Quantum Probabilistic Approach to Spectral Analysis of Star Graphs

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A general method for obtaining a vacuum spectral distribution of the adjacency matrix of a star graph is established within the framework of quantum probability theory. The spectral distribution tends asymptotically to the Bernoulli distribution as the number of leaves of a star graph tends to the infinity.

KEYWORDS: adjacency matrix, Boolean independence, interacting Fock space, quantum decomposition, spectral distribution, spectral gap, star graph

1. Introduction

To investigate spectral properties of the adjacency matrix of a graph is a fundamental problem in many branches of mathematics and physics. Among various generalizations we focus in this paper on the spectral distribution \( \langle A^m \rangle = \int_{\mathbb{R}} x^m \mu(dx), \quad m = 0, 1, 2, \ldots \), where \( \langle \cdot \rangle \) is the mean value with respect to a certain state.

In recent years a new method for computing the spectral distribution \( \mu \) has been developed in a series of papers. Motivated by the "semi-classical" argument of Hora [11], we first introduced in Hashimoto, Obata and Tabei [10] the method of quantum decomposition and obtained asymptotic spectral distributions for growing families of Hamming graphs as quantum central limit theorems. The situation observed therein has been abstracted by Hashimoto [8], where a growing family of Cayley graphs was discussed. More generally, a sufficient condition for a growing family of regular graphs to fall within this scheme has been sharpened in Hashimoto, Hora and Obata [9] and Hora and Obata [15, 16]. The above mentioned works are concerned with asymptotic properties with respect to the vacuum state and the results are formulated as quantum central limit theorems. Asymptotics with respect to another states has been studied by Hora [12–14].

On the other hand, our method can be applied to a single graph too. In fact, our prototype is a homogeneous tree to which our method is directly applicable, see e.g., Accardi and Obata [3], and our method yields an alternative derivation of the Kesten measure [17]. Recall that a homogeneous tree is related to the free independence of Voiculescu [26]. Lu [19] and Muraki [20] discovered another concept of independence, which is now called monotone independence. In the recent paper of Accardi, Ben Ghorbal and Obata [1], we found that the adjacency matrix of a comb graph is decomposed into a sum of monotone independent random variables. Thus, the spectral distribution can be computed very efficiently with the help of quantum probabilistic techniques established by Muraki [21, 22]. Moreover, comb graphs provide an interesting family of physical models which describe the Bose–Einstein condensation in low dimension, see Burioni et al. [6, 7].

In this paper, we focus on star graphs which are also discussed by Burioni et al. [6, 7] as models of the Bose–Einstein condensation. A star graph is obtained by gluing together the common origin of a finite number of copies of a graph, see Fig. 1.1. Again our method is effectively applied and the \(-t\)-transform of Bożejko and Wysoczanski [4, 5] appears. We shall show a general method for calculating the spectral distribution of a star graph and show concrete examples. In particular, for a star lattice of degree \( N \geq 3 \) we observe a "spectral gap," which indicates a relation to the hidden spectrum discussed by Burioni et al. [6, 7]. It is also noteworthy that our argument is free from the finite volume approximation (infinite volume limit) employed by them. A further detailed study is now in progress.

Lenczewski [18] pointed out that construction of a star graph, say "star product" of graphs, is related to the Boolean independence which is another concept of independence known as one of the five independences of Muraki [23]. As for the monotone independence, we have already investigated a relation to a comb graph in Accardi, Ben Ghorbal and Obata [1]. This situation suggests that various concepts of independence in quantum probability theory might be realized as special structures of graphs. This line of research is, however, somehow beyond the scope of this paper and will be discussed elsewhere.

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2. General Theory

2.1 Graph and its adjacency matrix

A graph is a pair \( G = (V, E) \), where \( V \) is a non-empty set and \( E \) is a subset of \( \{ \{x, y\}; x, y \in V, x \neq y \} \). Elements of \( V \) and of \( E \) are called a vertex and an edge, respectively. Two vertices \( x, y \in V \) are called adjacent if \( \{x, y\} \in E \), and in that case we write \( x \sim y \). For a graph \( G = (V, E) \) we define the adjacency matrix \( A = (A_{xy})_{x,y \in V} \) by

\[
A_{xy} = \begin{cases} 
1, & x \sim y, \\
0, & \text{otherwise}.
\end{cases}
\]

Obviously, (i) \( A \) is symmetric; (ii) an element of \( A \) takes a value in \( \{0, 1\} \); (iii) a diagonal element of \( A \) vanishes. Conversely, for a non-empty set \( V \), a graph structure is uniquely determined by such a matrix indexed by \( V \).

