > \alpha/q$. Therefore, we can prove that $\succsim$ is a continuous ordering by letting $n = q$ and applying the same argument as in the corresponding proof in Theorem 6 presented for the case where $\Omega = \mathbb{R}$. Further, letting $n = q$, the proof that $\succsim$ is increasing in its first argument is analogous to the corresponding proof in Theorem 6.

The proof of part (b) is analogous. ■

Adding translation invariance and scale invariance, respectively, to the axioms of Theorem 15, we obtain characterizations of the top income gap inequality ordering and the top income share inequality ordering.

Theorem 16. *Let $\Omega = \mathbb{R}$. R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, the composite transfer principle for top quantile, and translation invariance* if and only if $R = R_t^a$.*

Proof. Let $n \in \mathbb{N}$ and $x \in \Omega^{nq}$. Translation invariance with $\delta = -\mu(x)$ requires that

$$(x_1 - \mu(x), \dots, x_{nq} - \mu(x))Ix$$

and, by Theorem 15,

$$(\mu_q(x) - \mu(x), 0) \sim (\mu_q(x), \mu(x)).$$

Now let $n, m \in \mathbb{N}$, $x \in \Omega^{nq}$, and $y \in \Omega^{mq}$. Analogously to the proof of Theorem 7, using Theorem 15, it follows that

$$\begin{aligned} xRy &\Leftrightarrow (\mu_q(x) - \mu(x), 0) \succsim (\mu_q(y) - \mu(y), 0) \\ &\Leftrightarrow \mu_q(x) - \mu(x) \geq \mu_q(y) - \mu(y) \\ &\Leftrightarrow xR_t^a y. \blacksquare \end{aligned}$$

Theorem 17. *Let $\Omega = \mathbb{R}_{++}$. R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, the composite transfer principle for top quantile, and scale invariance* if and only if $R = R_t^r$.*

Proof. Let $n \in \mathbb{N}$ and $x \in \Omega^{nq}$. Scale invariance with $\lambda = 1/\mu(x)$ requires that

$$\left(\frac{x_1}{\mu(x)}, \dots, \frac{x_{nq}}{\mu(x)}\right) Ix$$

and, by Theorem 15,

$$\left(\frac{\mu_q(x)}{\mu(x)}, 1\right) \sim (\mu_q(x), \mu(x)).$$

Now let $n, m \in \mathbb{N}$, $x \in \Omega^{nq}$, and $y \in \Omega^{mq}$. Analogously to the proof of Theorem 8, using Theorem 15, we obtain

$$\begin{aligned} xRy &\Leftrightarrow \left(\frac{\mu_q(x)}{\mu(x)}, 1\right) \succsim \left(\frac{\mu_q(y)}{\mu(y)}, 1\right) \\ &\Leftrightarrow \frac{\mu_q(x)}{\mu(x)} \geq \frac{\mu_q(y)}{\mu(y)} \\ &\Leftrightarrow xR_t^r y. \blacksquare \end{aligned}$$

3.5 Bottom income gaps and shares

We characterize the bottom income share inequality ordering and the mean-bottom inequality ordering using the *composite transfer principle for bottom quantile,* which is an axiom dual to the composite transfer principle for top quantile. The composite transfer principle for bottom quantile requires the same property as the composite transfer principle for bottom income but the property applies only to income distributions for q persons.

Composite transfer principle for bottom quantile. For all $x, y \in \Omega^q$ with $x_\ell \leq x_{\ell+1}$ and $y_\ell \leq y_{\ell+1}$ for all $\ell \in \{1, \ldots, q-1\}$, if there exist $\delta, \varepsilon \in \mathbb{R}_{++}$ and $i, j \in \{2, \ldots, q\}$ with $i < j$ such that $x - y = \delta(e^1 - e^i) + \varepsilon(e^j - e^i)$, then yPx.

In analogy to the previous section, we characterize all inequality orderings that satisfy the axioms of the previous subsection when the composite transfer principle for top quantile is replaced with the composite transfer principle for bottom quantile. We begin with the following preliminary result.

Theorem 18. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$ and suppose that R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, and the composite transfer principle for bottom quantile. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$, if $\mu_1(x) = \mu_1(y)$ and $\mu(x) = \mu(y)$, then xIy.*

Proof. From Theorem 13, there exists a continuous and S-convex ordering $\succsim^*$ on S^* that satisfies (4). Thus, it remains to show that, for any $x, y \in S^*$, if $x_1 = y_1$ and $\mu(x) = \mu(y)$, then $x \sim^* y$. Analogously to the proof of Theorem 14, we can prove this claim by letting $n = q$ and applying the same argument as in Steps 1, 2, and 3 of the proof of Theorem 9. ■

Theorem 19. *Let $\Omega \in \{\mathbb{R}, \mathbb{R}_{++}\}$. R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, and the composite transfer principle for bottom quantile if and only if there exists a continuous ordering $\succsim$ on $S = \{(\alpha, \beta) \in \Omega^2 \mid \alpha \geq \beta\}$ such that*

(i) for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,

$$xRy \;\Leftrightarrow\; (\mu(x), \mu_1(x)) \succsim (\mu(y), \mu_1(y));$$

(ii) $\succsim$ is decreasing in its second argument.

Proof. 'If.' Suppose that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii). From properties (i) and (ii), R satisfies the composite transfer principle for bottom quantile. The proof that R satisfies the other axioms is analogous to the proof of Theorem 15.

'Only if.' The proof of the existence of the binary relation $\succsim$ on S satisfying property (i) is analogous to the proof of Theorem 15.

