Gradient Estimates of Harmonic Functions and the Asymptotics of Spectral Gaps on Path Spaces

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We show that the spectral gap of the Dirichlet form on the path space \( P_t(M) \rightarrow M; y(0) = x \) goes to 0 exponentially, when \( T \rightarrow \infty \). Here, \( M \) is a compact negatively curved manifold. This contrasts with the case of positive curvature. It is proved by using a gradient estimate of bounded harmonic functions on negatively curved manifolds.

KEYWORDS: gradient estimate, harmonic function, path space, Ricci curvature

1. Introduction

In this note, we study the asymptotic behaviour of the spectral gap \( C_G(SG) \) of the Dirichlet form on the path space \( P_t(M) \rightarrow M; y(0) = x \) as \( T \rightarrow \infty \), where

\[
C_G(SG) = \inf_{F(\omega) \in \mathcal{F}} \frac{\int_{P_t(M)} \|\nabla F(y)\|^2 dP_{x,T}}{\int_{P_t(M)} (F(y) - \langle F \rangle_y)^2 dP_{x,T}},
\]

\( M \) is a compact Riemannian manifold and \( P_{x,T} \) denotes the Brownian motion measure.

The definition of the derivative \( \nabla \) will be given in the next section. In the case of usual Wiener space, \( C_G(SG) = 1 \) holds. S. Fang [13] has used the Clark-Haussmann Ocone formula to prove the existence of the spectral gap, which depends on the Ricci curvature of the manifold. By using his result, in the case where the Ricci curvature is strictly positive, \( C_G(SG) \) remains positive even if \( T \rightarrow \infty \). However, we note that in the negative Ricci case, his estimate just gives the lower bound of the spectral gap, which goes to 0 exponentially as \( T \rightarrow \infty \). In the present paper, we construct a function which shows the spectral gap actually tends to 0 exponentially if a certain Riemannian covering manifold \( \bar{M} \) of \( M \) admits a nonconstant bounded harmonic function \( u \). This is shown by using a gradient estimate of \( u \). If the Ricci curvature satisfies the relationship, \( -\infty < -b^2 \leq \text{Ric} \leq -\bar{a}^2 < 0 \), then it holds that

\[
\|\bar{P}_t\|_{\nabla u}^2 \leq b^2 e^{-\bar{a}t} \|u\|_{L}^2,
\]

where \( \bar{P}_t = e^{t\bar{\Delta}/2} \) is a diffusion semigroup on \( \bar{M} \). Note that the author does not know whether \( C_G(SG) \) is an eigen value or whether or not its eigen function can be obtained by harmonic functions or not.

Clearly the estimate (1.2) is closely related to Yau’s celebrated estimate [20]. Our proof of (1.2) is another proof of his inequality in a certain special case. We refer the reader to Section 3. Our proof just uses the Ito formula and the commutation relation between the Laplacian and the covariant derivative: \( [\bar{\Delta} - \text{Ric}]\nabla = \nabla \Delta \). Attension is drawn to the Lemma and its application to prove Theorem 3.2. Note also the interesting work of the probabilistic treatment of the gradient estimate of harmonic functions using coupling by Cranston [3].

Remark 1.1 (Note added in press.) Wang [19] has obtained a better estimate than Theorem 3.2(1):

\[
\|\nabla u\|_{L}^2 \leq \frac{b^2}{2\pi} \|u\|_{L}^2,
\]

under the same assumption as in Theorem 3.2. Professor Wang used the method of coupling, as did Cranston.

2. Fang’s Estimate of the Spectral Gap

Fang proved the following extension of the Clark-Haussman-Ocone formula in the paper [13]. To state the theorem, we prepare some notations. Let \( \tau_t \) be a stochastic parallel translation operator along the Brownian curve \( \gamma_t, \tau_t: T_xM \rightarrow T_{\gamma_t}M \). Here, we adopt the Levi-Civita connection and define the derivative over path space
according to Driver [8] and Léandre [16]. We denote by \( \mathcal{P}_\infty^\oplus(P_x(M)_x) \) the space of smooth cylindrical functions. For \( F(\gamma) = f(\gamma, \cdots, \gamma_n) \), we define
\[
\nabla F(\gamma)_i = \sum_{j=1}^n \tau_j^{-1} \partial_j f(\gamma_i) \eta_i \wedge \tau_i
\]
where \( \partial_i \) denotes the derivative with respect to the \( i \)-th coordinate. We denote \( \nabla_i f(\gamma_i) = \tau_i^{-1} \partial_i f(\gamma_i) \). The norm is as follows:
\[
|\nabla F(\gamma)|^2 = |\nabla F(\gamma)|^2_{H_t} = \sum_{i=1}^n (t_{i+1} - t_i) \sum_{j=i+1}^n |\nabla_i f(\gamma_i)|^2.
\]
This expression was used to prove the log-Sobolev inequality on path spaces by Hsu [15].

For the tensor \( T \in F(\otimes^p T^*M \otimes^q T^*M) \), we denote
\[
\bar{T}(\gamma)_i = \tau_i^{-1} T(\gamma_i)(\eta_i) \in F(\otimes^p T^*M \otimes^q T^*M).
\]
Using this notation, let us consider a compact operator in \( L^2([0, T] \to T_xM) \) which depends on the sample path \( \gamma \):
\[
R(\gamma)\psi(t) = \frac{1}{2} \text{Ric}(\gamma) \int_0^t \psi(s) ds.
\]
In what follows, \( I \) denotes the identity operator and \( (I + R)^{-1} \) denotes the adjoint operator on \( L^2([0, T] \to T_xM) \). Also we denote by \( b(t) \) the Ito-Cartan development of \( \gamma_i \) into \( T_xM \). Finally, we consider a \( \sigma \)-field \( \bar{\mathcal{P}}_i = \sigma(\gamma(s): 0 \leq s \leq t) \). Now we can state Fang’s theorem.