The degree or valency of a vertex \( x \in V \) is defined by

\[
\kappa(x) = |\{y \in V; y \sim x\}|,
\]

where \(| \cdot |\) denotes the cardinality. A finite sequence \( x_0, x_1, \ldots, x_n \in V \) is called a walk of length \( n \) (or of \( n \) steps) if \( x_{i-1} \sim x_i \) for all \( i = 1, 2, \ldots, n \). In a walk some vertices may occur repeatedly. Unless otherwise stated, we always assume that a graph under discussion satisfies:

(a) (connectedness) any pair of distinct vertices are connected by a walk;

(b) (local boundedness) \( \kappa(x) < \infty \) for all \( x \in V \);

In fact, the examples in this paper satisfy the following condition which is stronger than (b):

(b') (uniform boundedness) \( \sup_{x \in V} \kappa(x) < \infty \).

Let \( \ell^2(V) \) denote the Hilbert space of \( \mathbb{C} \)-valued square-summable functions on \( V \). With each \( x \in V \) we associate a “delta function” defined by

\[
\delta_x(y) = \begin{cases} 
1 & \text{if } y = x, \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( \{\delta_x; x \in V\} \) becomes a complete orthonormal basis of \( \ell^2(V) \). The adjacency matrix is considered as an operator acting in \( \ell^2(V) \) in such a way that

\[
A \delta_x = \sum_{y \sim x} \delta_y, \quad x \in V.
\]

Then, \( A \) becomes a selfadjoint operator equipped with a natural domain. As is easily checked, (b') is a necessary and sufficient condition for \( A \) to be a bounded operator on \( \ell^2(V) \).

2.2 Stratification

For \( x \neq y \) let \( \partial(x, y) \) be the length of the shortest walk connecting \( x \) and \( y \). By definition \( \partial(x, x) = 0 \) for all \( x \in V \). The graph becomes a metric space with the distance function \( \partial \). Note that \( \partial(x, y) = 1 \) if and only if \( x \sim y \).

We fix a point \( o \in V \) as an origin of the graph. Then, the graph is stratified into a disjoint union of strata:

\[
V = \bigcup_{n=0}^{\infty} V_n, \quad V_n = \{x \in V; \partial(o, x) = n\},
\]

see Fig. 2.1. Note that \( V_n = \emptyset \) may occur for some \( n \geq 1 \). In that case we have \( V_n = V_{n+1} = \ldots = \emptyset \).

With each stratum \( V_n \) we associate a unit vector in \( \ell^2(V) \) defined by
\[ \Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x. \]  
(2.2)

The closed subspace of \( \ell^2(V) \) spanned by \( \{\Phi_0, \Phi_1, \ldots\} \) is denoted by \( \Gamma(\mathcal{G}) \). Since \( \{\Phi_0, \Phi_1, \ldots\} \) becomes a complete orthonormal basis of \( \Gamma(\mathcal{G}) \), we often write

\[ \Gamma(\mathcal{G}) = \sum_{n=0}^{\infty} \oplus C\Phi_n. \]

If \( V_n = \emptyset \) occurs for some \( n \), among such \( n \)'s we take the smallest one, say \( n_0 \) (\( \geq 1 \)), and define only \( \{\Phi_0, \ldots, \Phi_{n_0-1}\} \). In that case \( \Gamma(\mathcal{G}) \) becomes an \( n_0 \) dimensional subspace of \( \ell^2(V) \).

2.3 Quantum decomposition

Let \( A \) be the adjacency matrix of a graph \( \mathcal{G} = (V, E) \). According to the stratification (2.1), we define three matrices \( A^+, A^- \) and \( A^0 \) as follows: for \( x \in V_n \) we set

\[
\begin{align*}
(A^+)_{xy} &= \begin{cases} 
A_{xy} & \text{if } y \in V_{n+1}, \\
0 & \text{otherwise,}
\end{cases} \\
(A^-)_{xy} &= \begin{cases} 
A_{xy} & \text{if } y \in V_{n-1}, \\
0 & \text{otherwise,}
\end{cases} \\
(A^0)_{xy} &= \begin{cases} 
A_{xy} & \text{if } y \in V_n, \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Or equivalently, for \( x \in V_n \),

\[
A^+\delta_x = \sum_{y \in V_{n+1}, y \sim x} \delta_y, \quad A^-\delta_x = \sum_{y \in V_{n-1}, y \sim x} \delta_y, \quad A^0\delta_x = \sum_{y \in V_n, y \sim x} \delta_y,
\]
(2.3)

see Fig. 2.2.