In either case (that is, $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}_{++}$), for any $(\alpha, \beta) \in S$ and for any $q \in \mathbb{N}$ with $q \geq 3$, the vector $x \in \mathbb{R}^q$ defined by

$$x_1 = \beta \;\text{ and }\; x_\ell = \frac{q\beta - \alpha}{q - 1} \;\text{ for all } \ell \in \{2, \ldots, q\}$$

satisfies $x \in \Omega^n$, $\mu_1(x) = \beta$, and $\mu(x) = \alpha$. Therefore, we can prove that $\succsim$ is a continuous ordering satisfying property (ii) by letting $n = q$ and applying the same argument as in the corresponding proof in Theorem 10. ■

Adding translation invariance and scale invariance, respectively, to the axioms of Theorem 19, we obtain characterizations of the bottom income gap inequality ordering and the bottom income share inequality ordering.

Theorem 20. *Let $\Omega = \mathbb{R}$. R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, the composite transfer principle for bottom quantile, and translation invariance* if and only if $R = R_b^a$.*

Proof. From Theorem 19, it follows that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in Theorem 19. Applying the same argument as in the proof of Theorem 16 using $\delta = -\mu(x)$, we obtain that, for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,

$$xRy \Leftrightarrow (0, \mu_1(x) - \mu(x)) \succsim (0, \mu_1(y) - \mu(x)).$$

Since $\succsim$ is decreasing in its second argument, this is equivalent to

$$\begin{aligned} xRy &\Leftrightarrow \mu_1(x) - \mu(x) \leq \mu_1(y) - \mu(x) \\ &\Leftrightarrow \mu(x) - \mu_1(x) \geq \mu(y) - \mu_1(y) \\ &\Leftrightarrow xR_b^a y. \ \blacksquare \end{aligned}$$

Theorem 21. *Let $\Omega = \mathbb{R}_{++}$. R satisfies S-convexity*, continuity*, replication invariance*, transfer neutrality within quantiles, the composite transfer principle for bottom quantile, and scale invariance* if and only if $R = R_b^r$.*

Proof. From Theorem 19, it follows that there exists a continuous ordering $\succsim$ on S satisfying properties (i) and (ii) in Theorem 19. Applying the same argument as in the proof of Theorem 17 using $\lambda = 1/\mu(x)$, we obtain that, for all $n, m \in \mathbb{N}$, for all $x \in \Omega^{nq}$, and for all $y \in \Omega^{mq}$,

$$xRy \Leftrightarrow \left(1, \frac{\mu_1(x)}{\mu(x)}\right) \succsim \left(1, \frac{\mu_1(y)}{\mu(y)}\right).$$

Since $\succsim$ is decreasing in its second argument, this is equivalent to

$$\begin{aligned} xRy &\Leftrightarrow \frac{\mu_1(x)}{\mu(x)} \leq \frac{\mu_1(y)}{\mu(y)} \\ &\Leftrightarrow \frac{\mu(x)}{\mu_1(x)} \geq \frac{\mu(y)}{\mu_1(y)} \\ &\Leftrightarrow xR_b^r y. \ \blacksquare \end{aligned}$$

4 Empirical considerations

The measures characterized in this paper are easily understood and computed. They can be considered somewhat coarse, and the purpose of this empirical section is to explore their linear correlation with more standard indices of inequality. We employ a strategy that is inspired by Leigh (2007) and, in particular, we estimate the equations

$$SIneq_{i,t} = \alpha + \beta Ineq_{i,t} + \varepsilon_{i,t}, \tag{5}$$
$$SIneq_{i,t} = \alpha + \beta Ineq_{i,t} + \gamma_i + \varepsilon_{i,t}, \tag{6}$$
$$SIneq_{i,t} = \alpha + \beta Ineq_{i,t} + \gamma_i + \delta_t + \varepsilon_{i,t}, \tag{7}$$

where $SIneq_{i,t}$ is one of several standard indices of inequality in country i in year t that we compute to perform our comparisons. These alternative indices are given by (i) the absolute Gini coefficient, the variance, and the Kolm index with parameter values of 10^{-4} and $5 \cdot 10^{-4}$ in the absolute case, and (ii) the Gini coefficient and the Atkinson index with inequality-aversion parameter values of 0.5 and 1 for the relative measures. The variable $Ineq_{i,t}$ indicates one of the inequality measures characterized in this paper. Equation (6) also includes a country-specific term γ, while (7) controls for the year fixed effect δ in addition.

We use all the waves of the Luxembourg Income Study (LIS) datasets that are available as of May 2019, retaining the countries for which at least four years for the period 1974–2016 are covered. This leaves us with 36 countries in total and a global sample of 299 observations; for the countries retained, see Table 5. We follow the LIS rules for their provision of the key figures since we wish to be of guidance for researchers that decide to use the indices already available from LIS. In particular, in this specification, (i) the income measure is disposable household income equivalized by means of the square root equivalence scale; (ii) the unit of analysis is the individual; (iii) incomes are bottom-coded at 1% of equivalized mean income and top-coded at ten times mean income; (iv) missing and zero incomes are excluded. As an alternative, to test the sensitivity of our results to the LIS top-coding rules, we also provide the results without top coding and include all the observations on the right tail of the income distribution. The incomes are expressed in 2011 constant US dollars. All variables are standardized to Z-scores (that is, to a mean of zero and a standard deviation of one) to facilitate comparisons of the estimated coefficients. As a result of this standardization, the slope β of the regression line in (5) is Pearson's correlation coefficient among the independent and dependent variables. This equivalence does not hold in the other two estimated models since these are multivariate regressions. Again, the reference value is one because an increase in one standard deviation of one index is associated with an increase of one standard deviation in the other.