**Theorem 2.1** (S. Fang [13]) Let \( F(\gamma) = \int_{P_x(M)_x} F(\gamma) dP_x \), it holds that
\[
F(\gamma) - \int_{P_x(M)_x} F(\gamma) dP_x = \int_0^T \left( E[(I + R)^{-1} \nabla F(\gamma)|_{\bar{\mathcal{P}}_i}, dB_i] \right),
\]
where \( E[\cdot | \bar{\mathcal{P}}_i] \) denotes a conditional expectation.

Using this theorem, we get

**Theorem 2.2** Let us assume that \( \text{Ric} \geq c_1 |\text{Ric}| \leq c (c_1 \in \mathbb{R}, c \geq 0) \). Then we have
\[
C_f(SG) \geq \left( 1 + \frac{c}{2|c_1|} \sqrt{e^{-\alpha t} - 1 + c_1 T} \right)^{-2}.
\]
This theorem is essentially due to Fang, although he did not give the explicit bounds as expressed in the above theorem.

**Proof.**
Since
\[
(I + R)^{-1} h(t) = h(t) - \frac{1}{2} \text{Ric}(\gamma) \int_0^t v_t v_t^{-1} h(s) ds,
\]
where \( v_t \) is the solution to the following ODE,
\[
\dot{v}_t = -\frac{1}{2} \text{Ric}(\gamma) v_t, \quad v_0 = \text{id}.
\]
First of all, we shall prove that
\[
\|(I + R)^{-1}\|_{\text{op}} \leq \left( 1 + \frac{c}{2|c_1|} \sqrt{e^{-\alpha t} - 1 + c_1 T} \right).
\]
Here \( \| \cdot \|_{\text{op}} \) denotes the operator norm. Noting
\[
\|(I + R)^{-1} h\|_{L^2} \leq \|h\|_{L^2} + \left\{ \int_0^T \left[ \frac{1}{2} \text{Ric}(\gamma) \int_0^t v_t v_t^{-1} h(s) ds \right]^2 dt \right\}^{1/2},
\]
\[
\|v_t v_t^{-1}\|_{\text{op}} \leq e^{-\alpha (t-\beta)/2},
\]
we have
\begin{align*}
\int_0^T \left| \frac{1}{2} \operatorname{Ric}(y), v_t^{-1} h(s) ds \right|^2 dt &\leq \int_0^T \frac{r^2}{4} \left| \int_0^\gamma v_t^{-1} h(s) ds \right|^2 dt \\
n &\leq \int_0^T \frac{r^2}{4} \left( \int_0^\gamma e^{-c(t-s)} ds \int_0^\gamma |h(s)|^2 ds \right) dt \\
n &\leq \frac{r^2}{4} \left( \int_0^\gamma \left( 1 + e^{-cT} + cT \right) - \left( cT + c_1 |h(s)|^2 \right) ds \right) dt \\
n &\leq \frac{r^2}{4} \left( \int_0^\gamma (e^{-cT} + cT - 1) \int_0^\gamma |h(s)|^2 ds. \right)
\end{align*}

This shows (2.2). Using this, we have
\begin{align*}
\| E_t[F - E_t[F]^2] \|^2 &\leq \int_0^T E_t\left[ |I + R|^{-1} \right. \cdot \left. \mathcal{F}(y), |h(s)|^2 \right] dt \\
n &\leq \left( 1 + \frac{r}{2c_1} \cdot \sqrt{e^{-cT} - 1} \cdot cT \right) E_t[\mathcal{F}(y)^2]. \quad \blacksquare
\end{align*}

The following corollary is an immediate and rough consequence of Theorem 2.2.

**Corollary 2.3** Assume that $-\infty < -b^2 \leq \operatorname{Ric} \leq -\alpha < 0$. Then we have
\begin{align*}
C_T(SG) \geq \frac{4}{9} e^{-bT}.
\end{align*}

In the next section, we give an upper bound to $C_T(SG)$ by using a gradient estimate of harmonic functions. By a method similar to Theorem 2.2, we can prove the following theorem. This theorem shows the existence of the spectral gap even if we take an infinite volume limit $T \to \infty$ for a positive Ricci curvature.