Since \( x \in V_n \) and \( x \sim y \) imply that \( y \in V_{n-1} \cup V_n \cup V_{n+1} \), where we tacitly understand that \( V_{-1} = \emptyset \), one sees immediately that

\[ A = A^+ + A^- + A^0. \]
(2.4)

This is called the quantum decomposition of \( A \) associated with the stratification (2.1). Note also that

\[
(A^+)^* = A^-, \quad (A^-)^* = A^+,
\]

which are verified easily.

2.4 Problem and method

The vector state corresponding to \( \delta_0 = \Phi_0 \), where \( o \in V \) is the fixed origin, is analogous to the vacuum state in Fock space. It is our main problem to investigate a Borel probability measure \( \mu \) on \( \mathbb{R} \) satisfying
By the local boundedness condition (b) the “moment” sequence \( \{ \langle A^m \rangle \}_{m=0}^{\infty} \) is well-defined. Then the existence of a probability measure satisfying (2.5) is a consequence of Hamburger’s theorem, see e.g., Shohat and Tamarkin [24, Theorem 1.2]. Moreover, if \( A \) is bounded, i.e., if the uniform boundedness condition \((b')\) is fulfilled, then \( \mu \) is unique and has a compact support.

**Remark 2.1.** Obviously, \( \langle A^m \rangle \) coincides with the number of \( m \)-step walks starting and terminating at \( o \). Hence \( \langle A \rangle = 0 \) and \( \langle A^2 \rangle = \kappa(o) \).

Inserting the quantum decomposition (2.4) into (2.5), we obtain

\[
\langle A^m \rangle = \sum_{\epsilon_1, \ldots, \epsilon_n \in \{+,-,o\}} \langle \Phi_0, A_1^{\epsilon_1} \ldots A_n^{\epsilon_n} \Phi_0 \rangle.
\]

Hence, if \( \Gamma(\vec{g}) \) is invariant under the actions of the quantum components \( A^\epsilon, \epsilon \in \{+,-,o\} \), then \( \langle A^m \rangle \) is computed only within \( \Gamma(\vec{g}) \).

**Lemma 2.2.** If \( \Gamma(\vec{g}) \) is invariant under the quantum components \( A^\epsilon, \epsilon \in \{+,-,o\} \), then there exist two sequences \( \{\omega_n\}_{n=1}^{\infty} \) and \( \{\alpha_n\}_{n=1}^{\infty} \) such that

\[
A^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1}, \quad n \geq 0, \tag{2.6}
\]
\[
A^- \Phi_0 = 0, \quad A^- \Phi_n = \sqrt{\alpha_n} \Phi_{n-1}, \quad n \geq 1, \tag{2.7}
\]
\[
A^o \Phi_n = \alpha_{n+1} \Phi_n, \quad n \geq 0. \tag{2.8}
\]

In particular,

\[
\omega_1 = |V_1| = \kappa(o), \quad \alpha_1 = 0. \tag{2.9}
\]

**Proof.** In view of (2.2) and (2.3) we have

\[
|V_n|^{1/2} A^+ \Phi_n = \sum_{x \in V_n} \sum_{x \in V_{n+1}} \sum_{y \sim x} \delta_y = \sum_{y \in V_{n+1}} \kappa_-(y) \delta_y,
\]

where \( \kappa_-(y) = |\{x \in V_n; x \sim y\}| \) for \( y \in V_{n+1} \). Then

\[
A^+ \Phi_n = |V_{n+1}|^{-1/2} \sum_{y \in V_{n+1}} \frac{|V_{n+1}|^{1/2}}{|V_n|^{1/2}} \kappa_-(y) \delta_y. \tag{2.10}
\]

Since \( A^+ \Phi_n \in \Gamma(\vec{g}) \) by assumption, (2.10) must be a scalar multiple of \( \Phi_{n+1} \). Hence \( \kappa_-(y) \) does not depend on \( y \in V_{n+1} \) and, setting

\[
\sqrt{\omega_{n+1}} = \frac{|V_{n+1}|^{1/2}}{|V_n|^{1/2}} \kappa_-(y), \quad y \in V_{n+1}, \quad n = 0, 1, 2, \ldots,
\]

we obtain (2.6). Since \( A^- = (A^+)^* \) on \( \ell^2(V) \) and \( \Gamma(\vec{g}) \) is invariant under both \( A^\pm \), a simple duality argument implies (2.7). As for \( A^o \), by a similar argument as above we come to
where \( \kappa_\epsilon(y) = |\{x \in V_n; x \sim y\}| \) for \( y \in V_n \). Hence \( \kappa_\epsilon(y) \) does not depend on \( y \in V_n \) and, setting
\[
\alpha_{n+1} = \kappa_\epsilon(y), \quad y \in V_n, \quad n = 0, 1, 2, \ldots,
\]
we obtain (2.8). Finally, (2.9) is already shown.