Table 1 displays the results for the absolute inequality indices, and Table 2 contains those for the relative case following the LIS rules, while Tables 3 and 4 contain those without top-coding. Owing to the presence of high collinearity among the inequality indices (measured by a Variance Inflation Factor exceeding the reference value of ten by a large margin), we cannot include all of them simultaneously in the regression. To avoid lengthy tables, we report the estimation results of pairs of the classical and our inequality measures in a single column. The classical measure we consider is indicated in the top row and the inequality measures we characterize are listed in the first column. The equation numbers (5), (6), and (7) in the top row indicate the three regression models without fixed effects, with country fixed effects, and with country and year fixed effects, respectively.

All coefficients are positive and significant in the LIS specification, while some coefficients lose significance without top coding and also in one case for the Atkinson index. Let us first focus on the discussion of the results with the full application of the LIS rules. We observe many correlation coefficients among the indices above 0.9. For the absolute case, the lowest observed correlation is never below 0.352 (between the Kolm index with parameter value of 10^{-4} and the absolute mean-min indices in Table 1). The correlation coefficients for the relative measures range between 0.18 (observed for the Gini and the relative mean-min indices in Table 2) and 0.983 (between the Gini and the top income share indices).

The linear associations between the absolute indices are surprisingly high; see Table 1. Values very close to one are observed in all three models between all the absolute standard measures and the absolute mean-min, top 10% gap, and bottom 10% gap indices; the only exception is the correlation coefficient with the Kolm index with parameter value 10^{-4}, reported in the first column of the table. The results improve with the introduction of country and year fixed effects.

For the relative case in all models (Table 2), the value closest to one is observed for the 10% top income share index, followed by the 10% bottom income share index. The remaining indices do not perform that well, especially when year and country fixed effects are incorporated. It is worth noting that the values of the coefficient of determination (R-squared) are always above 0.9 as soon as the country dummies are introduced in the model.

As expected, the full consideration of the highest incomes (see Tables 3 and 4) has an effect on the results, lowering the correlation coefficients between the standard measures and the coarser indices, apart from the two that exclude the maximum income from their definitions (the top 10% gap and share and the bottom 10% gap and share). The absolute and relative mean-min indices perform well, especially in the absolute case with the absolute Gini coefficient and the two versions of the Kolm index.

Table 1: Standard absolute inequality measures and our absolute inequality measures

Dependent variable	Absolute Gini			Variance			Kolm (parameter 10^{-4})			Kolm (parameter $5 \cdot 10^{-4}$)		
	(5)	(6)	(7)	(5)	(6)	(7)	(5)	(6)	(7)	(5)	(6)	(7)
Absolute range	0.838***	0.606***	0.332***	0.809***	0.675***	0.429***	0.493***	0.555***	0.400***	0.869***	0.642***	0.406***
	(0.037)	(0.038)	(0.041)	(0.055)	(0.048)	(0.049)	(0.054)	(0.062)	(0.088)	(0.025)	(0.044)	(0.057)
	0.702	*0.910*	*0.959*	*0.654*	*0.875*	*0.927*	*0.243*	*0.687*	*0.759*	*0.755*	*0.879*	*0.932*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Absolute max–mean	0.823***	0.577***	0.306***	0.789***	0.647***	0.402***	0.498***	0.530***	0.374***	0.856***	0.612***	0.375***
	(0.038)	(0.038)	(0.040)	(0.056)	(0.047)	(0.048)	(0.053)	(0.061)	(0.085)	(0.026)	(0.044)	(0.055)
	0.677	*0.904*	*0.958*	*0.638*	*0.869*	*0.926*	*0.249*	*0.683*	*0.757*	*0.732*	*0.873*	*0.930*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Absolute mean–min	0.925***	1.167***	1.022***	0.829***	1.155***	0.965***	0.352***	1.031***	0.983***	0.917***	1.244***	1.211***
	(0.027)	(0.026)	(0.042)	(0.049)	(0.063)	(0.086)	(0.060)	(0.089)	(0.160)	(0.026)	(0.033)	(0.060)
	0.855	*0.984*	*0.988*	*0.688*	*0.913*	*0.939*	*0.124*	*0.737*	*0.777*	*0.841*	*0.965*	*0.971*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Top 10% gap	0.991***	1.011***	0.930***	0.923***	1.004***	0.890***	0.420***	0.848***	0.724***	0.904***	1.027***	0.938***
	(0.008)	(0.014)	(0.024)	(0.041)	(0.058)	(0.074)	(0.051)	(0.079)	(0.145)	(0.021)	(0.037)	(0.062)
	0.982	*0.990*	*0.993*	*0.851*	*0.921*	*0.945*	*0.176*	*0.726*	*0.765*	*0.817*	*0.950*	*0.958*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Bottom 10% gap	0.993***	1.060***	1.032***	0.923***	1.071***	1.030***	0.414***	0.880***	0.773***	0.926***	1.078***	1.045***
	(0.008)	(0.012)	(0.019)	(0.042)	(0.056)	(0.070)	(0.053)	(0.080)	(0.153)	(0.019)	(0.034)	(0.063)
	0.986	*0.996*	*0.997*	*0.852*	*0.934*	*0.953*	*0.171*	*0.727*	*0.765*	*0.858*	*0.957*	*0.962*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299

Notes. Each column presents the results of the corresponding pairs of classical and our inequality measures according to the estimated equations (5) without fixed effects, (6) with country fixed effects, and (7) with country and year fixed effects, respectively. Robust standard errors in parentheses. ***, **, and * denote $p < 0.01$, $p < 0.05$, and $p < 0.1$, respectively. R-squared is in italic.