**Theorem 2.4** Assume that $0 < c_1 \leq \operatorname{Ric} \leq c_2 < \infty$. Then it holds that
\begin{align*}
C_T(SG) \geq \left( 1 + \frac{c_2}{c_1} \cdot \sqrt{1 - e^{-cT/2}} \right)^{-2} \geq \left( 1 + \frac{c_2}{c_1} \right)^{-2}.
\end{align*}

**Proof.** We will estimate
\begin{align*}
J = \int_0^T \left| \frac{1}{2} \operatorname{Ric}(y), v_t^{-1} h(s) ds \right|^2 dt
\end{align*}
in the following way. For a nonnegative function $f$, it holds that
\begin{align*}
\left( \int_0^\gamma f(s) g(s) ds \right)^2 \leq \int_0^\gamma f(s) ds \int_0^\gamma f(s) g(s)^2 ds.
\end{align*}

Hence, taking $f(s) = e^{-c(t-s)^2}$, we see that
\begin{align*}
J &\leq \frac{c_1}{2c_1} \int_0^T \left( 1 - e^{-cT/2} \right) \left( \int_0^\gamma e^{-c(t-s)^2} |h(s)|^2 ds \right) dt \\
n &\leq \frac{c_1}{2c_1} \int_0^T \left( -2e^{-cT/2} + e^{-cT} \right) \left( \int_0^\gamma e^{cT/2} |h(s)|^2 ds \right) dt \\
n &\leq \frac{c_1}{2c_1} \int_0^T \left( -2e^{-cT/2} + e^{-cT/2} + 2 - e^{-cT/2} \right) |h(t)|^2 dt \\
n &\leq \left( \frac{c_1}{c_1} \right)^2 \left( 1 - e^{-cT/2} + e^{-cT/2} - \frac{e^{-cT/2}}{2} \right) \int_0^T |h(t)|^2 dt \\
n &\leq \left( \frac{c_1}{c_1} \right)^2 \left( 1 - e^{-cT/2} \right) \int_0^T |h(t)|^2 dt.
\end{align*}

The proof is completed by proceeding as for the proof of Theorem 2.2. \quad \blacksquare

3. Gradient Estimates of Harmonic Functions

We will begin by proving Theorem 3.2. Note that in this section, we do not assume the compactness of Rie-
manian manifold \((M, g)\). First, note the following Yau's estimate:

**Theorem 3.1** (S. T. Yau [20]) Assume that the Ricci curvature is bounded below:

\[
\text{Ric} \geq -b^2 > -\infty.
\]  

(3.1)

Then for positive harmonic function \(u\) on \(M\), we have

\[
|\nabla u(x)|^2 \leq (n - 1)b^2 |u(x)|^2,
\]

where \(n\) is the dimension of \(M\).

Using this estimate, we prove the following theorem. Note that actually we need just the boundedness of \(\nabla u\) and we can prove Theorem 3.2 without using Yau’s estimate but using Lemma below. Currently, the author does not know whether or not the Lemma holds generally. However the Lemma can be proven in very simple situation of rotationally symmetric Riemannian metric, and so we see Theorem 3.2 without using the boundedness of \(\nabla u\) (see Proposition 3.3). Below, we also denote by \(\nabla u\) the derivative of function \(u\) on \(M\).

**Theorem 3.2** (1) Assume that the Ricci curvature is bounded below:

\[
\text{Ric} \geq -b^2 > -\infty.
\]

Then for any bounded harmonic function \(u\), we have

\[
\|\nabla u\|_\infty^2 \leq b^2 \|u\|_\infty^2.
\]  

(3.3)

(2) Furthermore, if the Ricci curvature is bounded above:

\[-\infty < -b^2 \leq \text{Ric} \leq -a^2 < 0,
\]

then we have

\[
\|P_t \nabla u\|_\infty^2 \leq b^2 e^{-a^2 t} \|u\|_\infty^2.
\]  

(4.4)

**Proof.** First note that the Brownian motion on \(M\) does not explode because of another Yau’s theorem. By the Ito formula,

\[
u(y) = u(x) + \int_0^\infty \langle \nabla u(y), db(s) \rangle.
\]

Setting the stopping time: \(\tau_R(y) = \inf\{t > 0|\gamma(t) \notin B(x, R)\}\), we have

\[
E_{\nu}(|u(y_{\tau_R}) - u(x)|^2) = E_x \left[ \int_0^{\tau_R} |\nabla u(y)_s|^2 ds \right],
\]

where \(B(x, R)\) denotes a ball of radius \(R\) centered at \(x\). Hence, on first letting \(R \to \infty\) and next \(t \to \infty\), we obtain

\[
\int_0^{\infty} E_x(|\nabla u(y)|^2)dt \leq \lim_{t \to \infty} E_x(|u(y_t) - u(x)|^2) \leq \|u\|_\infty^2.
\]  

(3.5)

Here we apply the Ito formula to the square of gradient of \(\nabla u\) and get

\[
dt |\nabla u(y)|^2 = 2<\nabla^2 u(y), \nabla u(y)> + \Delta u(y)dt + |\nabla^2 u(y)|dt.
\]

By using the commutation relation \((\Delta - \text{Ric})\nabla = \nabla \Delta \) and \(\Delta u = 0\), we get

\[
\|\nabla u(y)|^2 = 2<\nabla^2 u(y), \nabla u(y)> + (\text{Ric}(y)\nabla u(y), \nabla u(y))dt + |\nabla^2 u(y)|^2dt.
\]  

(3.6)

By using the stopping time \(\tau_R\), we see

\[
E_x(|\nabla u(y_{\tau_R})|^2) = |\nabla u(x)|^2 + E_x \left[ \int_0^{\tau_R} (\text{Ric}(y)\nabla u(y), \nabla u(y))ds \right] + E_x \left[ \int_0^{\tau_R} |\nabla^2 u(y)|^2 ds \right].
\]  

(3.7)