The pair of sequences \( \{\omega_n\}, \{\alpha_n\} \) in Lemma 2.2 is referred to as Szegő–Jacobi sequences derived from \( A \).

**Remark 2.3.** It can happen that \( i \Gamma(\vec{g}) \) is invariant under \( A^+ \) but not invariant under \( A^- \). A monotone tree is a noticeable example. Let \( N \geq 1 \) be a fixed integer and consider the set \( V \) of all ordered sequences \( x = (x_1 < x_2 < \cdots < x_n) \), where \( x_i \in \{1, 2, \ldots, N\} \). The empty sequence \( \emptyset \) is also a member of \( V \). We introduce a graph structure in \( V \) by defining the adjacent points as follows: \( x = (x_1 < x_2 < \cdots < x_n) \) and \( y = (y_1 < y_2 < \cdots < y_m) \) are adjacent if \( n = m + 1 \) and \( (x_2 < \cdots < x_n) = (y_1 < y_2 < \cdots < y_m) \). By definition the adjacent points of \( \emptyset \) are (1), (2), \ldots, (\( N \)). In this case
\[
A^+ \Phi_n = \frac{|V_{n+1}|^{1/2}}{|V_n|^{1/2}} \Phi_{n+1}, \quad n = 0, 1, \ldots, N - 1.
\]

But \( A^- \Phi_n \) is not a constant multiple of \( \Phi_{n-1} \).

During the proof of Lemma 2.2, we have established the following

**Proposition 2.4.** Let \( \vec{g} = (V, E) \) be a graph with a fixed origin \( o \in V \). According to the stratification we set
\[
\kappa_\epsilon(y) = |\{x \in V_{n+\epsilon}; x \sim y\}|, \quad y \in V_n, \quad \epsilon \in \{+, -, o\}.
\]
(Here \( n + \epsilon = n + 1, n - 1, n \) according as \( \epsilon = +, -, o \).) Then \( i \Gamma(\vec{g}) \) is invariant under the quantum components \( A^\epsilon \), \( \epsilon \in \{+, -, o\} \), if and only if \( \kappa_\epsilon(y) \) does not depend on \( y \in V_n \) (and depends only on \( n \)). In that case, the Szegő–Jacobi sequences derived from \( A \) is given by
\[
\alpha_n = \frac{|V_n|}{|V_{n-1}|} \kappa_\epsilon(y)^2, \quad y \in V_n; \quad \omega_n = \kappa_\epsilon(y), \quad y \in V_{n-1},
\]
where \( n = 1, 2, \ldots \). In particular, \( \omega_1 = \kappa(o) \) and \( \alpha_1 = 0 \).

Lemma 2.2 says that \( (\Gamma(\vec{g}), A^+, A^-) \) is an interacting Fock space with Jacobi parameter \( \{\omega_n\}_{n=1}^\infty \) in the sense of Accardi and Bożejko [2]. Moreover, since
\[
A = A^+ + A^- + A^o = A^+ + A^- + \alpha_{N+1},
\]
where \( N \) is the number operator defined by \( N \Phi_n = n \Phi_n \), we may apply the canonical isomorphism from the interacting Fock space onto the closed linear span of the orthogonal polynomials determined by the Szegő–Jacobi sequences \( \{\omega_n\}, \{\alpha_n\} \). More precisely, the probability measure \( \mu \) under question is characterized by the property of orthogonalizing the polynomials \( \{P_n\} \) defined recurrently by
\[
\begin{align*}
P_0(x) &= 1, \\
P_1(x) &= x - \alpha_1, \\
xP_n(x) &= P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_nP_{n-1}(x), \quad n \geq 1.
\end{align*}
\]
If such a probability measure is unique (\( e.g. \), if the uniform boundedness condition \( (b') \) is fulfilled), the probability measure \( \mu \) is determined by the identity:
\[
G_\mu(z) = \int_R \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1 - \frac{\omega_1}{z-\alpha_2 - \frac{\omega_2}{z-\alpha_3 - \frac{\omega_3}{z-\alpha_4 - \cdots}}}}, \quad (2.11)
\]
where \( G_\mu(z) \) is called the Stieltjes transform. The probability measure \( \mu \) is obtained by calculating the continued fraction in (2.11) and by the Stieltjes inversion formula (\( e.g. \), [24, Introduction]). For a more detailed account of the method for computing \( \mu \), see Hora and Obata [15, 16].
3. Spectral Analysis of Star Graphs

