Some Topics on Hyperbolic Geometry and Heegaard Splittings of 3-Manifolds

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This article is a brief survey of some recent topics of Essential lamination theory and Heegaard theory.

The purpose of this paper is to survey briefly some recent topics on 3-dimensional topology of hyperbolic 3-manifolds and Heegaard splittings for 3-manifolds. In particular, we focus on works of Gabai, Kazez, Calegari, ... and those of Hempel [26]. We attach importance to the stream of several works. The readers who want to know precisely should read their papers. See References.

Embedded surfaces in 3-manifolds are important clues to investigate the 3-manifolds. For example, incompressible surfaces and Heegaard surfaces play important roles. 3-manifolds which contain incompressible surfaces have been researched from several viewpoints, and we know now many features of such 3-manifolds.

About 15 years ago, Gabai and Oertel introduced the notion ‘Essential lamination’ [19]. This is a generalization of incompressible surface and important clue to study 3-manifolds, and then we know the 3-manifolds which contain essential laminations have similar features to those which contain incompressible surfaces recently. In Part 1, we focus on this topic. In Part 2, a topic on Heegaard splittings is stated. Refer [7, 19, 25, 27] for basic notions and definitions.

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Part 1. Topology of Hyperbolic 3-Manifolds

1. Thurston’s Hyperbolization Conjecture

The next is Thurston’s Hyperbolization conjecture.

Conjecture 1.1. Let $M$ be a closed, irreducible 3-manifold with $\pi_1M$ infinite and $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_1M$, then $M = \mathbb{H}^3/\Gamma$ where $\Gamma \subset \text{Isom}(\mathbb{H}^3)$.

He proved this conjecture for Haken manifolds. See [28] for the detail.

Theorem 1.2 ([44]). If $M$ is Haken, then this conjecture is true.

The Haken manifolds and hyperbolic 3-manifolds are classified in the sense of the following.

Theorem 1.3 ([45]). Suppose that $N$ is Haken. If $f : M \to N$ is a homotopy equivalence where $M$ is irreducible 3-manifold, then $f$ is homotopic to a homeomorphism.

Theorem 1.4 ([18]). Suppose that $N$ is hyperbolic. If $f : M \to N$ is a homotopy equivalence where $M$ is irreducible 3-manifold, then $f$ is homotopic to a homeomorphism.

Since $M$ is a $K(\pi, 1)$ space, this theorem reduces Conjecture 1.1 to the next one.

Conjecture 1.5. If $M$ is a closed, connected, aspherical 3-manifold and $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_3(M)$, then $\pi_1M$ is isomorphic to $\pi_1N$ where $N$ is a hyperbolic 3-manifold.

The next conjecture is called Group negative curvature conjecture.

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Conjecture 1.6. If $M$ is a closed, connected, aspherical 3-manifold and $\mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M)$, then $\pi_1(M)$ is word hyperbolic.

The definition of word hyperbolic is in Sect. 3. The next conjecture is due to Cannon.

Conjecture 1.7. If $M$ is a closed, connected, aspherical 3-manifold and $\mathbb{Z}/C_8 \subseteq \mathbb{Z}/C_6/C_2$ of a closed hyperbolic 3-manifold.

Recently, 3-manifolds with ‘essential (genuine) lamination’ are studied as follows. Note that the family of these manifolds includes that of Haken manifolds, see Sect. 2.

Theorem 1.8 ([17]). Let $M$ be an atroidal 3-manifold with genuine lamination, then $\pi_1 M$ is word hyperbolic.

Theorem 1.9 ([5]). Let $\mathcal{L}$ be an essential lamination of an atroidal 3-manifold. Then either $\mathcal{L}$ contains a proper genuine lamination, or $\mathcal{L}$ admits a transverse genuine lamination.

By [19], the universal cover of a closed 3-manifold containing an essential lamination is homeomorphic to $\mathbb{R}^3$, that is, its fundamental group is infinite.

Corollary 1.10. If a closed atroidal 3-manifold $M$ contains an essential lamination, then $\pi_1 M$ is infinite word hyperbolic.

Theorem 1.11 ([37]). There exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain a Reebless foliation.

This includes that there are infinitely many hyperbolic 3-manifolds which does not admit a taut foliation. (Any taut foliation is an essential lamination.) See also Theorem 2.8. We should note a problem proposed by Gabai [15]: Do all closed hyperbolic 3-manifolds contain essential laminations? Note [1] in References. Recently, an algorithm to detect whether a 3-manifold contains an essential lamination is found, see [2].