Using the boundedness of \(\nabla u\) and letting \(R \to \infty\) and next \(t \to \infty\), and using the fact that there exists a sequence \(\{\tau_n\} \to \infty\) such that \(\lim_{t \to \infty} E_x(|\nabla u(y_{\tau_n})|^2) = 0\), we see that \(\int_0^{\infty} E_x(|\nabla^2 u(y)|^2)<\infty\) and

\[
E_x(|\nabla u(y)|^2) = |\nabla u(x)|^2 + \int_0^{\infty} E_x(|(\text{Ric}(y)\nabla u(y), \nabla u(y))|)ds + \int_0^{\infty} E_x(|\nabla^2 u(y)|^2)dt.
\]  

(3.8)

By the assumption on the Ric and the inequality (3.5), we have

\[
\int_0^{\infty} E_x(|\nabla^2 u(y)|^2)dt \leq b^2 \|u\|_\infty^2 - |\nabla u(x)|^2.
\]  

(3.9)

This proves Theorem 3.2 (1). Second we prove Theorem 3.2 (2). By (3.8) and (3.9), we have
\[ E_x[|\nabla u(y_t)|^2] \leq b^2\|u\|_2^2 - a^2 \int_0^t E_x[|\nabla u(y_t)|^2]ds. \]

Using the Gronwall inequality, this implies (3.4). □

If the following lemma is true, then we can prove the boundedness of \( \nabla u \) using a modification of the above simple argument. In the lemma below, we do not put any assumptions on the curvature.

**Lemma** Let \((M, g)\) be a complete Riemannian manifold which has a Green function \( G(x, y) = \int e^{-p_t(x, y)} dt < \infty \). We assume that the Brownian motion is nonexplosive. For a nonnegative function \( f \), assume that there exists \( x \in M \), such that
\[ E_x\left[ \int_0^\infty f(y_t) dt \right] \leq \int_M G(x, y) f(y) dy < \infty. \]

Then there exists an exhaustion sequence of domains of \( M \), \( D_n \subset M \) and a dense subset \( K \) such that
1. \( D_n \) is compact in \( M \),
2. the boundaries, \( \partial D_n \), are smooth,
3. \( D_n \subset D_{n+1} \) and \( \bigcup D_n = M \),
4. \( \lim_{n \to \infty} E_x[f(y_{\tau_{D_n}})] = 0 \), where \( y \in K \) and \( \tau_{D_n} \) is a hitting time of \( y \) to \( D_n \).

Note that in this Lemma, \( \tau_{D_n} < \infty \) a.s. holds. If the Lemma holds, by the Harnack inequality, the solution of the Dirichlet problem, \( E_x[f(y_{\tau_{D_n}})] \) actually goes to 0 uniformly on compact sets with respect to \( z \).

**Proof of Theorem 3.2** When applied to Theorem 3.2, the above lemma (if it holds) is sufficient to prove (3.3). We may assume that \((M, g)\) has a nonconstant bounded harmonic function. Then, as is well known, \((M, g)\) has a Green function \( G(x, y) \). Next notice that we can take \( f(x) = |\nabla u(x)|^2 \) because of (3.1). Instead of \( \tau_{\mathcal{R}_n} \), we take \( \tau_{D_n} \). The equation (3.6) is valid for these stopping times. At this stage, taking the limit \( t \to \infty \) and next \( n \to \infty \), then by using (4) in the Lemma, we get for \( y \in K \),
\[ E_x\left[ \int_0^\infty |\nabla^2 u(y_t)|^2 dt \right] = -E_x\left[ \int_0^\infty \langle \text{Ric}(y_t)\nabla u(y_t), \nabla u(y_t) \rangle dt \right] - |\nabla u(y_t)|^2 \leq b^2\|u\|_2^2 - |\nabla u(y_t)|^2 < \infty. \]

Using the denseness of \( K \), (3.10) implies (3.3). □

In this paper, the author cannot prove the above Lemma unconditionally. However, if we restrict ourselves to the following simple cases (1) and (2), the Lemma clearly holds.

1. \((M, g)\) has a pole \( x \in M \).
   Under this assumption, we have the geodesic polar coordinate \( y = \exp (r \theta) \), where \( r = d(x, y) \), \( \theta \in S^{d-1} \) (\( d \) is a dimension of \( M \) and \( |\theta| = 1 \)) and \( \exp \) denotes the exponential map.
2. The Riemannian metric is rotationally symmetric, i.e., the Riemannian metric \( ds^2 \) is expressed as follows:
\[ ds^2 = dr^2 + f(r)d\theta^2. \]

Under (1) and (2), \( p_t(x, y) \), \( G(x, y) \) are functions of \( r \), so we denote them by \( p_t(r) \), \( G(r) \) respectively.

**Proposition 3.3** The above lemma holds for the manifolds which satisfy the above (1) and (2). In particular, the Lemma holds for the Euclidean space whose dimension is greater than 2 and for the hyperbolic space \( H^{n+1}(-1) \).

