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Reduction of Finite Topological Spaces

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In this paper, we define two reductions of finite topological spaces. Our reductions are the processes to decrease the number of points of a finite topological space without changing the homotopy groups of the space. Indeed, there is a weak homotopy equivalence from the original space to its reduction.

KEYWORDS: finite topological space, reduction, weak homotopy equivalence, T_0-space

1 Introduction

McCord ([2]) showed that exactly the same singular homology groups and homotopy groups can occur in finite topological spaces as in finite simplicial complexes. The following theorem are easily induced from McCord ([2], Th. 2).

Theorem 1.1. (McCord [2]). *There exists a correspondence that assigns to each finite topological T_0-space X a finite simplicial complex $\mathscr{K}(X)$, whose vertices are the points of X, and a weak homotopy equivalence f_X: $|\mathscr{K}(X)| \to X$. Each continuous map φ: $X \to Y$ between finite topological T_0-spaces corresponds to a simplicial map $|\mathscr{K}(\varphi)|: |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$, and $\varphi \circ f_X = f_Y \circ |\mathscr{K}(\varphi)|$.*

Stong ([4]) also showed that each finite topological T_0-space X has a subspace X_0, called a core, which is a deformation retract of X. Finite topological T_0-spaces have the same homotopy type if and only if they have homeomorphic cores. So we can classify the homotopy types of finite topological T_0-spaces by using the homeomorphic types of their cores.

We shall show in this paper that $\mathscr{K}(X)$ collapses simplicially to $\mathscr{K}(X_0)$ for any finite topological T_0-space X and its core X_0 (see Theorem 3.3). In particular any finite topological T_0-space corresponding to a non collapsible simplicial complex is a core.

The purpose of the present paper is to define the process to reduce the number of points of a core without changing its homotopy groups. For this purpose, we prove the following our main theorems (Theorem 1.2 and 1.3) and define two reductions called open reduction and closed reduction.

Theorem 1.2. *Let X be a finite topological T_0-space. Let $x \in X$ be a point such that the intersection of U_x (cf. §2) and U_y is path-connected and the homotopy groups of the intersection vanish in dimensions greater than zero for all $y \in X$. Then the identification p: $X \to X/U_x$ is a weak homotopy equivalence.*

The process of making X/U_x from X is called *open reduction* and described by $or(x)$.

Theorem 1.3. *Let X be a finite topological T_0-space. Let $x \in X$ be a point such that the intersection of C_x (cf. §2) and C_y is path-connected and the homotopy groups of the intersection vanish in dimensions greater than zero for all $y \in X$. Then the identification p: $X \to X/C_x$ is a weak homotopy equivalence.*

The process of making X/C_x from X is called *closed reduction* and described by $cr(x)$.

It is unknown whether by a sequence of the reductions *or* and *cr*, each finite topological T_0-space can be reduced to the smallest space having the same homotopy groups. For example, however, we obtain such smallest spaces for S^1 and S^2 (see §4). In §2, we quote the McCord's results in [2] and Stong's results in [4]. In §3, we study further properties of finite topological T_0-spaces in relation to finite simplicial complexes. In particular, we will see that the homotopy type classification of finite topological T_0-spaces is not a good approach to investigate the homotopy types of the corresponding simplicial complexes. In §4, we introduce two reductions *or* and *cr*. Finally, we give examples of the smallest finite topological T_0-space obtained from a core by using the reductions *or* and *cr*.

2 Properties of Finite Topological Spaces

In this paper, we consider finite topological T_0-spaces, where a finite topological T_0-space means a finite topological space satisfying the T_0 separation axiom: for each pair of distinct points, there exists an open set containing one but not the other. It is clear that if a finite topological space is T_1-space then it is in fact discrete. T_0-spaces would be more interesting.

Dedicated to the memory of Professor Katsuo KAWAKUBO

In this section, we give a notation for a finite topological space, some definitions and theorems which will be used in subsequent sections.

Let X be a finite topological T_0-space. For each $x \in X$, let U_x be the intersection of all open sets of X containing x, and let C_x be the closure of x. The collection of all U_x, $x \in X$, forms a minimal base for the topology of X. For T_0-spaces, the partial order by inclusion on the minimal base induces a partial order, so that for points x, y of X, $x \leq y$ is equivalent to $U_x \subset U_y$ (or $x \in U_y$). Then X is a T_0-space if and only if $x \leq y$ and $y \leq x$ implies $y = x$. Let X and Y be finite topological T_0-spaces. Let $f: X \to Y$ be a function. f is continuous if and only if $x \leq y$ implies $f(x) \leq f(y)$.