There are some papers related to Theorems 1.3 and 1.4. See [4] and [16], for example.

2. Essential Lamination and Genuine Lamination

In this section, we review the notion of essential lamination and genuine lamination. Refer to [6, 17, 19] for the detail.

A (2-dimensional) lamination $\mathcal{L}$ in a 3-manifold $M$ is a foliation of a closed subset $M$. A component of $M - \mathcal{L}$ is called a complementary region. A closed complementary region is a component $C$ of $M - \mathcal{L}$ metrically completed with respect to the induced path metric.

A disk-with-end $D$ is a disk with a closed arc removed from its boundary. The closed complementary region $C$ is end-incompressible if for every proper map $f : D^2 - x \to C, x \in \partial D$ such that $f(\partial D^2 - x) \subseteq L$ a leaf of $\mathcal{L}$, then there exists a proper map $g : D^2 - x \to L$ such that $g(\partial D^2 - x) = f(\partial D^2 - x)$.

Definition 2.1. A lamination $\mathcal{L}$ is essential in $M$ if it satisfies the following conditions:
(i) The inclusion of leaves of $\mathcal{L}$ into $M$ induces a injection on $\pi_1$;
(ii) $C$ is irreducible;
(iii) $\mathcal{L}$ has no sphere leaves;
(iv) $\mathcal{L}$ is end-incompressible.

According to [19], we may see these notions from a view point of ‘branched surface’.

Definition 2.2. A branched surface $B$ is a union of finitely many compact smooth surfaces glued together to form a compact subspace (of $M$) locally modelled on Fig. 1.
Given a branched surface \( B \) embedded in a 3-manifold \( M \), we denote by \( N(B) \) a regular neighborhood of \( B \), as shown in Fig. 1. One can regard \( N(B) \) as an interval bundle over \( B \), with a foliation \( \tau \) by intervals. We denote \( \pi : N(B) \to B \) the projection that collapses every interval fiber of \( \tau \) to a point. The boundary of \( N(B) \) is divided into \( \partial_N(B) \) (the horizontal boundary) and \( \partial_v(N(B)) \) (the vertical boundary). That is, \( \partial N(B) = \partial_N(N(B)) \cup \partial_v(N(B)) \) where an interval fiber of \( N(B) \) meets \( \partial_N(N(B)) \) transversely and intersects \( \partial_v(N(B)) \) in one or two closed intervals in the interior of this fiber. See Fig. 1. The shaded part is \( \partial_v(N(B)) \). A lamination is fully carried by \( B \) if it intersects every interval fiber.

**Theorem 2.3 ([19]).** 1. Every essential lamination is fully carried by a branched surface with the following properties.
   1. \( \partial_v(N(B)) \) is incompressible in \( M - \hat{N}(B) \), and no component of \( \partial_v(N(B)) \) is a sphere.
   2. \( M - B \) is irreducible.
   3. There is no monogon in \( M - \hat{N}(B) \), that is, no disk \( D \subset M - \hat{N}(B) \) with \( \partial D = D \cap N(B) = \alpha \cup \beta \), where \( \alpha \subset \partial_N(N(B)) \) is in an interval fiber of \( \partial_v(N(B)) \) and \( \beta \subset \partial_v(N(B)) \).
   4. \( B \) has no disk of contact, that is, no disk \( D \subset N(B) \) such that \( D \) is transverse to the I-fiber of \( N(B) \) and \( \partial D \subset \partial_v(N(B)) \). See Fig. 2.

   ![Fig. 2.](image)

(5) \( B \) does not carry a torus that bounds a solid torus in \( M \).

2. If a branched surface with the properties above fully carries a lamination, then it is an essential lamination.

**Remark 2.4.** Haken manifolds are compact orientable irreducible 3-manifolds that contain an embedded incompressible surface. Such a surface \( S \) is always an essential lamination and is a ‘genuine lamination’ (Definition 2.10) except in the case that its closed complementary region is an I-bundle.

There are many non-Haken manifolds with essential laminations. Thus the notion of essential lamination is a generalization of that of incompressible surface. In [46], Wu showed the following. We denote by \( K(r) \) the 3-manifold obtained by Dehn surgery on a knot \( K \) along the slope \( r \).