**Proof.** Clearly the harmonic measure \( Q_t(x, dy) \) on the boundary \( S_t(x) \) of \( B(x, r) \) starting from \( x \) is a uniform probability measure on the sphere because of the rotationally symmetric argument on the metric. So we get
\[ \int_M G(x, y)f(y)dy = \int_0^\infty \{ \int_{S_t(x)} f(y)Q_t(x, dy) \}G(r)V(r)dr < \infty, \]
where \( V(r) \) denotes the volume of the ball \( B(x, r) \). Since the Brownian motion is nonexplosive, \( \int_M G(x, y)dy = \infty \) holds. Noting that the hitting distribution of \( y_t \) is merely the harmonic measure, we see that there exists \( R_n \to \infty \) (\( n \in \mathbb{N} \)) such that
\[ \lim_{n \to \infty} E_x[f(y_{\tau_{R_n}})] = \lim_{n \to \infty} \left[ \int_{S_t(x)} f(y)Q_{R_n}(x, dy) \right] = 0. \]

Note that for \( 0 < r < R_n \), we have
\[ E_x[f(y_{\tau_{R_n}})] = \int_{S_t(x)} E_y[f(y_{\tau_{R_n}})Q(x, dy)]. \]

Hence, using a diagonal process, we see that for \( K = \bigcup_{n \geq 0} K_n \), the Lemma holds. Here, \( Q^+ \) denotes the set of
4. Spectral Gap Estimate on Path Spaces on Negatively Curved Manifolds

In this section, we prove the exponential decay of the spectral gap $C_T(SG)$. Before doing so, we prove the following lemma. In what follows, we fix a Riemannian cover of $M$, $\tilde{M}$ and fix a starting point $\tilde{x}(\pi(\tilde{x}) = x)$ in $M$. Also, let us denote the lift of Brownian motion $\gamma$ to $\tilde{M}$ by $\tilde{\gamma}$ ($\tilde{\gamma}_0 = x$). Finally, we will denote the derivative on the path space on $\tilde{M}$ by $\tilde{\nabla}$ and the diffusion semigroup on $\tilde{M}$ by $\tilde{P}_t$.

**Lemma 4.1** Let $f$ be a $C^1$ function on $\tilde{M}$ with $\|\partial f\|_{\infty} < \infty$. Then $f(\tilde{\gamma}_t) \in D(\partial)$ and

$$\nabla(f(\tilde{\gamma}_t)) = \pi \ast \tilde{\nabla}(f(\tilde{\gamma}_t)), \quad (4.1)$$

where $\tilde{\nabla}$ is the derivative on the path space on $\tilde{M}$. Especially

$$E_x[|\nabla(f(\tilde{\gamma}_t))|^2] = t \tilde{P}_t[|\nabla f|^2]. \quad (4.2)$$

**Proof.** We may assume that $M$ is isometrically embedded in a Euclidean space $\mathbb{R}^d$ and let us consider a function $\Phi$ for the Hölder continuous path $\gamma$

$$\Phi_{\alpha,l}(\gamma) = D_{\alpha,l} \int_0^{T} \int_0^{T} \frac{|\gamma(t) - \gamma(s)|^{2l}}{|t - s|^{1 + 4\alpha}} ds dt,$$

where $l (\geq 4)$ denotes an integer and $\alpha$ is in $(0, 1/5)$ with $2\alpha l \geq 1$ and we choose $D_{\alpha,l}$ so that the following bound holds:

$$\|\gamma\|_{C^\alpha} \leq \Phi_{\alpha,l}(\gamma)^{1/2l}. \quad (4.3)$$

Here $\| . \|_{C^\alpha}$ denotes the $\alpha$-Hölder norm. (See for example, Stroock and Varadhan [18].) Note that, actually, the equation (4.3) holds for a positive number $l$ satisfying the above condition. Next let us take a smooth cutoff function $\phi$ such that

$$\phi(x) = \begin{cases} 1, & (|x| \leq 1/2) \\ 0, & (|x| \geq 1). \end{cases}$$

Let us set $\phi_{\alpha,l}(\gamma) = \phi(\Phi_{\alpha,l}(\gamma)/n)$. We prove the following claims:

1. $\phi_{\alpha,l}(\gamma) \in D(\partial)$ and $\lim_{n \to \infty} \Phi_{\alpha,l}(\gamma) = 1$ in $\mathcal{D}'$ sense.

2. For $n \in \mathbb{N}$ there exists a sequence $\{0 = t_1 < t_2 < \ldots < t_{mn(n)} = t\}$ and a smooth function $g$ on $\tilde{M} \times \cdots \times \tilde{M}$ such that $g(\gamma_1, \ldots, \gamma_m(n)) \Phi_{\alpha,l}(\gamma) = f(\tilde{\gamma}) \Phi_{\alpha,l}(\gamma)$.

Consequently $f(\tilde{\gamma}) \Phi_{\alpha,l}(\gamma)$ is $D(\partial)$.

3. It holds that

$$\nabla(f(\tilde{\gamma}) \Phi_{\alpha,l}(\gamma)) = \pi \ast \tilde{\nabla}f(\tilde{\gamma}) \Phi_{\alpha,l}(\gamma) + f(\tilde{\gamma}) \nabla \Phi_{\alpha,l}(\gamma).$$

Hence, these claims proves the lemma. We prove (1). As is well-known, it holds that there exist constants $A$ and $B$

$$P(\Phi_{\alpha,l}(\gamma) \geq r) \leq A e^{-Br^2}.$$

This shows the $L^2$ convergence of $\Phi_{\alpha,l}(\gamma)$ to 1.

