博士学位論文要約（令和２年３月）

彩色遷移問題とその一般化に関する研究

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Coloring Reconfiguration Problems and Their Generalizations

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Recently, the framework of reconfiguration problem is studied intensively in the field of theoretical computer science. The framework of reconfiguration deals with the problem where we wish to find a step-by-step transformation between initial and target configurations while preserving a constraint of some combinatorial search problem, and each step must respect a fixed reconfiguration rule. In this thesis we study a generalization of a well-studied reconfiguration problem, k-COLORING RECONFIGURATION. In k-COLORING RECONFIGURATION, we are given two feasible k-colorings of a graph G, and asked to determine whether one coloring can be transformed into the other by recoloring one vertex at a time, while always maintaining a feasible k-coloring. In this thesis we generalize the reconfiguration rule of k-coloring reconfiguration by restricting recolorable pair of colors, in the form of a (directed/undirected) graph whose vertex set is the color set $\{1, 2, \ldots, k\}$. We then give a precise analysis of the complexity status of the generalized problem with respect to some classes of graphs whose vertices are colors.

1. Introduction

Recently, the framework of (*combinatorial*) *reconfiguration* is studied intensively in the field of theoretical computer science. This framework deals with a situation where we wish to find a step-by-step transformation sequence between given two feasible states of a system, called initial and target states, while maintaining the feasibility of the system. Reconfiguration problem can be considered a problem which asks the reachability of *solution graph*, whose vertex set is a set of feasible states of the system, and whose edge set is induced by the adjacency relationship between two states. Reconfiguration problem has been studied for several combinatorial search problems, such as COLORING, INDEPENDENT SET, SHORTEST PATH, BOOLEAN SATISFIABILITY, and so on.

1.1 Coloring reconfiguration

In this thesis we study COLORING RECONFIGURATION problem which is a typical reconfiguration problem. Let G be a graph whose vertex set is $V(G)$ and edge set $E(G) \subseteq \{vw \mid v, w \in V(G)\}$. Let C be a *color* set of size k, then k-*coloring* of a graph G is a mapping $f : V(G) \to C$ such that $f(v) \neq f(w)$ for any edge $vw \in E(G)$. We denote the size of vertex set and edge set of graph G by n and m respectively. Two colorings f and f' are called *adjacent* if $|\{v \in V(G) \mid f(v) \neq f'(v)\}| = 1$. For two given colorings f_0 and f_r, k-COLORING RECONFIGURATION asks whether there exists a sequence of k-colorings $\langle f_0, f_1, \ldots, f_t \rangle$ of G such that,

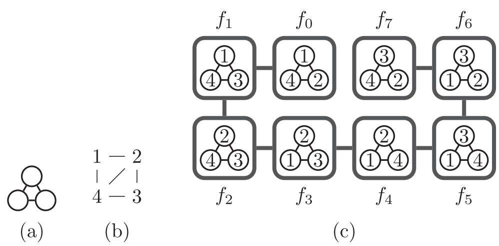

Figure 1. (a) A graph G. (b) A recolorability graph R, (c) A reconfiguration sequence from f_0 to f_r.

- $f_t = f_r$, and
- f_i and f_{i+1} are adjacent for all $i \in \{0, 1, \ldots, t-1\}$.

Such a sequence is called a *reconfiguration sequence* from f_0 to f_r. Figure 1(c) shows an example of a reconfiguration sequence $\langle f_0, f_1, \ldots, f_7 \rangle$ of 4-COLORING RECONFIGURATION where $f_0, \ldots, f_7$ are 4-colorings of the graph shown in Figure 1(a).

From the viewpoint of the number of colors, k-COLORING RECONFIGURATION is known to be linear-time solvable if the number k is at most three [3], while it is known to be PSPACE-complete for any fixed number k of colors at least four [1].

For the case of 3-COLORING RECONFIGURATION, we can also compute the shortest length of reconfiguration sequence between two given 3-colorings in linear time [3].