For a finite topological spaces $X = \{1, 2, \cdots, n\}$, the *relation matrix* $M = (m_{ij})$ of X is defined by

$$\begin{cases} m_{ij} = 1 & j \in C_i \\ m_{ij} = 0 & \text{otherwise.} \end{cases}$$

For instance, if X is the set $\{1, 2\}$ with the system of open sets $\{0, \{1\}, \{2\}, \{1, 2\}\}$, then the relation matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Another example is the set $X = \{1, 2\}$ with the system of open sets $\{0, \{1\}, \{1, 2\}\}$, whose the relation matrix is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If $M = (m_{ij})$ is the relation matrix of a finite topological T_0-space $X = \{1, 2, \cdots, n\}$ and if $B_j = \{i \in X | m_{ij} = 1\}$, then $B_j = U_j$ (see [3], Th. 1).

A continuous map $f: X \to Y$ is a *weak homotopy equivalence* if the induced maps

$$f_*: \pi_i(X, x) \to \pi_i(Y, f(x))$$

are isomorphisms for all x in X and all $i \geq 0$ (for $i = 0$, "isomorphism" simply means a bijection). In the following, if K is a simplicial complex, $|K|$ always denotes the underlying polyhedron. All maps of spaces are assumed to be continuous.

The following two theorems are easily induced from McCord ([2], Th. 2 and Th. 3).

Theorem 2.1. *There exists a correspondence that assigns to each finite topological T_0-space X a finite simplicial complex $\mathscr{K}(X)$, whose vertices are the points of X, and a weak homotopy equivalence f_X: $|\mathscr{K}(X)| \to X$. Each continuous map $\varphi: X \to Y$ between finite topological T_0-spaces corresponds to a simplicial map $|\mathscr{K}(\varphi)|: |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$ such that $\varphi \circ f_X = f_Y \circ |\mathscr{K}(\varphi)|$.*

Theorem 2.2. *There exists a correspondence that assigns to each finite simplicial complex K a finite T_0-space $\mathscr{X}(K)$, which consists of the barycenters of the simplices of K, and a weak homotopy equivalence f_K: $|K| \to \mathscr{X}(K)$. Furthermore, to each simplicial map $\psi: K \to L$, we can associate a map $\mathscr{X}(\varphi)$: $\mathscr{X}(K) \to \mathscr{X}(L)$ such that $\mathscr{X}(\varphi) \circ f_K$ is homotopic to $f_L \circ |\psi|$.*

It will be readily seen from the construction below that both $\mathscr{K}$ and $\mathscr{X}$ are covariant functors. In fact, the correlation of the functors $\mathscr{K}$ and $\mathscr{X}$ are represented by the equation $K' = \mathscr{K}(\mathscr{X}(K))$, where K' is the first barycentric subdivision of the simplicial complex K.

Definition of $\mathscr{K}(X)$. Let X be a finite topological T_0-space. The vertices of the complex $\mathscr{K}(X)$ are the points of X. The simplices of $\mathscr{K}(X)$ are the totally ordered subsets of X.

If Y is a subspace of the finite topological T_0-space X, then $\mathscr{K}(Y)$ is a full subcomplex of $\mathscr{K}(X)$.

Definition of the map f_X: $|\mathscr{K}(X)| \to X$. If $u \in |\mathscr{K}(X)|$ (where X is a finite topological T_0-space), then u is contained in a unique open simplex $(x_0, x_1, \cdots, x_r)$ where $x_0 < x_1 < \cdots < x_r$ in X. We let $f_X(u) = x_0$.

Definition of $\mathscr{X}(K)$. Let K be a finite simplicial complex. For each simplex σ of K, let $b(\sigma)$ be the barycenter of σ. Let $\mathscr{X}(K) = \{b(\sigma) | \sigma \in K\}$. Now $\mathscr{X}(K)$ has a partial order defined by $b(\sigma) \leq b(\sigma')$ if $\sigma \subset \sigma'$. Thus $\mathscr{X}(K)$ becomes a finite topological T_0-space.

Definition of the map f_K: $|K| \to \mathscr{X}(K)$. Let the map f_X be simply the map $f_{\mathscr{X}(K)}$.

In the remainder of this section, we review the homotopy theory of finite topological spaces from [4].