3.1 Star graphs

Let \( \mathcal{G} = (V, E) \) be a graph and \( A \) the adjacency matrix. Assume that \( \mathcal{G} \) is equipped with a fixed origin \( o \in V \). For \( N \geq 1 \) an \( N \)-fold star power or simply a star graph, denoted by \( \mathcal{G}^N \), is obtained by gluing together the common origin of \( N \) copies of \( \mathcal{G} \), see Fig. 1.1. The initial graph \( \mathcal{G} \) is naturally considered as a subgraph of \( \mathcal{G}^N \), which we call a leaf. The adjacency matrix of \( \mathcal{G}^N \) is denoted by \( A^N \). Taking the glued vertex as an origin of \( \mathcal{G}^N \), we introduce a stratification and the quantum decomposition:

\[
A^N = (A^N)^+ + (A^N)^- + (A^N)^0.
\]

**Lemma 3.1.** Assume that \( \Gamma(\mathcal{G}) \) is invariant under the quantum components \( A^\epsilon \), \( \epsilon \in \{+, -, 0\} \), and let \( \{\omega_n, \alpha_n\} \) be the Szegö–Jacobi sequences derived from \( A \). Then, \( \Gamma(\mathcal{G}^N) \) is also invariant under \( (A^N)^\epsilon \), \( \epsilon \in \{+, -, 0\} \), and the Szegö–Jacobi sequences derived from \( A^N \) is given by

\[
\{(N\omega_1, \omega_2, \omega_3, \ldots), \{\alpha_1, \alpha_2, \alpha_3, \ldots\}\}. \tag{3.1}
\]

**Proof.** Let \( V \) and \( V^{(N)} \) be the sets of vertices of \( \mathcal{G} \) and \( \mathcal{G}^N \), respectively. Let

\[
V = \bigcup_{n=0}^{\infty} V_n, \quad V^{(N)} = \bigcup_{n=0}^{\infty} V^{(N)}_n
\]

be their stratifications. Set

\[
\kappa(y) = |[x \in V_{n+2}; x \sim y]|, \quad y \in V_n, \quad \epsilon \in \{+, -, 0\},
\]

\[
\kappa^{(N)}(y') = |[x' \in V^{(N)}_{n+2}; x' \sim y']|, \quad y' \in V^{(N)}_n, \quad \epsilon \in \{+, -, 0\}.
\]

Then by Proposition 2.4 and assumption, the Szegö–Jacobi sequences derived from \( A \) is given by

\[
\omega_n = \frac{|V_n|}{|V_{n-1}|} \kappa_-(y)^2, \quad y \in V_n, \tag{3.2}
\]

\[
\alpha_n = \kappa(y), \quad y \in V_{n-1}, \tag{3.3}
\]

where \( \omega_n \) and \( \alpha_n \) are defined independently of the choice of \( y \). On the other hand, we see from construction of the star graph that

\[
\kappa^{(N)}(y') = \kappa(y), \quad y' \in V^{(N)}_n, \quad y \in V_n, \tag{3.4}
\]

\[
\kappa^{(N)}_+(y') = \kappa_+(y'), \quad y' \in V^{(N)}_{n-1}, \quad y \in V_{n-1}. \tag{3.5}
\]

It then follows from Proposition 2.4 that \( \Gamma(\mathcal{G}^N) \) is invariant under the quantum components \( (A^N)^\epsilon \), \( \epsilon \in \{+, -, 0\} \).

Let \( \{\omega_n^{(N)}, \alpha_n^{(N)}\} \) be the Szegö–Jacobi sequences derived from \( A^N \). Then, by (3.2) and (3.4) we see that for \( n = 2, 3, \ldots \),

\[
\omega_n^{(N)} = \frac{|V^{(N)}_n|}{|V^{(N)}_{n-1}|} \kappa^{(N)}_-(y')^2 = \frac{N|V_n|}{N|V_{n-1}|} \kappa_-(y)^2 = \omega_n,
\]

where \( y' \in V^{(N)}_n \) and \( y \in V_n \). And for \( n = 1 \),

\[
\omega_1^{(N)} = \frac{|V^{(N)}_1|}{|V_0|} \kappa^{(N)}_-(y')^2 = \frac{N|V_1|}{|V_0|} \kappa_-(y)^2 = N\omega_1,
\]

where \( y' \in V^{(N)}_1 \) and \( y \in V_1 \). Similarly, by (3.3) and (3.5) we obtain for \( n = 1, 2, \ldots \),

\[
\alpha_n^{(N)} = \kappa^{(N)}_+(y') = \kappa(y) = \alpha_n,
\]

where \( y' \in V^{(N)}_{n-1} \) and \( y \in V_{n-1} \). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( \mathcal{G} = (V, E) \) be a graph satisfying conditions (a) and (b'). Assume that \( \mathcal{G} \) is equipped with a fixed origin \( o \in V \) and that \( \Gamma(\mathcal{G}) \) is invariant under the quantum components of \( A \). Let \( \{\omega_n, \alpha_n\} \) be the Szegö–Jacobi sequences derived from \( A \). Then the spectral distribution \( \mu_N \) of the \( N \)-fold star power \( \mathcal{G}^N \) is uniquely determined by
\[
G_N(z) = \int_R \frac{\mu_N(dx)}{z-x} = \frac{1}{N\omega_1} - \frac{\omega_2}{z-\alpha_2} - \frac{\omega_3}{z-\alpha_3} - \frac{\omega_4}{z-\alpha_4} - \ldots
\]