1.2 Our contribution

In this thesis, we investigated the complexity status of COLORING RECONFIGURATION from viewpoints of the restriction of adjacency relationship between colorings. We also study the complexity status of EDGE-COLORING RECONFIGURATION, which is equivalent to COLORING RECONFIGURATION on the line graphs.

1.2.1 Recolorability

To describe our restriction of adjacency, we introduce *recolorability graph* which is an undirected graph whose vertex set is the color set $C = \{1, 2, \ldots, k\}$. For a recolorbility graph R, two k-colorings f and f' of a graph G are *adjacent under* R if

- $|\{v \in V(G) \mid f(v) \neq f'(v)\}| = 1$, and
- $f(v)f'(v) \in E(R)$ if $f(v) \neq f'(v)$, $v \in V(G)$.

For given two k-colorings f_0 and f_r, COLORING RECONFIGURATION UNDER R-RECOLORABILITY is a problem which asks whether there exists a sequence of k-colorings $\langle f_0, f_1, \ldots, f_t\rangle$ of G such that,

- $f_t = f_r$, and
- f_i and f_{i+1} are adjacent under R for all $i \in \{0, 1, \ldots, t-1\}$.

Figure 1(c) shows an example of COLORING RECONFIGURATION UNDER R-RECOLORABILITY where R is shown in Figure 1(b).

Notice that the ordinary k-coloring reconfiguration problem is equivalent to COLORING RECONFIGURATION PROBLEM UNDER R-RECOLORABILITY where R is a complete graph K_k. In this thesis we assume that the recolorability graph R is connected since if it has two disconnected compoenents R_1 and R_2, solving an instance of COLORING RECONFIGURATION UNDER R-RECOLORABILITY is equivalent to solve two subproblems whose recolorability graphs are R_1 and R_2. In this thesis we analyze the complexity status of COLORING RECONFIGURATION UNDER R-RECOLORABILITY with respect to the structure of the recolorability graph R.

1.2.2 Irreversible rules

We introduce further restriction of adjacency, by extending the recolorability graph to directed graphs. *Directed recolorability graph* denoted by $\overrightarrow{R}$, is a directed graph whose vertex set is the color set $\{1, 2, \ldots, k\}$ of size k. We denote the arc set of $\overrightarrow{R}$ by $A(\overrightarrow{R})$. Then, for given two k-colorings f_0 and f_r of a graph G, COLORING RECONFIGURATION UNDER $\overrightarrow{R}$ asks whether there exists a sequence $\langle f_0, f_1, \ldots, f_t\rangle$ of k-colorings such that

- $f_t = f_r$, and
- f_i and f_{i+1} are adjacent under the underlying graph of $\overrightarrow{R}$ for all $i \in \{0, 1, \ldots, t-1\}$, and
- if $f_i(v) \neq f_{i+1}(v)$, $v \in V(G)$ then $(f_i(v), f_{i+1}(v)) \in A(\overrightarrow{R})$ for all $i \in \{0, 1, \ldots, t-1\}$.

1.2.3 Edge-coloring reconfiguration

In this thesis we also study EDGE-COLORING RECONFIGURATION. An *edge-coloring* of a graph G is a map $f : E(G) \to C$ which assigns colors in a color set $C = \{1, 2, \ldots, k\}$ to each edge $e \in E(G)$ so that every adjacent two edges have different colors. Two edge colorings f and f' are *adjacent* if $|\{e \in E(G) \mid f(e) \neq f'(e)\}| = 1$. Then EDGE-COLORING RECONFIGURATION is defined analogously to k-COLORING RECONFIGURATION.

We also study a variant of EDGE-COLORING RECONFIGURATION, LIST EDGE-COLORING RECONFIGURATION. A *list* $L : E(G) \to 2^C$ is a mapping which maps a subset of the color set for each edge. Then *list edge-coloring* f of a graph G is a edge-coloring satisfying $f(e) \in L(e)$ for each edge $e \in E(G)$. For two list edge-colorings f_0 and f_r of a graph G and a list L, LIST EDGE-COLORING RECONFIGURATION asks whether there exists a reconfiguration sequence from f_0 to f_r in which all edge-colorings are list edge-colorings.