Let X be a finite topological T_0-space. We say that $x \in X$ is *linear* if there exists $y \in X$ with $y > x$ such that $z > x$ ($z \in X$) implies $z \geq y$, *colinear* if there exists $y \in X$ with $y < x$ such that $z < x$ ($z \in X$) implies $z \leq y$. A finite topological T_0-space X (resp. (X, p)) is called a *core* if neither linear points nor colinear points lie on X (resp. $X \backslash \{p\}$). A core of a finite topological T_0-space X (resp. (X, p)) is a subspace X_0 of X (resp. (X, p)) such that X_0 (resp. (X_0, p)) is a core and such that X_0 (resp. (X_0, p)) is a strong deformation retract of X (resp. (X, p)). For (X, p), the point p is called the base point of X. By the statement that (X_0, p) is a strong deformation retract of (X, p), we mean the existence of a strong deformation retraction of X to X_0 preserving p.

Theorem 2.3. *A finite topological T_0-space X (resp. (X, p)) always has a core.*

Theorem 2.4. *Let X (resp. (X, p)) be a core. Then any map $f: X \to X$ (resp. $f: X \to X$ preserving p) which is homotopic to the identity (resp. identity relative to base points) is the identity.*

Theorem 2.5. *Let X, Y (resp. (X, p), (Y, q)) be finite topological T_0-spaces with cores X_0, Y_0. Then X (resp. (X, p)) is homotopically equivalent to Y (resp. (Y, q)) if and only if X_0 (resp. (X_0, p)) is homeomorphic to Y_0 (resp. (Y_0, q)).*

By the statement (X, p) is homeomorphic to (Y, q), we mean the existence of a homeomorphism of X onto Y which maps p to q.

Proposition 2.6. *Both U_x and C_x are contractible.*

Proposition 2.7. *Connectedness and path-connectedness are equivalent for finite topological T_0-spaces.*

The following example shows that the homotopical classifications of finite topological T_0-spaces are too strict to investigate the homotopy type of simplicial complexes. This is why we introduce the concept of reduction.

Example 2.8. *Let X, Y be finite topological T_0-spaces, where X consists of four points and Y consists of six points. The relation matrices of X and Y are the following.*

$$\overset{X}{\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{Y}{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}.$$

Both $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are the triangulations of S^1. In particular, $|\mathscr{K}(X)|$ is homeomorphic to $|\mathscr{K}(Y)|$, though X is not homeomorphic to Y. Moreover X is not homotopically equivalent to Y because X and Y are themselves cores.

3 Further Properties

In this section, we investigate more about finite topological T_0-spaces and the associated simplicial complexes. By Theorem 3.3, we will see that the simplicial complex $\mathscr{K}(X)$ associated with a finite topological T_0-space X simplicially collapses to the simplicial complex $\mathscr{K}(X_0)$ associated with a core X_0 of X.

Proposition 3.1. *For a finite topological T_0-space X, we have the following:*
(i) $U_x = \{y \in X \mid y \leq x\}$.
(ii) $C_x = \{y \in X \mid y \geq x\}$.

Proof. By the definition of the order of X, (i) is straightforward. To see (ii), put $D_x = \{y \in X \mid y \geq x\}$. Then $U_z \cap D_x = \emptyset$ for all $z \in X \backslash D_x$. This means that D_x is closed. Hence C_x is included by D_x. For contradiction, assume that $z \in D_x \backslash C_x$. Then $U_z \cap C_x = \emptyset$. But U_z includes x because $z \geq x$. This is a contradiction. Thus $D_x \backslash C_x x = \emptyset$. □

Let X, X' be finite topological T_0-spaces such that X and X' coincide set-theoretically, and that X and X' have inverse order. Then the identity map $id: X \to X'$ is not always continuous. Therefore, X is not necessarily homeomorphic to X' and the same thing is true also for homotopy equivalence. However, the following theorem holds for such spaces.

Theorem 3.2. *Let X and X' be the spaces just as stated above. Then $\mathscr{K}(X)$ and $\mathscr{K}(X')$ are isomorphic.*

Proof. By the definition of the X', it is clear that the vertices of $\mathscr{K}(X)$ are the vertices of $\mathscr{K}(X')$. Furthermore, let σ be a simplex of $\mathscr{K}(X)$. Consider the corresponding totally ordered subset S of X. Clearly, S is also totally ordered in X', because X' has the inverse order of X. Hence, σ is again a simplex of $\mathscr{K}(X')$. □

Before stating the next theorem, we recall the definitions of an *elementary simplicial collapse* and a *simplicial collapse*.