Table 2: Standard relative inequality measures and our relative inequality measures

Dependent variables	Gini (5)	(6)	(7)	Atkinson (parameter 0.5) (5)	(6)	(7)	Atkinson (parameter 1) (5)	(6)	(7)
Relative range	0.327***	0.123***	0.078***	0.335***	0.108***	0.066***	0.341***	0.112***	0.070**
	(0.044)	(0.019)	(0.023)	(0.042)	(0.018)	(0.022)	(0.043)	(0.021)	(0.027)
	0.107	*0.957*	*0.967*	*0.112*	*0.954*	*0.964*	*0.116*	*0.941*	*0.953*
Obs.	299	299	299	299	299	299	299	299	299
Relative max–mean	0.334***	0.093***	0.051***	0.342***	0.078***	0.038*	0.331***	0.070***	0.025
	(0.050)	(0.020)	(0.023)	(0.048)	(0.018)	(0.022)	(0.050)	(0.021)	(0.026)
	0.111	*0.954*	*0.966*	*0.117*	*0.951*	*0.963*	*0.110*	*0.937*	*0.952*
Obs.	299	299	299	299	299	299	299	299	299
Relative mean–min	0.180***	0.097***	0.059***	0.180***	0.090***	0.054***	0.208***	0.110***	0.073**
	(0.025)	(0.018)	(0.018)	(0.022)	(0.016)	(0.016)	(0.022)	(0.019)	(0.020)
	0.032	*0.954*	*0.967*	*0.032*	*0.952*	*0.964*	*0.043*	*0.941*	*0.954*
Obs.	299	299	299	299	299	299	299	299	299
Top 10% share	0.983***	0.822***	0.763***	0.969***	0.814***	0.770***	0.965***	0.888***	0.857***
	(0.010)	(0.040)	(0.039)	(0.016)	(0.045)	(0.046)	(0.018)	(0.049)	(0.056)
	0.965	*0.981*	*0.987*	*0.939*	*0.979*	*0.984*	*0.932*	*0.972*	*0.978*
Obs.	299	299	299	299	299	299	299	299	299
Bottom 10% share	0.892***	0.547***	0.471***	0.918***	0.620***	0.554***	0.927***	0.671***	0.615***
	(0.118)	(0.149)	(0.141)	(0.108)	(0.144)	(0.141)	(0.099)	(0.155)	(0.156)
	0.792	*0.971*	*0.979*	*0.843*	*0.976*	*0.982*	*0.860*	*0.968*	*0.976*
Obs.	299	299	299	299	299	299	299	299	299

Notes. Each column presents the results of the corresponding pairs of classical and our inequality measures according to the estimated equations (5) without fixed effects, (6) with country fixed effects, and (7) with country and year fixed effects, respectively. Robust standard errors in parentheses. ***, **, and * denote $p < 0.01$, $p < 0.05$, and $p < 0.1$, respectively. R-squared is in italic.

Table 3: Standard absolute inequality measures and our absolute inequality measures (without top-coding)

Dependent variable	Absolute Gini (5)	(6)	(7)	Variance (5)	(6)	(7)	Kolm (parameter 10^{-4}) (5)	(6)	(7)	Kolm (parameter $5 \cdot 10^{-4}$) (5)	(6)	(7)
Absolute range	0.025	0.064***	0.015	0.562***	0.576***	0.561***	0.088**	0.065***	0.003	0.067	0.063***	0.011
	(0.035)	(0.024)	(0.012)	(0.084)	(0.069)	(0.067)	(0.044)	(0.016)	(0.020)	(0.043)	(0.019)	(0.012)
	0.001	*0.775*	*0.937*	*0.316*	*0.547*	*0.648*	*0.008*	*0.584*	*0.735*	*0.005*	*0.736*	*0.906*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Absolute max–mean	0.023	0.064***	0.015	0.561***	0.576***	0.561***	0.087**	0.064***	0.003	0.066	0.062***	0.011
	(0.034)	(0.024)	(0.012)	(0.084)	(0.069)	(0.067)	(0.044)	(0.016)	(0.020)	(0.041)	(0.019)	(0.011)
	0.001	*0.775*	*0.937*	*0.315*	*0.547*	*0.648*	*0.008*	*0.584*	*0.735*	*0.004*	*0.736*	*0.906*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Absolute mean–min	0.917***	1.179***	1.044***	0.467***	0.872***	0.829***	0.355***	1.025***	0.974***	0.917***	1.240***	1.207***
	(0.028)	(0.028)	(0.045)	(0.050)	(0.162)	(0.203)	(0.060)	(0.088)	(0.159)	(0.026)	(0.032)	(0.059)
	0.841	*0.981*	*0.986*	*0.218*	*0.385*	*0.473*	*0.126*	*0.739*	*0.778*	*0.840*	*0.965*	*0.971*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Top 10% gap	0.981***	1.016***	0.904***	0.431***	0.584***	0.237	0.413***	0.852***	0.720***	0.901***	1.033***	0.929***
	(0.013)	(0.020)	(0.034)	(0.033)	(0.100)	(0.187)	(0.051)	(0.080)	(0.144)	(0.021)	(0.038)	(0.064)
	0.963	*0.979*	*0.985*	*0.186*	*0.339*	*0.445*	*0.170*	*0.726*	*0.766*	*0.811*	*0.947*	*0.956*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299
Bottom 10% gap	0.990***	1.070***	1.053***	0.498***	0.812***	0.882***	0.420***	0.870***	0.756***	0.927***	1.070***	1.031***
	(0.009)	(0.013)	(0.021)	(0.040)	(0.143)	(0.201)	(0.052)	(0.079)	(0.149)	(0.019)	(0.034)	(0.061)
	0.981	*0.995*	*0.996*	*0.248*	*0.399*	*0.483*	*0.176*	*0.728*	*0.766*	*0.859*	*0.957*	*0.962*
Obs.	299	299	299	299	299	299	299	299	299	299	299	299