LIST EDGE-COLORING RECONFIGURATION is polynomial-time solvable if the number of all the colors in $\bigcup_{e \in E(G)} L(e)$ is at most three, on the other hand it is known to be PSPACE-complete if the number of colors is at least six [2].

2. Our results

In the remaining of this thesis we show our result of three kinds of problems described in Section 1.2.1,1.2.2 and 1.2.3 in Section 2.1, 2.2 and 2.3 respectively.

2.1 Complexity status with recolorability

In this subsection we show the results of COLORING RECONFIGURATION UNDER R-RECOLORABILITY with respect to the structure of the recolorability graph R. From the viewpoint of tractability, we show the following theorem.

Table 1. Computational complexities of COLORING RECONFIGURATION UNDER R-RECOLORABILITY

	maximum number of cycles contained in one connected component of R	
maximum degree $\Delta(R)$ of R	at most one	at least two
$\Delta(R) \leq 2$	Linear	(Such a graph does not exist)
$\Delta(R) = 3$	Some results	PSPACE-c
$\Delta(R) \geq 3$	PSPACE-c	PSPACE-c

Theorem 1. COLORING RECONFIGURATION UNDER R-RECOLORABILITY *is solvable in* $O(k+n+m)$ *time if* R *is of maximum degree at most two.*

Since K_3 has maximum degree two, this generalizes the known result of 3-coloring reconfiguration which is linear-time solvable [3]. We also generalize the known result that the shortest length of reconfiguration sequence can be computed in linear time [3] as follows.

Theorem 2. *The shortest length of reconfiguration sequence of* COLORING RECONFIGURATION UNDER R-RECOLORABILITY *can be computed in* $O(k+n+m)$ *time if* R *is of maximum degree at most two.*

On the complexity side, we show that computational complexity is closed under supergraph relationship of recolorability graphs in the sense of polynomial-time reducibility as follows.

Theorem 3. COLORING RECONFIGURATION UNDER R-RECOLORABILITY *can be reduced to* COLORING RECONFIGURATION UNDER R'-RECOLORABILITY *in polynomial-time if* R' *is a supergraph of* R.

We show that COLORING RECONFIGURATION UNDER R-RECOLORABILITY is PSPACE-complete if R is a star $K_{1,4}$ with five vertices. Theorem 3 extends this result to any recolorability graph R of maximum degree at least four.

Theorem 4. COLORING RECONFIGURATION UNDER R-RECOLORABILITY *is PSPACE-complete if* R *is of maximum degree at least four.*

We also show that COLORING RECONFIGURATION UNDER R-RECOLORABILITY is PSPACE-complete if R has two cycles.

Theorem 5. COLORING RECONFIGURATION UNDER R-RECOLORABILITY *is PSPACE-complete if* R *has two cycles.*

Table 1 summarises our results. For the case where R is of maximum degree three and R has at most one cycle, we also show complexity status for some typical small graphs.

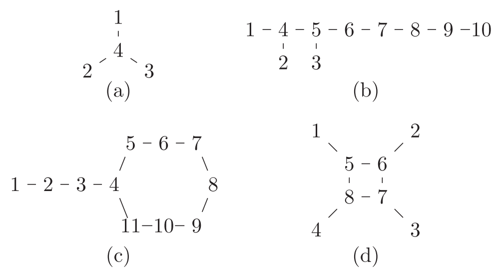

Figure 2. (a) A claw graph. (b) R_1. (c) R_2. (d) R_3.