**Theorem 2.5 ([46]).** Let \( K \) be an arborescent knot. If \( K \) is not a Montesinos knot of length at most 3, then \( K(r) \) contains an essential lamination for all non-trivial slopes.

**Example 2.6.** Figure 3 indicates an essential branched surface which Wu constructed in the complement of the knot \( \beta_1 \). This surface is based on a tangle decomposition of the knot. Wu showed this type of essential branched surface carries an essential lamination, and proved the next theorem.

**Theorem 2.7 ([46]).** All but finitely many integral surgeries on a ‘type II’ arborescent knot produce non-Haken manifolds that contain essential laminations.

Similarly, essential laminations are in fact more prevalent comparing with taut foliations. Compare with the problem stated in the last of Sect. 1.

**Theorem 2.8 ([37]).** There exist infinitely many closed orientable hyperbolic 3-manifolds which contain neither a Reebless foliation nor a transversely oriented essential lamination but which do contain essential laminations.

Let \( X \) be a codimension 0 submanifold of a 3-manifold \( M \). As in the definition of the boundary of a regular neighborhood of a branched surface, we name the boundary of \( X \) in \( M \). The horizontal boundary of \( X \) is defined to be \( X \cap \partial M \) and is denoted \( \partial_hX \). The vertical boundary of \( X \) is defined to be the closure in \( \partial X \) of \( \partial X - \partial_hX \) and denoted \( \partial_vX \). The pair \((X, \partial_vX)\) is called an I-bundle if \( X \) is the total space of an I-bundle over a surface \( S \) in such a way that \( \partial_vX \) consists of the boundary points of the I-bundle fibers.

A product disk is a disk \((I \times I, \partial I \times I, I \times \partial I)\) properly embedded in \((X, \partial_vX, \partial_hX)\) such that each component of \( \partial I \times I \) is an essential arc in \( \partial_vX \). A product disk is essential if it is not parallel to a disk in \( \partial_vX \) keeping \( \partial I \times I \) in \( \partial_vX \) and \( I \times \partial I \) in \( \partial_hX \).
Lemma 2.9 ([17]). Let $X$ be the disjoint union of the closed complementary regions of an essential lamination $\mathcal{L}$. There exists a unique (up to isotopy in $X$) finite collection $\mathcal{A} = A_1 \cup \ldots \cup A_n$ of properly embedded annuli in $X$ satisfying following conditions:

1. $X = \mathcal{G} \cup I$ where $\mathcal{G} \cap I = \partial_\mathcal{G} = \partial_\mathcal{I} = \partial_\mathcal{A}$.
2. $(\mathcal{I}, \partial_\mathcal{I})$ is an $I$-bundle over a possibly noncompact or disconnected surface. No component of $(\mathcal{I}, \partial_\mathcal{I})$ is an $I$-bundle over a compact surface with non-empty boundary.
3. $(\mathcal{G}, \partial_\mathcal{G})$ is compact, has no components homeomorphic to $(D^2 \times I, D^2 \times \partial I)$ and contains no essential product disk.

See Figs. 1.1 and 1.2 in [17] for abstract figures.

Definition 2.10. The space $\mathcal{G}$ is called the guts of $\mathcal{L}$. A genuine lamination is an essential lamination in $M$ with non-empty guts.

3. Word Hyperbolic Group

Following [23], we introduce word hyperbolic group.

Definition 3.1. Let $M$ be an $n$-dimensional compact manifold (possibly with boundary) with a Riemannian metric. Suppose $n \geq 3$. The fundamental group is word hyperbolic if, for some positive constant $c$, each homotopically trivial simple closed curve $K$ bounds a disk $D$ with:

$$\frac{\text{length}(K)}{\text{area}(D)} \geq c.$$ 

This inequality is called a linear isoperimetric inequality.

The atmosphere is as follows. See Fig. 4. Let $\Delta$ be the regular triangle with the length $r$ edge in flat space. Then $\text{area}(\Delta)$ is equal to $\sqrt{3}r^2/4$, so

$$\frac{\text{length}(\partial \Delta)}{\text{area}(\Delta)} = \frac{4\sqrt{3}}{r} \to 0 \quad (r \to \infty).$$

On the other hand, suppose $\Delta$ is a triangle in the negative curved space. Then the triangle $\Delta$ becomes thin, when $\partial \Delta$ becomes longer because of the Gauss–Bonnet theorem that says
\[ \int_{\Delta} \kappa = (\text{the sum 3 angles of } \Delta) - \pi. \]

If \( \kappa = -1 \) for example, \(-\text{area}(\Delta) = (\text{the sum 3 angles of } \Delta) - \pi \), namely, \( \text{area}(\Delta) < \pi \).