Let $K_0 \subset K$ be simplicial complexes. Suppose A and aA are not in K_0 but are simplices of K, where a is a vertex of K_0, and suppose further that $K = K_0 \cup \{A\} \cup \{aA\}$. Here, aA denotes the join of a and A. Then we say that K collapses to K_0 by an *elementary simplicial collapse*, and we write $K \searrow^{es} K_0$. We say that K *collapses simplicially* to K_0, denoted by $K \searrow^{s} K_0$, if there is a finite sequence $K = K_r \searrow^{es} K_{r-1} \searrow^{es} \cdots \searrow^{es} K_0$ (see [1]).

Theorem 3.3. *Let X be a finite topological T_0-space and X_0 be a core of X. Then $\mathscr{K}(X)$ collapses simplicially to $\mathscr{K}(X_0)$.*

Proof. In the proof of Theorem 2.3, Stong ([4], Th. 2) repeatedly used the fact that if x is linear or colinear then $X \backslash \{x\}$ is a strong deformation retract of X. For the proof of Theorem 3.3, it suffices to show that $\mathscr{K}(X)$ collapses simplicially to $\mathscr{K}(X \backslash \{x\})$. We only treat the case where x is linear. Because if x is colinear, the proof is straightforward from Theorem 3.2 and the case where x is linear. Let y be a point of X with $y > x$ such that $z > x$ implies $z \geq y$. Let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be the simplices of $\mathscr{K}(X)$ having the vertex x. The suffix is always chosen in such a way that $i < j$ implies $\dim \sigma_i \leq \dim \sigma_j$. Clearly σ_k has the vertex y. Let n be the dimension of σ_k. Let $z_1 < z_2 < \cdots < z_i < x < y < z_{i+1} < \cdots < z_{n-1}$ be the vertices of σ_k. Let $\sigma_{k,y}$ be the $(n-1)$-simplex which has the vertices $z_1 < z_2 < \cdots < z_i < x < z_{i+1} < \cdots < z_{n-1}$. Then $\sigma_{k,y}$ is included by just one n-simplex σ_k. The reason is the following. A simplex which includes $\sigma_{k,y}$ must have the vertices $z_1 < z_2 < \cdots < z_i < x < y' < z_{i+1} < \cdots < z_{n-1}$. But y' must coincide with y by the linearity of x. Hence $\sigma_{k,y}$ is included just by σ_k. This means that the process of removing simplices $\{\sigma_k, \sigma_{k,y}\}$ is an elementary simplicial collapse.

This process can be applied to all n-simplices of $\sigma_1, \sigma_2, \cdots, \sigma_k$. So let K_1 be the simplicial complex obtained by removing all such n-simplices from $\mathscr{K}(X)$ by such elementary simplicial collapses. Some $(n-1)$-simplices are also removed.

Similarly let $\tau_1, \tau_2, \cdots, \tau_l$ be the simplices of K_1 which has the vertex x. Then $\dim \tau_l = n-1$. All these simplices have the vertex y. The reason is the following. The $(n-1)$-simplex τ_i $(1 \leq i \leq l)$ which does not have the vertex y is included in the n-simplex $y\tau_i$. Then $\{\tau_i, y\tau_i\}$ is already removed by the previous process. In particular, every $(n-1)$-simplex τ_i has the vertex y. Let $\tau_{l,y}$ be the $(n-2)$-face of τ_l which does not have the vertex y. Clearly $\tau_{l,y}$ is included only in τ_l. Therefore the process of removing simplices $\{\tau_l, \tau_{l,y}\}$ is an elementary simplicial collapse.

Repeating this process, we obtain the simplicial complex K_n which contains no simplices having the vertex x. Then $\mathscr{K}(X\backslash\{x\})$ is equal to K_n, because $\mathscr{K}(X\backslash\{x\})$ is the full subcomplex. So $\mathscr{K}(X) \searrow^s \mathscr{K}(X\backslash\{x\})$. Immediately we have $\mathscr{K}(X) \searrow^s \mathscr{K}(X_0)$. □

Corollary 3.4. *Let X and Y be finite topological T_0-spaces which are homotopically equivalent. Then $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple-homotopy type.*

Proof. By Theorems 3.3 and 2.5, the proof is straightforward. □

Cor. 3.4 shows that the homotpy type classification of finite topological T_0-spaces is finer than the simple homotopy type classification of the corresponding simplicial complexes. Thus it is too strict to investigate the homotopy types of the corresponding simplicial complexes. The following example shows that two triangulations of the same manifold S^1, which are not collapsible, correspond to finite topological spaces which are not homotopically equivalent.