Theorem 6. COLORING RECONFIGURATION UNDER R-RECOLORABILITY *is*

- *polynomial-time solvable if R is claw (shown in Figure 2(a)),*
- *NP-hard if R is R_1 (shown in Figure 2(b)),*
- *NP-hard if R is R_2 (shown in Figure 2(c)), and*
- *PSPACE-complete if R is R_3 (shown in Figure 2(d)).*

2.2 Complexity status with irreversible rules

In this subsection we show complexity status of COLORING RECONFIGURATION UNDER $\overrightarrow{R}$-RECOLORABILITY with respect to the structure of a directed recolorability graph $\overrightarrow{R}$. We mainly deal with the case where $\overrightarrow{R}$ is a directed acyclic graph (DAG). If $\overrightarrow{R}$ is a DAG, any vertex of G can be recolored at most $k-1$ times from initial coloring. Therefore if a reconfiguration sequence from a given initial coloring to a given target coloring exists, its length is at most $n(k-1)$. Therefore the following lemma follows.

Lemma 1. *If* $\overrightarrow{R}$ *is a DAG,* COLORING RECONFIGURATION UNDER $\overrightarrow{R}$-RECOLORABILITY *is in NP.*

We show results of hardness and algorithm of our problem. First, we show NP-completeness of our problem for a particular directed recolorability graph.

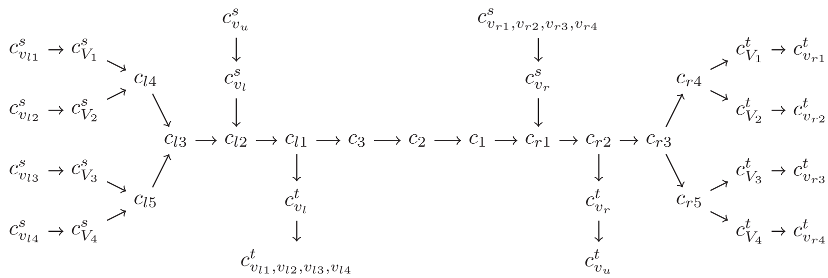

Figure 3. A directed recolorability graph $\overrightarrow{R_4}$.

Theorem 7. Coloring reconfiguration under $\overrightarrow{R}$-recolorability *is NP-complete if* $\overrightarrow{R}$ *is* $\overrightarrow{R_4}$ *(shown in Figure 3).*

On the other hand, we show that coloring reconfiguration under $\overrightarrow{R}$-recolorability can be solved in polynomial time if $\overrightarrow{R}$ is a *rooted tree* which is a directed graph whose underlying graph is a tree, and having a special vertex called *root* such that for any vertex v there is a directed path from root to v.

Theorem 8. Coloring reconfiguration under $\overrightarrow{R}$-recolorability *is polynomial-time solvable if* $\overrightarrow{R}$ *is a rooted tree.*

2.3 Edge-coloring reconfiguration

In this subsection we show PSPACE-completeness of edge-coloring reconfiguration and list edge-coloring reconfiguration. PSPACE-completeness of both problems are proved by polynomial-time reduction from a known PSPACE-complete problem, nondeterministic constraint logic (NCL). NCL is defined on an undirected graph called *constraint graph* which has weights on each vertices and edges. NCL asks whether two orientation of constraint graph can be reachable under specific changing rule. NCL is known to be PSPACE-complete even if the constraint graph is planar, bounded bandwidth, and maximum degree three [4]. Polynomial-time reduction from NCL to list edge-coloring reconfiguration is done by replacing each vertex of constraint graph by several types of gadgets of constant size, therefore properties of constraint graph are inherited.

Theorem 9. *For any fixed number of colors at least four,* list edge-coloring reconfiguration *is PSPACE-complete even if the graph is planar and has bounded bandwidth and maximum degree three.*

For edge-coloring reconfiguration, the sizes of gadgets depend on the number of colors, therefore bandwidth and maximum degree depend on the number of colors.

Theorem 10. *For any fixed number of colors at least five,* edge-coloring reconfiguration *is PSPACE-complete even if the graph is planar and has bounded bandwidth and bounded maximum degree.*

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