**Example 3.2.** The fundamental group of a negatively curved Riemannian closed manifold is word hyperbolic.

Compare this example with Conjecture 1.7.

By the well-known Mostow rigidity, homotopy equivalence for closed hyperbolic manifolds in dimension bigger than 2 implies homeomorphism. Furthermore, Farrell and Jones \([10]\) shows that closed negatively curved manifolds of dimension bigger than 4 are homotopy equivalent if and only if they are homeomorphic (this is also conjectured to be true in lower dimensions).

For every finitely presented group \( G \), there exists a smooth bounded connected domain \( V \subset \mathbb{R}^n \) for every \( n \geq 5 \), such that \( \pi_1 V \) is isomorphic to \( G \). A standard example of such \( V \) is obtained as follows: Fix a finite presentation of \( G \) and let \( P \) be the 2-dimensional cell complex whose 1-cells correspond in the usual way to the generators and the 2-cells to the relations in \( G \), such that \( \pi_1(P) = G \). Embed \( P \) into \( \mathbb{R}^5 \) and take a regular neighborhood of \( P \) in \( \mathbb{R}^5 \) for \( V \). Thus, by using \( V \) and its fundamental group \( G \), the notion of word hyperbolic may be defined for any finitely presented group. See \([23]\) or \([36]\) for another definition of a word hyperbolic group.

The following examples are known \([12]\).

**Example 3.3.** A free group is word hyperbolic. In particular, \( \mathbb{Z} \) is word hyperbolic.

**Example 3.4.** A finite group is word hyperbolic.

**Example 3.5.** \( \mathbb{Z} \oplus \mathbb{Z} \) is not word hyperbolic.

A group which contains \( \mathbb{Z} \) as a subgroup of finite index is also word hyperbolic. These groups and finite groups are said to be elementary. A free group whose rank is greater than 1 is not elementary, and the fundamental group of a closed negative curvature manifold is not elementary.

The next theorem is called Gabai’s Ubiquity theorem.

**Theorem 3.6 ([14]).** Let \( K \subsetneq B^3 \) be a smooth simple closed curve in a closed, atroidal, irreducible 3-manifold \( M \). There exists constants \( c_1 \) and \( c_2 \) such that if \( D \) is a least area disk with \( \partial D \cap K = \emptyset \) and \( \text{length}(\partial D)/\text{area}(D) < c_1 \), then \( \text{wr}(\partial D, K)/\text{area}(D) > c_2 \).

Here the wrapping number \( \text{wr}(\partial D, K) \) is the minimal geometric intersection number between \( K \) and all immersed disks whose boundary is \( \partial D \).

This theorem plays an important role to prove Theorem 1.8. Set \( K = \text{a core of } \mathcal{A} \), and show \( K \subsetneq B^3 \) at first. (See Lemma 2.9 for the definition of \( \mathcal{A} \).) We assume there is a disk which shows that a linear isoperimetric inequality does not hold, and then apply this theorem.

See also \([29]\) on Theorem 3.6.
Part 2. Heegaard Splittings of 3-Manifolds

4. Heegaard Splitting

In this section, we review some basic notion of Heegaard splitting.

Definition 4.1. A compression body \(W\) is a cobordism rel \(\partial\) between surfaces \(\partial_s W\) and \(\partial_w W\) such that \(W\) is homeomorphic to \(\partial_s W \times I \cup 2\)-handles \(\cup 3\)-handles and \(\partial w W\) has no 2-sphere components. If \(\partial_s W\) is closed, connected and \(\partial_w W = \emptyset\), \(W\) is a handlebody.

In other words, a connected compression body \(W\) is a connected 3-manifolds obtained from a closed surface \(\partial w W\) by attaching \(1\)-handles along the components of \(\partial s W\) together along \(\partial s W \times \{1\} \subset \partial w W \times I\).

Definition 4.2. Suppose two compression bodies \(W_1\) and \(W_2\) have that \(\partial_s W_1\) is homeomorphic to \(\partial_s W_2\). Then glue \(W_1\) and \(W_2\) together along \(\partial s W_i = H\). The resulting compact 3-manifold \(M\) can be written \(M = W_1 \cup_H W_2\) and this structure is called a Heegaard splitting of \(M\).