Let us set for $0 < \kappa < T$,

$$\Phi_{\alpha,l,\kappa}(\gamma) = D_{\alpha,l} \int_0^{T} \int_0^{T} \frac{|\gamma(t) - \gamma(s)|^{2l}}{|t - s|^{1 + 4\alpha}} ds dt.$$

Then clearly $\Phi_{\alpha,l,\kappa}(\gamma) \in D(\partial)$ and

$$I_\kappa(\gamma) = |(\nabla \Phi_{\alpha,l,\kappa}(\gamma), h)| \leq C_{\alpha,l} \int_0^{T} \int_0^{T} |\gamma(t) - \gamma(s)|^{2l-1} |\tau_i h_i - \tau_i h(s)| \frac{ds dt}{|t - s|^{1 + 4\alpha}}, \quad (4.4)$$

where $C_{\alpha,l}$ is a constant. Let us take a Fourier series orthonormal basis

$$e_{2k,l} = \frac{T \sin(2\pi k t/T)}{2k\pi} e_i, \quad (k \geq 1)$$

$$e_{2k+1,l} = \frac{T \cos(2\pi k t/T)}{2k\pi} e_i, \quad (k \geq 1)$$

where $e_i$ is a basis of $\mathbb{R}$. Then

$$E_x[|\nabla(\phi_{\alpha,l})(\tilde{\gamma}_t)|^2] \leq C_{\alpha,l} \int_0^{T} \int_0^{T} |\gamma(t) - \gamma(s)|^{2l-1} |\tau_i h_i - \tau_i h(s)| \frac{ds dt}{|t - s|^{1 + 4\alpha}}, \quad (4.4)$$

where $C_{\alpha,l}$ is a constant. Let us take a Fourier series orthonormal basis
$$e_{i,i} = \frac{t}{\sqrt{T}} e_i,$$

where \(\{e_i\}\) denotes an orthonormal basis in \(T_x M\). Then, nothing that

$$\|\tau_t e_{i,i}(t) - \tau_t e_{i,i}(s)\| \leq \frac{\sqrt{2T}}{k\pi}$$

and by (4.4), we have

$$I_{\alpha,0}(\gamma)^2 \leq \frac{2T}{k^2 \pi^2} \Phi_{(\alpha/(-1/2))\alpha,0,0}^2 \Phi_{(\alpha/(-1/2))\alpha,0,1/2,\kappa}(\gamma)^2.$$ 

Hence

$$\|\nabla(\Phi_{\alpha,0}(\gamma) / n)\| \leq \text{const.} \frac{1}{n} \|\Phi_n^\prime\| \Phi_{(\alpha/(-1/2))\alpha,0,1/2,\kappa}(\gamma).$$

Taking the limit \(\kappa \to 0\), we get

$$\lim_{n \to \infty} \|\nabla(\Phi_{\alpha,0}(\gamma))\|_2 = 0.$$ 

Consequently, we have shown (1).

Next let us prove (2). For \(\gamma\) with \(\Phi_{\alpha,0}(\gamma) \neq 0\), it holds that \(\sup_{t \in [a]} \int_t^s f(s) - f(s) \leq n^{1/2} |t - s|^{1/2}\). Also note that there exists a positive number \(\delta\) such that for any \(x \in M\), there exists a one to one map \(\sigma : B(x, \delta) \to \tilde{M}\) satisfying \(\sigma \cdot \sigma = \text{id}\) and there exists a unique geodesic joining any two points in \(B(x, \delta)\). Let us consider an open subset of \(M \times \cdots \times M\) as follows:

$$PU_{\alpha,n}(x) = \{(x_1, x_2, \ldots, x_m) \in \tilde{M} \times \cdots \times \tilde{M} | x_0 = x, d(x_i, x_{i+1}) < \varepsilon, (i = 1, \ldots, m - 1)\}.$$ 

Let us fix \(\varepsilon\) so that \(\varepsilon < \delta\). Then, by considering the geodesic joining consecutive two points, we can associate a continuous path \(\gamma_x\) in \(M\) with a point \(x\) of \(PU_{\alpha,n}\), so that \(\gamma_x(tk/m) = x_k\). Considering the lift of \(\gamma_x\), we can obtain a unique map \(\sigma_{\alpha,n} : PU_{\alpha,n}(x) \to \tilde{M}\) such that \(\sigma_{\alpha,n}(x_k) = \tilde{\gamma}_t(t)\). By using an induction, it is easy to see that this is a \(C^n\)-map. Next we consider a discretization of the path \(\gamma\) in \(M\). Let us take a sequence \(\{0 < t_1 < t_2 < \ldots < t_l = t\}\) so that \(|t_i - t_i| < (\varepsilon/4)^{1/n}(1/n)^{1/2\alpha}\). Then, for a path \(\gamma\) satisfying \(\Phi_{\alpha,0}(\gamma) \neq 0\), we can associate a sequence \(\{\gamma_1, \ldots, \gamma_l\} \in PU_{\alpha,n}(x)\) and it holds that

$$\tilde{\gamma}_t = \sigma_{\alpha,n}(\gamma_1, \ldots, \gamma_l)$$

because the part of the path \(\gamma|_{t_{i-1} \leq t_i \leq t_i}\) lies in the \(\varepsilon/2\)-ball centered at \(\gamma_t\). Now we take a cut-off function \(\eta\) such that \(\eta(x) = 1 (x \in \text{a neighborhood of } PU_{\alpha,n}(x)\) and \(\eta(x) = 0 (x \in PU_{\alpha,n}(x)\). Then we have for \(g\) where

$$\eta(\gamma_1, \ldots, \gamma_l) = f(\sigma_{\alpha,n}(\gamma_1, \ldots, \gamma_l)) \eta(\gamma_1, \ldots, \gamma_l)$$