**Proof.** By Lemma 3.1 we know the Szegö-Jacobi sequences derived from \(A^N\). Then the statement is obtained by rephrasing the argument at the end of Subsection 2.4.

### 3.2 Simple examples

There is a unique graph consisting of two vertices. (Recall that a graph under consideration is connected.)

**Example 3.3.** (Fig. 3.1) Let \(G\) consist of two vertices. Then

\[
\omega_1 = 1, \quad \omega_2 = \omega_3 = \cdots = 0; \quad \alpha_n = 0,
\]

and

\[
G(z) = \frac{z}{z^2 - 1}, \quad \mu = \frac{1}{2} (\delta_{+1} + \delta_{-1}).
\]

For the \(N\)-fold star power \(G^*\) we have

\[
G_N(z) = \frac{z}{z^2 - N}, \quad \mu_N = \frac{1}{2} (\delta_{+\sqrt{N}} + \delta_{-\sqrt{N}}).
\]

Note also that \(E(\mu_N) = 0\) and \(V(\mu_N) = N\).

![Fig. 3.1](image)

There are two graphs consisting of three vertices.

**Example 3.4.** (Fig. 3.2) Let \(\bar{G}\) be a graph consisting of three vertices and the edges given as in Fig. 3.2. Then,

\[
\omega_1 = \omega_2 = 1, \quad \omega_3 = \cdots = 0; \quad \alpha_n = 0,
\]

and

\[
G(z) = \frac{z^2 - 1}{z^3 - 2z}, \quad \mu = \frac{1}{2} \delta_0 + \frac{1}{4} (\delta_{+\sqrt{2}} + \delta_{-\sqrt{2}}).
\]

For the \(N\)-fold star power \(G^*\) we have

![Fig. 3.2](image)
\[ G_N(z) = \frac{z^2 - 1}{z^3 - (N+1)z}, \]
\[ \mu_N = \frac{1}{N+1} \delta_0 + \frac{N}{2(N+1)} (\delta + \sqrt{N+1} + \delta - \sqrt{N+1}). \]

Note also that \( \mathbf{E}(\mu_N) = 0 \) and \( \mathbf{V}(\mu_N) = N. \)

**Example 3.5. (Fig. 3.3)** Let \( \mathcal{G} \) be a graph consisting of three vertices forming a loop. Then,
\[ \omega_1 = 2, \quad \omega_2 = \omega_3 = \cdots = 0; \quad \alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = \ldots = 0, \]
and
\[ G(z) = \frac{z - 1}{z^2 - z - 2}, \quad \mu = \frac{1}{3} \delta + \frac{2}{3} \delta. \]

For the \( N \)-fold power \( \mathcal{G}^N \) we have
\[ G_N(z) = \frac{z - 1}{z^2 - z - 2N} = \frac{z - 1}{(z - \alpha_i)(z - \alpha_{-i})}, \quad \alpha_\pm = \frac{1 \pm \sqrt{1 + 8N}}{2} \]
and
\[ \mu_N = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + 8N}} \right) \delta_{i+} + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 + 8N}} \right) \delta_{i-}. \]

Note that \( \mathbf{E}(\mu_N) = 0 \) and \( \mathbf{V}(\mu_N) = 2N. \)

3.3 Star lattice

Let \( \mathcal{G} = (V, E) \) be the half line of integers, i.e., \( V = \{0, 1, 2, \ldots\} \) and \( i \sim j \) if and only if \( |i - j| = 1. \) The \( N \)-fold star power \( \mathcal{G}^N \) is called a star lattice, see Fig. 3.4.

The Szegö–Jacobi sequences derived from the adjacency matrix \( A \) of \( \mathcal{G} \) is given by
\[ \omega_n = 1; \quad \alpha_n = 0. \]

Hence by Theorem 3.2 the spectral distribution \( \mu_N \) of \( A^N \) is obtained from the identity:
The right hand side is easily computed:
\[ G_N(z) = \frac{1}{2N - (2N - 2)x^2}. \]

Then, applying the Stieltjes inversion formula, we obtain the absolutely continuous part of the probability measure \( \mu_N \):
\[ \rho_N(x) = \frac{1}{2\pi N^2 - (N - 1)x^2}, \quad -2 \leq x \leq 2. \]

Note that \( \rho_N(x) \) for \( N \geq 2 \) is a constant multiple of the density function of a Kesten measure, see Appendix.