Notes. Each column presents the results of the corresponding pairs of classical and our inequality measures according to the estimated equations (5) without fixed effects, (6) with country fixed effects, and (7) with country and year fixed effects, respectively. Robust standard errors in parentheses. ***, **, and * denote $p < 0.01$, $p < 0.05$, and $p < 0.1$, respectively. R-squared is in italic.

Table 4: Standard relative inequality measures and our relative inequality measures (without top-coding)

Dependent variables	Gini (5)	(6)	(7)	Atkinson (parameter 0.5) (5)	(6)	(7)	Atkinson (parameter 1) (5)	(6)	(7)
Relative range	0.057	0.023*	0.012	0.079	0.021	0.014	0.064	0.014	0.009
	(0.116)	(0.013)	(0.012)	(0.127)	(0.015)	(0.015)	(0.118)	(0.013)	(0.014)
	0.003	*0.944*	*0.962*	*0.006*	*0.936*	*0.954*	*0.004*	*0.931*	*0.949*
Obs.	299	299	299	299	299	299	299	299	299
Relative max–mean	0.056	0.023*	0.011	0.077	0.021	0.013	0.062	0.013	0.008
	(0.115)	(0.013)	(0.012)	(0.125)	(0.015)	(0.015)	(0.116)	(0.012)	(0.014)
	0.003	*0.944*	*0.962*	*0.006*	*0.936*	*0.954*	*0.004*	*0.931*	*0.949*
Obs.	299	299	299	299	299	299	299	299	299
Relative mean–min	0.230***	0.100***	0.058***	0.229***	0.089***	0.049***	0.255***	0.112***	0.072***
	(0.026)	(0.018)	(0.018)	(0.023)	(0.016)	(0.018)	(0.023)	(0.018)	(0.075)
	0.053	*0.949*	*0.963*	*0.052*	*0.936*	*0.955*	*0.065*	*0.937*	*0.963*
Obs.	299	299	299	299	299	299	299	299	299
Top 10% share	0.948***	0.600***	0.529***	0.909***	0.519***	0.455***	0.927***	0.637***	0.528***
	(0.022)	(0.063)	(0.064)	(0.031)	(0.076)	(0.079)	(0.027)	(0.070)	(0.075)
	0.899	*0.962*	*0.973*	*0.827*	*0.949*	*0.962*	*0.858*	*0.951*	*0.963*
Obs.	299	299	299	299	299	299	299	299	299
Bottom 10% share	0.896***	0.564***	0.484***	0.916***	0.659***	0.583***	0.927***	0.673***	0.610***
	(0.117)	(0.146)	(0.138)	(0.109)	(0.147)	(0.143)	(0.100)	(0.152)	(0.152)
	0.803	*0.969*	*0.978*	*0.839*	*0.970*	*0.977*	*0.860*	*0.967*	*0.974*
Obs.	299	299	299	299	299	299	299	299	299

Notes. Each column presents the results of the corresponding pairs of classical and our inequality measures according to the estimated equations (5) without fixed effects, (6) with country fixed effects, and (7) with country and year fixed effects, respectively. Robust standard errors in parentheses. ***, **, and * denote $p < 0.01$, $p < 0.05$, and $p < 0.1$, respectively. R-squared is in italic.

5 Concluding remarks

In this paper, we characterize some inequality measures that are based on simple summary statistics such as the minimum, the maximum, and the mean of an income distribution. Although most of these indices are well-known, there do not appear to be any axiomatizations available. Our theoretical results are supplemented with an empirical analysis that is intended to show that there may be more to our contribution than merely filling a gap in the literature. Especially in the case of some absolute measures, it turns out that there are some strong correlations between these indices and inequality orderings that are of a more complex nature. This latter observation, along with Leigh's (2007) analysis, suggests that there is a surprisingly high level of agreement across indices when it comes to practical applications. Thus, it may be promising to extend our application to include a larger class of measures and additional data sets.

Appendix

Independence of the axioms in Theorems 2, 3, and 4

First, let $\Omega = \mathbb{R}$ and define the inequality ordering R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, if $\max\{x_1, \ldots, x_n\} = \min\{x_1, \ldots, x_n\}$ and $\max\{y_1, \ldots, y_m\} = \min\{y_1, \ldots, y_m\}$,

$$xRy \;\Leftrightarrow\; n \leq m,$$

and if $\max\{x_1, \ldots, x_n\} > \min\{x_1, \ldots, x_n\}$ or $\max\{y_1, \ldots, y_m\} > \min\{y_1, \ldots, y_m\}$,

$$xRy \;\Leftrightarrow\; xR^a_{xn}y.$$

We show that R is a well-defined ordering on D. To show that R is complete, let $n, m \in \mathbb{N}$, $x \in \Omega^n$, and $y \in \Omega^m$. First, suppose that $\max\{x_1, \ldots, x_n\} = \min\{x_1, \ldots, x_n\}$ and $\max\{y_1, \ldots, y_m\} = \min\{y_1, \ldots, y_m\}$. Then, it follows that xRy or yRx since $n \leq m$ or $n \geq m$. Next, suppose that $\max\{x_1, \ldots, x_n\} > \min\{x_1, \ldots, x_n\}$ or $\max\{y_1, \ldots, y_m\} > \min\{y_1, \ldots, y_m\}$. Then, it follows from the completeness of R^a_{nx} that xRy or yRx.