(2) holds. Again, using the induction argument, it is easy to see that

$$\eta(\gamma_t, h) = (\delta f(\tilde{\gamma}_t), \tilde{h}_t)$$

for \(\eta\) with \(\Phi_{\alpha,0}(\gamma) \neq 0\). This proves (3). ■

Let us denote by \(L | P(M)\) the generator of the Dirichlet form \(\sigma(M)\). We denote the spectral set by \(\sigma(L | P(M)_{\alpha}\). Then it follows from Lemma 4.1 that:

**Corollary 4.2** \(\sigma(L | P(M)_{\alpha}) = \sigma(L | P(M)_{\alpha}\).\)

Now we are in a position to prove the following estimate:

**Theorem 4.3** (1) Let \((M, g)\) be a compact Riemannian manifold. Assume that a Riemannian cover \(\tilde{M}\) of \(M\) has a nonconstant bounded harmonic function \(u\). Then it holds that

$$\int_1^\infty \frac{C_T(SG)}{T} dT < \infty.$$  

(4.5)

In particular, we have \(\lim_{T \to \infty} C_T(SG) = 0\).

(2) In addition to the assumption in (1), we assume that the Ricci curvature satisfies that \(-\infty < -b^2 \leq \text{Ric} \leq -a^2 < 0\). Then it holds that for \(T \geq 1\)

$$C_T(SG) \leq c(u)T\beta e^{-\beta T},$$

where \(c(u)\) denotes a positive constant depending on \(u\).

**Proof.** (1) Put \(u_T = u(T)\). Then, by the Lemma 4.1, \(u_T \in L^p\). Also, we have
\[
\frac{\mathcal{E}(u_T, u_T)}{E_s[u_T]^2 - E_s[u_T]} = \frac{T\bar{P}_T|\nabla u|^2(\bar{x})}{\bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2}.
\]

Note that
\[
\int_0^T \bar{P}_T|\nabla u|^2(\bar{x}) \, ds = E_s[u(\bar{y})^2] - u(\bar{x})^2.
\]

Hence this integral converges as \( t \to \infty \). Therefore, we have
\[
\int_1^\infty \frac{C_T(SG)}{T} \, dT \leq \int_1^\infty \frac{\bar{P}_T|\nabla u|^2(\bar{x})}{\bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2} \, dT
\]

\[
\leq \bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2 < \infty.
\]

(2) By Theorem 3.2, we have
\[
\mathcal{E}(u_T, u_T) = \frac{T\bar{P}_T|\nabla u|^2(\bar{x})}{E_s[u_T]^2 - E_s[u_T]} = \frac{\bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2}{b^2 T e^{-\theta_T \|u\|_0^2}}
\]

\[
\leq \frac{\bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2}{\bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2}.
\]

Note that \( \bar{P}_T|u|^2(\bar{x}) - u(\bar{x})^2 \) is an increasing function of \( T \) and converges to
\[
\int_0^\infty \bar{P}_T|\nabla u|^2(\bar{x}) \, dT
\]
as \( T \to \infty \). This completes the proof. ■

As is well-known for pinched negative curvature manifolds whose sectional curvature \( K \) satisfy that
\[
-\infty < -B^2 \leq K \leq -A^2 < 0,
\]  

(4.7) there are many bounded harmonic functions ([I2], [17]). So in this case it holds that
\[
C_T(SG) \leq c(n)(n - 1)B^2 T \exp \{- (n - 1)B^2 T\},
\]  

(4.8) where \( n \) is the dimension of the manifold.

**Corollary 4.4** Assume that (4.7) holds. Then there is no spectral gap on \( C([0, \infty) \to M; \gamma(0) = x) \).

5. Log-Sobolev Inequality on Path Spaces and the Spectral Gap Estimate on Free Path Spaces

Let us consider the free path space \( P(M)_T = C(0, T] \to M) \), where we assume there are no constraints on \( \gamma(0) \) or \( \gamma(T) \). Recall that \( P_s, T \) denotes the Brownian measure starting at \( x \). On \( P(M)_T \), we have a probability measure \( dP_M, T = dP_s, T \otimes dv(x) \) corresponding to the reversible diffusion process generated by the Dirichlet form on \( M \):

\[
\mathcal{E}(f, f) = \int_M \langle \nabla f(x), \nabla f(x) \rangle dv(x).
\]

Here, \( dv(x) \) denotes the normalized Riemannian volume corresponding to the Riemannian metric. Let us define the derivative for \( F(\gamma) = f(y_0, y_1, \ldots, y_n) \in \mathcal{E} \otimes \mathcal{E}(P(M)_T) \) \( (t_0 = 0, t_n = T) \) as follows:

\[
\nabla F(\gamma) = \partial_0 f(y_0) + \sum_{i=1}^n \nabla f(y_t) t_i \wedge t \in T_y M
\]  

(5.1)

and define the norm:

\[
|\nabla F(\gamma)|^2 = |\nabla f(y_0)|^2 + \sum_{i=1}^{n-l} (t_{i+1} - t_i) \sum_{j=i+1}^n |\nabla f(y_t_i)|^2.
\]  

(5.2)

See Section 2 for the notation \( \nabla f(y_t_i) \).