By using a results in [32], we have:

Theorem 4.3. Any compact 3-manifold whose boundary has no 2-sphere components has Heegaard splitting.

Definition 4.4. A Heegaard splitting \(W_1 \cup_H W_2\) is reducible if there are essential disks \(D_i\) properly embedded in \(W_i\) so that \(\partial D_1 \cap \partial D_2 = \emptyset\) in \(H\).

The next notion is introduced by Casson and Gordon [8].

Definition 4.5. A Heegaard splitting \(W_1 \cup_H W_2\) is weakly reducible if there are essential disks \(D_i\) properly embedded in \(W_i\) so that \(\partial D_1 \cap \partial D_2 = \emptyset\) in \(H\).

A Heegaard splitting which is not weakly reducible is said to be strongly irreducible. By using this idea, Scharlemann and Thompson introduced the notion of ‘Thin position of Heegaard splitting’ or ‘Generalized Heegaard splitting’ [41]. It is useful to treat incompressible annuli properly embedded in 3-manifolds with boundaries, and then to study the behavior of tunnel number under connected sum [33, 34, 39, 40].

The next notion is defined by Thompson. See [43].

Definition 4.6. A Heegaard splitting \(W_1 \cup_H W_2\) is said to have the disjoint curve property if there exist essential simple closed curves \(\ell\), \(\partial D_1\) and \(\partial D_2\) on \(H\) such that \(\ell \cap (\partial D_1 \cup \partial D_2) = \emptyset\) and \(\partial D_i\) bounds a disk \(D_i\) in \(W_i\).

The topology of 3-manifolds which admit the above Heegaard splittings will be written in Sect. 6.

5. Curve Complex and Distance of Heegaard Splitting

The notions and results in this section is contained in [26]. The readers who want to know minutely should read [26]. The idea of curve complex appears in many subjects of topology. See [31], for instance.

Let \(H\) be a closed, connected, oriented surface of genus \(g \geq 2\).

Definition 5.1. The curve complex of \(H\), denoted \(C(H)\), is the complex whose vertices are the isotopy classes of essential simple closed curves in \(H\), and where distinct vertices \(x_0, x_1, \ldots, x_n\) determines an \(n\)-simplex of \(C(H)\) if they are represented by pairwise disjoint simple closed curves.

A simplex \(X\) of \(C(H)\) determines a compression body

\[
W_X = H \times [0, 1] \cup_{X \times 1} \text{2-handles } \cup \text{3-handles},
\]

obtained by attaching 2-handles along the components of \(X \times 1\) and filling in any resulting 2-sphere boundary with 3-balls.

A pair \(X, Y\) of simplexes of \(C(H)\) determine a Heegaard splitting:

\[
W_X \cup_H W_Y = M_{X,Y}
\]

for which \((H; X, Y)\) is a Heegaard diagram.

There is a subcomplex \(K_X \subset C(H)\) consisting of those simplexes \(X'\) (and their faces) with \((W_{X'}, H) = (W_X, H)\). See Lemma 1.2 in [26]. Thus the pair \(K_X, K_Y\) of subcomplexes of \(C(H)\) describes the different diagrams for a fixed splitting.

The geodesic distance function \(d\) is defined on the 0-skeleton of \(C(H)\) by...
$d(x, y) =$ the minimal number of 1-simplexes in a simplicial path joining $x$ to $y$.

**Example 5.2.** For simple closed curves $x$ and $y$, $d(x, y) \leq 1$ if and only if $x \cap y = \emptyset$. $d(x, y) \leq 2$ if and only if there is a simple closed curve $z$ such that $x \cap z = \emptyset$ and $y \cap z = \emptyset$.

**Definition 5.3.** The distance of the Heegaard splitting $(H; W_X, W_Y)$ is defined by

$$d(K_X, K_Y) = \text{the minimal distance between their respective vertices}$$

We can observe the following; this distance is a natural generalization of several known notions:

**Proposition 5.4.** The Heegaard splitting $(H; W_X, W_Y)$ of a compact 3-manifold $M_{X,Y}$ is (has)

1. reducible if and only if $d(x, y) = 0$.
2. weakly reducible if and only if $d(x, y) \leq 1$.
3. disjoint curve property if and only if $d(x, y) \leq 2$.

**Theorem 5.5 ([26]).** There are distance $n$ Heegaard splittings of closed orientable 3-manifolds for arbitrarily large $n$.