For \( N = 1 \) the spectral distribution \( \mu_1 \) is the semicircle law:
\[ \rho_1(x) = \frac{1}{\pi \sqrt{4 - x^2}}, \quad -2 \leq x \leq 2. \]

In fact, \( (\Gamma(\mathbb{G}), A^+, A^-) \) coincides with the free Fock space. For \( N = 2 \) we have
\[ \rho_2(x) = \frac{1}{\pi \sqrt{4 - x^2}}, \quad -2 \leq x \leq 2, \]

which is known as the arcsine law (with variance 2).

If \( N \geq 3 \) the total mass of \( \rho_N(x) \) is less than 1 and \( \mu_N \) contains a discrete measure. By a simple computation we have
\[ \mu_N(dx) = \rho_N(x)dx + \frac{N - 2}{2N - 2} \left( \delta_{-N/\sqrt{N-1}} + \delta_{N/\sqrt{N-1}} \right)(dx). \]

For \( 3 \leq N \leq 6 \), \( \rho_N(x) \) has two local maxima at
\[ x = \pm \sqrt{\frac{8 - (N - 4)^2}{N - 1}} \]

and for \( N \geq 7 \) it has just one at \( x = 0 \), see Appendix.

**Remark 3.6.** As is immediately seen from (3.6), for \( N \geq 3 \) there is a spectral gap:
\[ \gamma = \frac{N}{\sqrt{N - 1}} - 2 = \frac{(\sqrt{N - 1} - 1)^2}{\sqrt{N - 1}} > 0. \]

A finite volume approximation of \( \mathbb{G}^{+N} \) gives a discrete measure which approaches to \( \mu_N \) by taking the infinite volume limit. It is interesting to study asymptotic behavior of discrete spectrum between \([2, 2 + \gamma]\) along this approximation for the hidden spectrum of Burioni et al. [6, 7] might be related. This question will be discussed elsewhere.

### 3.4 Limit measure

The distributions \( \mu_N \) in Examples 3.3–3.5, after normalization to have variance one, tend to the Bernoulli distribution.
as $N \to \infty$. A similar phenomena is observed also in the case of star lattices. In fact, the spectral distributions of star graphs $\mathcal{G}^{xN}$ possess a common asymptotic feature.

**Theorem 3.7.** Let $\mathcal{G} = (V, E)$ be a graph satisfying conditions (a) and (b'). Assume that $\mathcal{G}$ is equipped with a fixed origin $o \in V$ and that $\Gamma(\mathcal{G})$ is invariant under the quantum components of $A$. Then we have

$$
\lim_{N \to \infty} \left( \Phi_0, \left( \frac{A^{xN}}{\sqrt{N\kappa(o)}} \right)^m \Phi_0 \right) = \frac{1}{2} \int_{\mathbb{R}} x^m (\delta_{-1} + \delta_{+1}) (dx), \quad m = 1, 2, \ldots.
$$

(3.7)

**Proof.** Let $([\omega_0], [\alpha_0])$ be the Szegő–Jacobi sequences derived from $A$. It then follows from Lemma 3.1 (or Theorem 3.2) that the Szegő–Jacobi sequences of $A^{xN}$ is given by

$$
[N\omega_1, \omega_2, \ldots], \quad [\alpha_1, \alpha_2, \ldots].
$$

Let $\mu_N$ be the corresponding probability measure:

$$
\left( \Phi_0, (A^{xN})^m \Phi_0 \right) = \int_{\mathbb{R}} x^m \mu_N (dx), \quad m = 1, 2, \ldots.
$$

Then,

$$
\left( \Phi_0, \left( \frac{A^{xN}}{\sqrt{N\kappa(o)}} \right)^m \Phi_0 \right) = \int_{\mathbb{R}} x^m \left[ S^*(\kappa(o))^{-1}; \mu_N \right] (dx), \quad m = 1, 2, \ldots,
$$

where the scaled measure $S^*_{\lambda} \mu_N$, $\lambda > 0$, is defined by

$$
\int_{\mathbb{R}} f(x) [S^*_{\lambda} \mu_N] (dx) = \int_{\mathbb{R}} f(\lambda x) \mu_N (dx).
$$