Example 3.5. *Let K_1 be a triangulation of S^1 which has just three 1-simplices and let K_2 be a triangulation of S^1 which has just four 1-simplices. The relation matrices of $\mathscr{X}(K_1)$ and $\mathscr{X}(K_2)$ are the following:*

$$\begin{array}{c} \mathscr{X}(K_1) \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathscr{X}(K_2) \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

$\mathscr{X}(K_1)$ and $\mathscr{X}(K_2)$ are cores, because K_1 and K_2 are not collapsible. Clearly, $\mathscr{X}(K_1)$ is not homeomorphic to $\mathscr{X}(K_2)$. Hence $\mathscr{X}(K_1)$ is not homotopically equivalent to $\mathscr{X}(K_2)$. However, $|K_1|$ is homotopically equivalent to $|K_2|$. Moreover $|K_1|$ is homeomorphic to $|K_2|$.

4 Reduction of Finite Topological Spaces

In the previous section, we study the relation between a homotopy of a finite topological T_0-space and a simplicial collapse of a finite simplicial complex. Cor. 3.4 in particular implies that the homotopy type classification of finite topological T_0-space is too strict to investigate the homotopy type of a simplicial complex. There is a case that finite topological T_0-spaces X and Y have the distinct hotomopy types, but simplicial complexes $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same homotpy type. In this section we will consider a weak homotopy equivalence between finite topological spaces.

A quotient space of a topological space X is a quotient set X' of X topologized by the topology coinduced by the projection map $X \to X'$. If $A \subset X$, then X/A will denote the quotient space of X obtained by identifying all of A to a single point. We call the natural projection $p: X \to X/A$ *the identification.*

Now, we have Theorem 1.2, one of our main theorems. Before getting into the proof, we induce a theorem from [2], Th. 6. An open cover $\mathscr{U}$ of a space X is said to be *basis-like* if $x \in U \cap V$ and $U, V \in \mathscr{U}$, there exists $W \in \mathscr{U}$ such that $x \in W \subset U \cap V$.

Theorem 4.1. (See KcCord [2]), Th. 6. *Let $p: E \to B$ be a map between finite topological T_0-spaces E and B*

for which there exists a basis-like open cover $\mathscr{U}$ of B satisfying the following condition: For each $U \in \mathscr{U}$, the restriction $p|p^{-1}(U)\colon p^{-1}(U) \to U$ is a weak homotopy equivalence. Then p itself is a weak homotopy equivalence.

Proof of Theorem 1.2. Let $\mathscr{U} = \{U_y | y \in X/U_x\}$. Then $\mathscr{U}$ is a basis-like open cover. It suffices to show that

$$p|p^{-1}(U_u)\colon p^{-1}(U_y) \to U_y$$

is a weak homotopy equivalence for all $y \in X/U_x$. If $U_y \cap p(x) = \emptyset$, then $p^{-1}(U_y) = U_y$, and the above mapping is a weak homotopy equivalence. The remainder is the case $U_y \cap p(x) \neq \emptyset$. In this case $p^{-1}(U_y) = U_x \cup U_y \subset X$. The proof is reduced to showing that $\pi_i(U_x \cup U_y) = 0$ $(i \geq 1)$ and that $U_x \cup U_y$ is path-connected, because $U_y \subset X/U_x$ is contractible.

Path-connectedness is clear, because U_x, U_y and $U_x \cap U_y$ are path-connected. We next observe that $\pi_i(U_x \cup U_y)$ is isomorphic to $\pi_i(|\mathscr{K}(U_x \cup U_y)|)$. We therefore consider $|\mathscr{K}(U_x \cup U_y)|$.

$$\mathscr{K}(U_x \cup U_y) = \mathscr{K}(U_x) \cup \mathscr{K}(U_y) \quad \text{and} \quad \mathscr{K}(U_x \cap U_y) = \mathscr{K}(U_x) \cap \mathscr{K}(U_y).$$

By van Kampen's theorem, we obtain $\pi_1(|\mathscr{K}(U_x \cup U_y)|) = 0$. By Mayer-Vietoris exact sequence, we see that the homology groups of $|\mathscr{K}(U_x \cup U_y)|$ are zero in dimensions greater than zero. Then we have $\pi_i(|\mathscr{K}(U_x \cup U_y)|) = 0$ $(i \geq 1)$ by the Hurewicz isomorphism theorem. This implies that $\pi_i(U_x \cup U_y) = 0$ $(i \geq 1)$. Thus $p|p^{-1}(U_y)$ is also a weak homotopy equivalence in this case. This completes the proof. □

This means that we can obtain a new finite topological T_0-space with fewer points than the original space and with the same homotopy groups. Remember that the new space is not always homotopically equivalent to the original space.