6. **Topology of 3-Manifolds from the Viewpoint of Heegaard Splittings with Curve Complex**

**Theorem 6.1 ([26]).** Let $M$ be a closed orientable 3-manifold which is Seifert fibered space or which contains an essential torus. Then any Heegaard splitting of $M$ is a distance $\leq 2$ splitting.

The converse is not true. Actually, there are many hyperbolic 3-manifolds with distance 2 splittings.

**Example 6.2.** Any Dehn surgery on a 2-bridge knot has a distance $\leq 2$ splitting.

Thus this distance is not enough in order to separate the family of 3-manifolds that consists of Seifert fibered spaces and 3-manifolds containing an essential torus from the others. Sakuma proposed the following questions.

**Question 1.** Can we find a good word of Heegaard splittings to separate them by improving the above distance?

**Question 2.** Can we construct hyperbolic structures on corresponding 3-manifolds by using the distance in Question 1?
Theorem 6.3 ([24]). Let \( M \) be a closed 3-manifold. A Heegaard splitting of \( M \) is reducible if and only if \( M \) is reducible.

Theorem 6.4 ([8]). Let \( (W, W') \) be a Heegaard splitting of a closed 3-manifold \( M \). If \( (W, W') \) is weakly reducible then either \( (W, W') \) is reducible or \( M \) contains an incompressible surface of positive genus.

7. Genus Two Heegaard Splittings and the Disjoint Curve Property

There are some works on genus 2 manifolds with the distance \( \leq 2 \).

Theorem 7.1 ([43]). A weakly reducible genus 2 Heegaard splitting is reducible.

Theorem 7.2 ([43]). Let \( (H; W_1, W_2) \) be a genus 2 Heegaard splitting of a 3-manifold \( M \). Suppose \( (H; W_1, W_2) \) has the disjoint curve property. Then either:

1. \( (H; W_1, W_2) \) is a stabilization;
2. \( M \) is reducible;
3. \( M \) is a small Seifert fibered space;
4. \( M \) contains an essential torus;
5. \( M \) is obtained by surgery on a \( (1, 1) \)-knot in a Lens space, \( S^3 \) or \( S^3 \times S^1 \).

See also Sect. 6 in [26] for more details. A 3-manifold with a knot case has been discussed in [21, 30].

We note that \((1, 1)\)-knots include 2-bridge knots. There are some works on \( (1, 1) \)-knots. See, for example, [9, 20, 35]. Further, this knot class is important from the view point of Dehn surgery. For example, it is conjectured that if a knot \( K \) in \( S^3 \) yields a lens space \( M \) by a Dehn surgery, then the core knot \( K^* \) of the Dehn surgery is a \( (1, 1) \)-knot in \( M \), see [3, 22].

There are some trials to Question 1 in Sect. 6.

Definition 7.3. A Heegaard splitting \((H; W_1, W_2)\) is said to have the double disjoint curve property if there are essential disks \( D_1 \subset W_1 \) and \( D_2 \subset W_2 \), and there are disjoint non-parallel simple closed curves \( \ell_1 \) and \( \ell_2 \) on \( H \) such that \( (\partial D_1 \cup \partial D_2) \cap (\ell_1 \cup \ell_2) = \emptyset \).

Theorem 7.4 ([21]). Let \( (H; W_1, W_2) \) be a genus 2 Heegaard splitting of a 3-manifold \( M \). Suppose \( (H; W_1, W_2) \) has the disjoint curve property. Then either:

1. \( (H; W_1, W_2) \) is weakly reducible;
2. \( M \) contains an incompressible torus;
3. \( M \) is obtained by surgery on a 2-bridge knot or link.

Theorem 7.5 ([43]). Let \( (H; W_1, W_2) \) be a genus 2 Heegaard splitting of a 3-manifold \( M \). Suppose there exist essential disks \( D_1 \) and \( D_2 \) in \( W_1 \) and \( W_2 \), respectively such that \( \partial D_1 \) intersects \( \partial D_2 \) in at most three points. Then

1. \( (H; W_1, W_2) \) has the disjoint curve property, and
2. \( M \) is not hyperbolic.

REFERENCES

[1] Recently, Fenley announced that there are infinitely many closed, hyperbolic 3-manifolds which do not admit essential laminations [11]. In order to approach Hyperbolization conjecture from this viewpoint, we might need a new notion. One candidate may be ‘loosess laminations’. See [6], 6.3.


[37] Sakuma, M., private communications.