One may define the derivative and the norm as follows:

\[
\nabla' F(\gamma) = \partial_0 f(y_T) + \sum_{i=1}^n \tau_i \partial_0 f(y_T)(T - t_i) \wedge (T - t) \in T_{\gamma(T)} M
\]  

(5.3)
\[ |\nabla' F(\gamma)|^2 = |\delta_s f(\gamma_T)|^2 + \sum_{i=1}^{n-1} (t_{i+1} - t_i) \sum_{j=0}^{i} \tau_{i,j} \partial_s f(\gamma_s)|^2. \]  
(5.4)

Here, \( \tau_{i,T} : T_{\gamma} M \to T_{\gamma T} M \) denotes the stochastic parallel translation operator along \( \gamma \). However, taking the reversibility of \( P_{M,T} \) into account, we see that for \( F(\gamma) = f(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \), it holds that \( F(\gamma) \) and \( \bar{F}(\gamma) \) have the same law and

\[ |\nabla' F(\gamma)|^2 = |\nabla \bar{F}(\gamma)|^2. \]

Hence it is sufficient to consider the case (5.1) and (5.2) to prove logarithmic Sobolev inequality:

\[ \int_{\mathcal{F}(M,T)} F^2(\gamma) \log (F(\gamma) / \|F\|^2) dP_{M,T} \leq \lambda \mathcal{E}(F, F) \]  
(5.5)

where

\[ \mathcal{E}(F, F) = \int_{\mathcal{F}(M,T)} |\nabla F(\gamma)|^2 dP_{M,T}. \]

To study the spectral gap problem on free path space to this Dirichlet form, we need logarithmic Sobolev inequality on path space:

\[ \int_{\mathcal{P}(M,T)} F^2(\gamma) \log (F(\gamma) / \|F\|^2) dP_{\gamma} \leq \lambda \int_{\mathcal{P}(M,T)} |\nabla F(\gamma)|^2 dP_{\gamma} \]  
(5.6)

which is stronger than the Poincaré inequality and due to Hsu [15] and Aida and Elworthy [1]. Let us introduce the log-Sobolev constant \( C_T(\mathcal{L}) \):

\[ C_T(\mathcal{L}) = \inf \{ \lambda > 0 \mid (5.6) \text{ holds for any } F \in \mathcal{F}(\mathcal{P}(M,T)) \}. \]

Note that there is a relation between \( C_T(\mathcal{S}) \) and \( C_T(\mathcal{L}) \) such that

\[ C_T(\mathcal{S}) \geq \frac{2}{C_T(\mathcal{L})}. \]  
(5.7)

For this, we refer to Deuschel and Stroock [7]. The following theorem is due to Hsu [15]. Actually, his estimate is a little different from but for the sake of completeness and for our purpose, we give a sketch of the proof.

**Theorem 5.1** (E. Hsu [15]) On \( \mathcal{P}(M,T) \), the following estimate is valid.

1. Assume that \( \text{Ric} \geq c_1, |\text{Ric}| \leq c (c_1 \in \mathbb{R}, c \geq 0). \) Then we have

\[ C_T(\mathcal{L}) \leq 2 \exp \left( \max (-c_1, 0) T \right) \left( 1 + \frac{c}{2|c_1|} \sqrt{e^{-\alpha T} - 1 + c_1 T} \right)^2. \]

2. Assume that \( 0 < c_1 \leq \text{Ric} \leq c_2 < \infty \). Then it holds that

\[ C_T(\mathcal{L}) \leq 2 \left( 1 + \frac{c_2}{c_1} \sqrt{1 + e^{-\alpha T}} \right)^2. \]

**Proof.** Hsu proved the following estimate: for any \( \lambda > 0 \),

\[ \int_{\mathcal{P}(M,T)} F^2(\gamma) \log (F(\gamma) / \|F\|^2) dP_{\gamma} \leq 2 \exp \left( \max (-c_1, 0) T \right) \left( 1 + \frac{\lambda}{\sqrt{1 + e^{-\alpha T}}} \right)^2 \]

\[ \left( 1 + \frac{1}{\lambda} \right) \left( 1 + \frac{c}{4\lambda} \sqrt{\int_0^T e^{-c_1(t-s)/2} g(s) \, ds} \right)^2 \]

(5.8)

where \( g(s) = \nabla F(\gamma)(s) \). Choosing

\[ \lambda = \frac{\sqrt{e^{-\alpha T} - 1 + c_1 T} c}{2|c_1|}, \]

we obtain (1). As for (2), it is easily obtained by using (5.8) and proceeding as for the proof of Theorem 2.4.

We denote the log-Sobolev constant to the Dirichlet form \( \mathcal{E}_M \) by \( c_M \). Since \( M \) is a compact manifold, \( c_M \) is a finite number. The following theorem is an immediate consequence of Theorem 5.1 and the Faris and Federbush additivity argument.

**Theorem 5.2** (1) Assume that \( \text{Ric} \geq c_1, |\text{Ric}| \leq c \). Then (5.5) holds for

\[ C_T(\mathcal{L}) \leq c_M \vee 2 \exp \left( \max (-c_1, 0) \right) \left( 1 + \frac{c}{2|c_1|} \sqrt{e^{-\alpha T} - 1 + c_1 T} \right)^2. \]
(2) Assume that $0 < c_1 \leq \text{Ric} \leq c_2$. Then we have
\[ C_T(\text{LS}) \leq c_M \sqrt{2 \left( 1 + \frac{c_2}{c_1} \sqrt{1 + e^{-c_1T/2}} \right)^2}. \]

This theorem implies that in the case of positive Ricci curvature, the spectral gap remains positive under taking an infinite volume limit $T \to \infty$. However, a modification of Theorem 4.3 shows that the spectral gap goes to 0 exponentially if the Ricci curvature is strictly negative and a nonconstant bounded harmonic function exists on certain Riemannian covers of $M$.

REFERENCES