On the other hand, by observing the recurrence formula for the orthogonal polynomials we see that the Szegő–Jacobi sequences of $S^*(\kappa(o))^{-1}; \mu_N$ is given by

$$
\left\{ \frac{1}{\sqrt{N\omega_1}}, \frac{\alpha_1}{\sqrt{N\omega_1}}, \ldots, \frac{\alpha_2}{\sqrt{N\omega_1}}, \ldots \right\}.
$$

(3.8)

where we used $\omega_1 = \kappa(o)$. Obviously, in the coordinatewise sense, (3.8) converge to

$$
\{ 1, 0, 0, \ldots \}, \quad \{ 0, 0, \ldots \},
$$

which is the Szegő–Jacobi sequences of the Bernoulli distribution $(\delta_{+1} + \delta_{-1})/2$. Since the $m$-th moment is expressed in terms of the first $m$ terms of the Szegő–Jacobi sequences (see e.g., the Accardi–Bożejko formula [2]), we obtain (3.7).

**Remark 3.8.** For a probability measure $\mu$ its $t$-transform, denoted by $U_{t\mu}$, is introduced by Bożejko and Wysoczanski [4, 5]. Let $([\omega_0], [\alpha_0])$ be the Szegő–Jacobi sequences of $\mu$. Then $U_{t\mu}$ is characterized by the Szegő–Jacobi sequences given by $([t\omega_1, \omega_2, \omega_3, \ldots], \{t\alpha_1, \alpha_2, \alpha_3, \ldots\})$. Now consider the situation of Theorem 3.2. Let $\mu$ and $\mu_N$ be the spectral distributions of $\Lambda$ and $\Lambda^{xN}$, respectively. Since $\alpha_1 = 0$ always holds (see e.g., Lemma 2.2), we have $\mu_N = U_{t\mu}$. Moreover, as was pointed out in Bożejko and Wysoczanski [5], $U_{t\mu}$ is the Boolean convolution power of $\mu$, for relevant discussion see also Speicher and Woroudi [25]. Thus, Theorem 3.7 is well understood in terms
of quantum central limit theorem associated with the Boolean independence. Detailed study in this direction will be published elsewhere.

Appendix: Kesten Measures

The following formula is elementary:

\[
\int_{0}^{a} \frac{\sqrt{a^2 - x^2}}{b^2 - x^2} \, dx = \frac{\pi}{2} \left( 1 - \frac{\sqrt{b^2 - a^2}}{b} \right), \quad 0 < a \leq b.
\]

By normalization we obtain a probability distribution.

**Definition A.1.** Let \(0 < a \leq b\). The probability measure with density function

\[
\rho_{a,b}(x) = \frac{1}{\pi} \left( 1 - \frac{\sqrt{b^2 - a^2}}{b} \right) \frac{a^2 - x^2}{b^2 - x^2} = \frac{b(b + \sqrt{b^2 - a^2})}{\pi a^2} \frac{a^2 - x^2}{b^2 - x^2}, \quad -a \leq x \leq a,
\]

is called the Kesten measure with parameter \(a, b\).

The Kesten measure \(\rho_{a,b}(x)dx\) is symmetric and supported by \([-a, a]\). The mean is obviously zero:

\[
E(\rho_{a,b}) = \int_{-a}^{a} x \rho_{a,b}(x)dx = 0.
\]

The variance is given by

\[
V(\rho_{a,b}) = \int_{-a}^{a} x^2 \rho_{a,b}(x)dx = \frac{b}{2} (b - \sqrt{b^2 - a^2}).
\]

Hence, the Kesten measure is normalized so as to have variance one if and only if \(a^2 + (4/b^2) = 4\). Local maxima of the density function \(\rho_{a,b}(x)\) are found easily from

\[
\frac{d}{dx} \left( \frac{\sqrt{a^2 - x^2}}{b^2 - x^2} \right) = \frac{x(2a^2 - b^2 - x^2)}{(b^2 - x^2)^2 \sqrt{a^2 - x^2}}.
\]

If \(\sqrt{2a > b}\), then \(x_\pm = \pm \sqrt{2a^2 - b^2}\) lies in the interval \((-a, a)\) and the density function has a local maximum at \(x_\pm\). If \(\sqrt{2a \leq b}\), the density function has just one local maximum at \(x = 0\).

**Remark A.2.** In the famous paper Kesten [17] obtained the density function of a certain transition probability of the homogeneous random walk on the free group on \(N\) generators, \(N \geq 1\). The density function is given by

\[
\rho_{\sqrt{2N-1/N},1}(x) = \frac{1}{\pi} \frac{\sqrt{2N - 1 - N^2x^2}}{1 - x^2}, \quad |x| \leq \frac{\sqrt{2N-1}}{N}.
\]

REFERENCES