Next, to show that R is transitive, let $n, m, \ell \in \mathbb{N}$, $x \in \Omega^n$, $y \in \Omega^m$, and $z \in \Omega^\ell$. Suppose that xRy and yRz. We distinguish two cases.

(i) $\max\{y_1, \ldots, y_m\} = \min\{y_1, \ldots, y_m\}$. If $\max\{z_1, \ldots, z_\ell\} > \min\{z_1, \ldots, z_\ell\}$, we obtain a contradiction to yRz. Thus, yRz implies

$$\max\{z_1, \ldots, z_\ell\} = \min\{z_1, \ldots, z_\ell\} \text{ and } m \leq \ell.$$

If $\max\{x_1, \ldots, x_n\} = \min\{x_1, \ldots, x_n\}$, xRy implies $n \leq m \leq \ell$ and we obtain xRz. If $\max\{x_1, \ldots, x_n\} > \min\{x_1, \ldots, x_n\}$, xRz follows since $xP^a_{xn}z$.

(ii) $\max\{y_1, \ldots, y_m\} > \min\{y_1, \ldots, y_m\}$. If $\max\{x_1, \ldots, x_n\} = \min\{x_1, \ldots, x_n\}$, we obtain a contradiction to xRy. Thus,

$$\max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\} \geq \max\{y_1, \ldots, y_m\} - \min\{y_1, \ldots, y_m\} > 0.$$

Since yRz implies

$$\max\{y_1, \ldots, y_m\} - \min\{y_1, \ldots, y_m\} \geq \max\{z_1, \ldots, z_\ell\} - \min\{z_1, \ldots, z_\ell\},$$

we obtain $xR^a_{nx}z$, which implies xRz since $\max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\} > 0$.

The inequality ordering R defined above satisfies the axioms of Theorem 2 and 3 except for equality indifference.

If $\Omega = \mathbb{R}_{++}$, define the inequality ordering R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, if $\max\{x_1, \ldots, x_n\} = \min\{x_1, \ldots, x_n\}$ and $\max\{y_1, \ldots, y_m\} = \min\{y_1, \ldots, y_m\}$,

$$xRy \;\Leftrightarrow\; n \leq m,$$

and if $\max\{x_1, \ldots, x_n\} > \min\{x_1, \ldots, x_n\}$ or $\max\{y_1, \ldots, y_m\} > \min\{y_1, \ldots, y_m\}$,

$$xRy \;\Leftrightarrow\; xR^r_{nx}y.$$

That R is an ordering on D can be proven by employing the same argument as that used in the case $\Omega = \mathbb{R}$. This inequality ordering satisfies the axioms of Theorem 2 and 4 except for equality indifference.

Second, let $\Omega = \mathbb{R}$ and define the ordering R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \ \Leftrightarrow \ x_1 - \min\{x_1, \dots, x_n\} \geq y_1 - \min\{y_1, \dots, y_m\}.$$

This inequality ordering satisfies the axioms of Theorems 2 and 3 except for anonymity.

If $\Omega = \mathbb{R}_{++}$, define the ordering R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \ \Leftrightarrow \ \frac{x_1}{\min\{x_1, \dots, x_n\}} \geq \frac{y_1}{\min\{y_1, \dots, y_m\}}.$$

This inequality ordering satisfies the axioms of Theorems 2 and 4 except for anonymity.

Third, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \ \Leftrightarrow \ yR^a_{xn}x.$$

This inequality ordering satisfies the axioms of Theorems 2 and 3 except for expansion dominance.

If $\Omega = \mathbb{R}_{++}$, define the ordering R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \ \Leftrightarrow \ yR^r_{xn}x.$$

This inequality ordering satisfies the axioms of Theorems 2 and 4 except for expansion dominance.

Fourth, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^a_{xn}y$ or

(ii) $xI^a_{xn}y$ and $\dfrac{\max\{x_1, \dots, x_n\} - \min\{x_1, \dots, x_n\}}{n} \geq \dfrac{\max\{y_1, \dots, y_m\} - \min\{y_1, \dots, y_m\}}{m}$.

This inequality ordering satisfies the axioms of Theorems 2 and 3 except for conditional independence.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^r_{xn}y$ or

(ii) $xI^r_{xn}y$ and $\left(\dfrac{\max\{x_1, \dots, x_n\}}{\min\{x_1, \dots, x_n\}}\right)^{\frac{1}{n}} \geq \left(\dfrac{\max\{y_1, \dots, y_m\}}{\min\{y_1, \dots, y_m\}}\right)^{\frac{1}{m}}$.

This inequality ordering satisfies the axioms of Theorems 2 and 4 except for conditional independence.

Fifth, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; (\max\{x_1, \ldots, x_n\})^2 - (\min\{x_1, \ldots, x_n\})^2 \geq (\max\{y_1, \ldots, y_m\})^2 - (\min\{y_1, \ldots, y_m\})^2.$$

This inequality ordering satisfies the axioms of Theorem 3 except for translation invariance.