Theorem 4.2. *Let X be a finite topological T_0-space. Let $x \in X$ be a point as in Theorem 1.2. Let $p\colon X \to X/U_x$ be the identification. Then the map $|\mathscr{K}(p)|\colon |\mathscr{K}(X)| \to |\mathscr{K}(X/U_x)|$ is a weak homotopy equivalence.*

Proof. The following diagram is commutative.

$$\begin{array}{ccc} |\mathscr{K}(X)| & \xrightarrow{|\mathscr{K}(p)|} & |\mathscr{K}(X/U_x)| \\ f_X \ \downarrow & & \downarrow \ f_{X/U_x} \\ X & \xrightarrow{p} & X/U_x \end{array}$$

This immediately implies $p_* \circ f_{X*} = f_{X/U_x*} \circ |\mathscr{K}(p)|_*$. By Theorems 2.1 and 1.2, we see that the maps p_*, f_{X*} and f_{X/U_x*} are isomorphisms, and so is $|\mathscr{K}(p)|_*$. □

The homotopy equivalence between $|\mathscr{K}(X)|$ and $|\mathscr{K}(X/U_x)|$ follows immediately from Theorem 4.2, because they are CW-complexes. This means that we can obtain a new simplicial complex with fewer vertices than the original complex and with the same homotopy type.

We have Theorem 1.3, the other one of our main theorems.

Proof of Theorem 1.3. Let X' be the set stated in Theorem 3.2. By Proposition 3.1, C_x is mapped to U_x by the map $id\colon X \to X'$. If $C_x \cap C_y \subset X$ satisfies a topological condition as above, then $U_x \cap U_y \subset X'$ also satisfies the condition. Hence this theorem follows from Theorem 1.2. □

For this theorem, the same property as Theorem 4.2 is also available.

Definition. The process of Theorem 1.2 (resp. 1.3) is called a *open* (resp. *closed*) *reduction* and is written as $or(x)$ (resp. $cr(x)$). The following are the examples of spaces which can be reduced to the smallest spaces by reductions or and cr.

Example 4.3. *The following are examples of such reductions applied to the 6-pointed space, which corresponds to the triangulation of S^1 with just three 1-simplices. In order to understand this easily, we number the columns of the relation matrices.*

$$\begin{array}{c} \mathscr{K}(K) \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \xrightarrow{or(4)} \begin{array}{c} \mathscr{K}(K)/U_4 \\ \begin{array}{cccc} 1 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \mathscr{K}(K) \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \xrightarrow{cr(2)} \begin{array}{c} \mathscr{K}(K)/U_4 \\ \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

As in the above diagram, the finite-topological T_0-space is reduced to 4-pointed space by both reductions. These 4-pointed spaces are the smallest finite topological T_0-spaces such that there exists a weak homotopy equivalence of S^1.

We easily see that every finite topological T_0-space corresponding to the triangulation of S^1 can be reduced to the 4 points space.

Example 4.4. *The following is an example of the reductions applied to the 14-pointed space, which corresponds to the triangulation of S^2 with just four 2-simplices. We again number the columns of the relation matrices.*

$$\begin{array}{c} \mathscr{K}(K) \\ \begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \xrightarrow{or(14)}$$

$$\begin{array}{c} \mathscr{K}(K)/U_{14} \\ \begin{array}{cccccccc} 1 & 14 & 5 & 6 & 8 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \xrightarrow{cr(5)} \begin{array}{c} (\mathscr{K}(K)/U_{14})/C_5 \\ \begin{array}{cccccc} 1 & 14 & 5 & 6 & 8 & 13 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The 6-pointed space is the smallest finite topological space such that there exists a weak homotopy equivalence of S^2. In this example, we need to use both reductions.

In both examples, we see that $|\mathscr{K}(X)|$, $|\mathscr{K}(X/U_x)|$ and $|\mathscr{K}(X/C_x)|$ are homeomorphic. So we have a conjecture.

Conjecture. Let X be a finite topological space such that $|\mathscr{K}(X)|$ is a manifold. For an open (resp. closed) reducible point $x \in X$, $|\mathscr{K}(X)|$ is homeomorphic to $|\mathscr{K}(X/U_x)|$ (resp. $|\mathscr{K}(X/C_x)|$).

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