Finally, define R as the restriction of R^a_{xn} to $\cup_{n \in \mathbb{N}} \mathbb{R}^n_{++}$. This ordering satisfies the axioms of Theorem 4 except for scale invariance.

Independence of the axioms in Theorems 6, 7, and 8

From Theorems 10, 11, and 12, the composite transfer principle for top income is independent of the other axioms in Theorems 6, 7, and 8. To prove that the axioms in Theorems 6, 7, and 8 are independent, consider the following examples.

First, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$\begin{aligned} xRy \Leftrightarrow\; & \max\{x_1, \ldots, x_n\} + \min\{x_1, \ldots, x_n\} - 2\mu(x) \\ & \geq \max\{y_1, \ldots, y_m\} + \min\{y_1, \ldots, y_m\} - 2\mu(y). \end{aligned}$$

This inequality ordering satisfies the axioms of Theorems 6 and 7 except for S-convexity.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \frac{\max\{x_1, \ldots, x_n\} \min\{x_1, \ldots, x_n\}}{\mu(x)^2} \geq \frac{\max\{y_1, \ldots, y_m\} \min\{y_1, \ldots, y_m\}}{\mu(y)^2}.$$

This inequality ordering satisfies the axioms of Theorems 6 and 8 except for S-convexity.

Second, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^a_{x\mu}y$ or

(ii) $xI^a_{x\mu}y$ and $xR^a_{\mu n}y$.

This inequality ordering satisfies the axioms of Theorems 6 and 7 except for continuity.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^r_{x\mu}y$ or

(ii) $xI^r_{x\mu}y$ and $xR^r_{\mu n}y$

This inequality ordering satisfies the axioms of Theorems 6 and 8 except for continuity.

Third, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; n(\max\{x_1, \ldots, x_n\} - \mu(x)) \geq m(\max\{y_1, \ldots, y_m\} - \mu(y)).$$

This inequality ordering satisfies the axioms of Theorems 6 and 7 except for replication invariance.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \left(\frac{\max\{x_1, \ldots, x_n\}}{\mu(x)}\right)^n \geq \left(\frac{\max\{y_1, \ldots, y_m\}}{\mu(y)}\right)^m.$$

This inequality ordering satisfies the axioms of Theorems 6 and 8 except for replication invariance.

Fourth, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \max\{x_1, \ldots, x_n\} + \mu(x) \geq \max\{y_1, \ldots, y_m\} + \mu(y).$$

This inequality ordering satisfies the axioms of Theorem 7 except for translation invariance.

Finally, consider the restriction of $R^a_{x\mu}$ to $\cup_{n \in \mathbb{N}} \mathbb{R}^n_{++}$. This ordering satisfies the axioms of Theorem 8 except for scale invariance.

Independence of the axioms in Theorems 10, 11, and 12

From Theorems 6, 7, and 8, the composite transfer principle for bottom income is independent of the other axioms in Theorems 10, 11, and 12. To prove that the other axioms in Theorems 10, 11, and 12 are independent, consider the following examples.

First, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$\begin{aligned} xRy \;&\Leftrightarrow\; \max\{x_1, \ldots, x_n\} + \min\{x_1, \ldots, x_n\} - 2\mu(x) \\ &\leq\; \max\{y_1, \ldots, y_m\} + \min\{y_1, \ldots, y_m\} - 2\mu(y). \end{aligned}$$

This inequality ordering satisfies the axioms of Theorems 10 and 11 except for S-convexity.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \frac{\max\{x_1, \ldots, x_n\} \min\{x_1, \ldots, x_n\}}{\mu(x)^2} \leq \frac{\max\{y_1, \ldots, y_m\} \min\{y_1, \ldots, y_m\}}{\mu(y)^2}.$$

This inequality ordering satisfies the axioms of Theorems 10 and 12 except for S-convexity.

Second, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^a_{\mu n}y$ or

(ii) $xI^a_{\mu n}y$ and $xR^a_{x\mu}y$.

This inequality ordering satisfies the axioms of Theorems 10 and 11 except for continuity.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$, xRy if and only if

(i) $xP^r_{\mu n}y$ or

(ii) $xI^r_{\mu n}y$ and $xR^r_{x\mu}y$.

This inequality ordering satisfies the axioms of Theorems 10 and 12 except for continuity.

Third, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; n(\mu(x) - \min\{x_1, \ldots, x_n\}) \geq m(\mu(y) - \min\{y_1, \ldots, y_m\}).$$

This inequality ordering satisfies the axioms of Theorems 10 and 11 except for replication invariance.

If $\Omega = \mathbb{R}_{++}$, define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \left(\frac{\mu(x)}{\min\{x_1, \ldots, x_n\}}\right)^n \geq \left(\frac{\mu(y)}{\min\{y_1, \ldots, y_m\}}\right)^m.$$

This inequality ordering satisfies the axioms of Theorems 10 and 12 except for replication invariance.

Fourth, let $\Omega = \mathbb{R}$ and define R as follows. For all $n, m \in \mathbb{N}$, for all $x \in \Omega^n$, and for all $y \in \Omega^m$,

$$xRy \;\Leftrightarrow\; \min\{x_1, \ldots, x_n\} + \mu(x) \leq \min\{y_1, \ldots, y_m\} + \mu(y).$$

This inequality ordering satisfies the axioms of Theorem 11 except for translation invariance.

Finally, consider the restriction of $R^a_{\mu n}$ to $\cup_{n\in\mathbb{N}}\mathbb{R}^n_{++}$. This ordering satisfies the axioms of Theorem 12 except for scale invariance.

Independence of the axioms in Theorems 13, 15, 16, 17, 19, 20, and 21

Transfer neutrality within quantiles is independent of the other axioms in Theorems 13, 15, 16, and 17 because the restriction of $R^a_{x\mu}$ to $\cup_{n\in\mathbb{N}}\mathbb{R}^{nq}$ (or $\cup_{n\in\mathbb{N}}\mathbb{R}^{nq}_{++}$) satisfy the other axioms of Theorems 13, 15, and 16 and the restriction of $R^r_{x\mu}$ to $\cup_{n\in\mathbb{N}}\mathbb{R}^{nq}_{++}$ satisfies the other axioms of Theorem 17. Using $R^a_{\mu n}$ and $R^r_{\mu n}$, the same argument applies to Theorems 19, 20, and 21.

The independence of the composite transfer principle for top quantile in Theorems 15, 16, and 17 follows from Theorems 19, 20, and 21.

The examples that show the independence of the other axioms of Theorems 15, 16, and 17 are analogous to those that we used for checking that the corresponding axioms of Theorems 6, 7, and 8 are independent. Specifically, the examples are given by replacing $\max\{x_1, \ldots, x_n\}$ (respectively $\min\{x_1, \ldots, x_n\}$) with $\mu_q(x)$ (respectively $\mu_1(x)$) in the previous examples for Theorems 6, 11, and 12.

Likewise, replacing $\min\{x_1, \ldots, x_n\}$ (respectively $\max\{x_1, \ldots, x_n\}$) with $\mu_1(x)$ (respectively $\mu_q(x)$) in the previous examples for Theorems 10, 11, and 12, the examples showing the independence of the other axioms of Theorems 19, 20, and 21 are analogous to those that we used for the corresponding axioms of Theorems 10, 11, and 12.

Data description

As noted in Section 4, we used all the waves of the LIS dataset and retained the countries for which at least four years for the period 1974–2016 are covered. The countries retained in the dataset are listed in Table 5.

Table 5: Countries and years covered in the dataset

Australia (1981, 1985, 1989, 1995, 2001, 2003, 2004, 2008, 2010, 2014)	Germany (1981, 1983, 1984, 1987, 1989, 1991, 1994, 1995, 1998, 2000–2015)	Poland (1986, 1992, 1995, 1999, 2004, 2007, 2010, 2013, 2016)
Austria (1987, 1994, 1995, 1997, 2000, 2004, 2007, 2010, 2013, 2016)	Greece (1995, 2000, 2004, 2007, 2010, 2013)	Republic of Korea (2006, 2008, 2010, 2012)
Belgium (1985, 1988, 1992, 1995, 1997, 2000)	Hungary (1991, 1994, 1999, 2005, 2007, 2009, 2012, 2015)	Russia (2000, 2004, 2007, 2010, 2013–2016)
Brazil (2006, 2009, 2011, 2013, 2016)	Ireland (1987, 1994–1996, 2000, 2004, 2007, 2010)	Serbia (2006, 2010, 2013, 2016)
Canada (1981, 1987, 1991, 1994, 1997, 1998, 2000, 2004, 2007, 2010, 2013)	Israel (1979, 1986, 1992, 1997, 2001, 2005, 2007, 2010, 2012, 2014, 2016)	Slovakia (1996, 2004, 2007, 2010, 2013)
Chile (1990, 1992, 1994, 1996, 1998, 2000, 2003, 2006, 2009, 2011, 2013, 2015)	Italy (1986, 1987, 1989, 1991, 1993, 1995, 1998, 2000, 2004, 2008, 2010, 2014)	Slovenia (1997, 1999, 2004, 2007, 2010, 2012)
Colombia (2004, 2007, 2010, 2013, 2016)	Luxembourg (1985, 1991, 1994, 1997, 2000, 2004, 2007, 2010, 2013)	Spain (1980, 1985, 1990, 1995, 2000, 2004, 2007, 2010, 2013, 2016)
Czech Republic (1996, 2002, 2004, 2007, 2010, 2013)	Mexico (1984, 1989, 1992, 1994, 1996, 1998, 2000, 2002, 2004, 2008, 2010, 2012)	Sweden (1981, 1987, 1992, 1995, 2000, 2005)
Denmark (1987, 1992, 1995, 2000, 2004, 2007, 2010, 2013)	Netherlands (1983, 1987, 1990, 1993, 1999, 2004, 2007, 2010, 2013)	Switzerland (1982, 1992, 2000, 2002, 2004, 2007, 2010, 2013)
Estonia (2000, 2004, 2007, 2010, 2013)	Norway (1979, 1986, 1991, 1995, 2000, 2004, 2007, 2010, 2013)	United Kingdom (1991, 1994, 1995, 1999, 2004, 2007, 2010, 2013, 2016)
Finland (1987, 1991, 1995, 2000, 2004, 2007, 2010, 2013, 2016)	Paraguay (2000, 2004, 2007, 2010, 2013, 2016)	United States (1974, 1979, 1986, 1991, 1994, 1997, 2000, 2004, 2007, 2010, 2013, 2016)
France (1978, 1984, 1989, 1994, 2000, 2005, 2010)	Peru (2004, 2007, 2010, 2013)	Uruguay (2004, 2007, 2010, 2013, 2016)

Note. The years covered in the dataset appear in parentheses.

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