Isoperimetric Constants of \((d, f)\)-Regular Planar Graphs

Yusuke HIGUCHI\(^1\) and Tomoyuki SHIRAI\(^2\)

\(^1\)Mathematics Laboratories, College of Arts and Sciences, Showa University, 4562 Kamiyoshida, Fujiyoshida, Yamanashi 403-0005, Japan
\(^2\)Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

E-mail: higuchi@cas.showa-u.ac.jp, shirai@math.titech.ac.jp


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For a \((d, f)\)-regular planar graph, which is an infinite planar graph embedded in the plane such that the degree of each vertex is \(d\) and the degree of each face is \(f\), we determine two kinds of isoperimetric constants in concrete form.

KEYWORDS: isoperimetric constants, planar graph, hyperbolic, discrete Laplacian

1. Introduction

Let \(G = (V(G), E(G))\) be a connected undirected graph without loops and multiple edges, where \(V(G)\) is the set of vertices and \(E(G)\) is the set of edges. For \(x \in V(G)\), the degree of \(x\) in \(G\), denoted by \(\deg_G(x)\), implies the number of edges incident with \(x\), and the neighbourhood of \(x\) in \(G\), denoted by \(N_G(x)\), implies the set of vertices adjacent to \(x\) in \(G\); thus \(\deg_G(x) = |N_G(x)|\). Definitions and notation on graphs which are not given in this note can be found in [1, 2]. A graph \(G\) is said to be a \((d, f)\)-regular planar graph if it satisfies the following:

1. \(G\) is planar and already embedded in the plane;
2. \(G\) is regular in the ordinary sense, that is, \(\deg_G(x) = d\) for every vertex \(x \in V(G)\) and \(d \geq 3\);
3. Every face \(R\) is an \(f\)-gon, that is, \(d(R) = f\) for every face \(R \in F(G)\), where \(F(G)\) is the set of faces of \(G\), \(d(R)\) is the number of edges of the boundary of \(R\) and \(f \geq 3\).

Set \(H(d, f) = 4 - (d - 2)(f - 2)\). It is well known that, if \(H(d, f) > 0\), \(G\) is one of the platonic graphs, which are finite regular polyhedra. If \(H(d, f) \leq 0\), \(G\) is an infinite graph. In this note, we deal with only infinite \((d, f)\)-regular planar graphs satisfying the following conditions:

Assumption 1.1.

(A1) If \(H(d, f) = 0\), then \(G\) is isomorphic to the graph which is obtained as a tessellation of the two-dimensional Euclidean space \(\mathbb{R}^2\) consisting of regular \(f\)-gons. In other words, if \((d, f) = (3, 6), (4, 4)\) or \((6, 3)\), then \(G\) is so-called the hexagonal lattice, the square one or the triangular one, respectively;

(A2) If \(H(d, f) < 0\), then \(G\) is isomorphic to the graph which is obtained as a tessellation of the Poincaré disc \(H^2\) consisting of regular \(f\)-gons with angle \(2\pi/d\). [We can derive such a tessellation by considering the canonical tessellation under a triangle group \(G(2, f, d)\), which is a Fuchsian group (cf. [13]).]

One of the main topics in spectral analysis of a discrete Laplacian is to give some information on its spectrum for a given graph; for example, a set of spectrum, a lower or upper bound of this set, and so on. Here, a discrete Laplacian is a bounded linear operator \(\Delta_G : \ell^2(V(G)) \to \ell^2(V(G))\) defined by

\[
\Delta_G \varphi(x) = \frac{1}{\deg_G(x)} \sum_{y \in N_G(x)} (\varphi(y) - \varphi(x))
\]

for a given general graph \(G\), where \(\ell^2(V(G)) = \{ \varphi : V(G) \to \mathbb{R} \mid \| \varphi \|_V < \infty \}\) with the inner product \(\langle \varphi_1, \varphi_2 \rangle_V = \sum_{x \in V(G)} \varphi_1(x) \overline{\varphi_2(x)} \deg_G(x)\). Let \(\text{Spec}(\Delta_G)\) be the spectrum of \(-\Delta_G\) and \(\lambda_0(G) = \inf \text{Spec}(\Delta_G)\). For some kinds of graphs, the spectra are determined exactly (for instance, [15, 16]).

Our final aim for a \((d, f)\)-regular planar graph \(G\) is to determine the spectrum of \(\Delta_G\) concretely when \(H(d, f) < 0\), but we do not succeed at this moment. In this note, we instead determine two kinds of isoperimetric constants of \(G\), which relate closely to a lower bound of \(\text{Spec}(\Delta_G)\). One, denoted by \(\alpha(\cdot)\), is an analogue of Cheeger’s constant, which can be defined for general graphs (cf. [4–6, 10, 11]); the other, denoted by \(\alpha^*(\cdot)\), is essentially the same as the one found in combinatorial group theory, which is defined only for planar graphs and gives a “hyperbolic criterion for an infinite.
planar graph” ([4, 7, 8, 10, 11]). These $\alpha(G)$ and $\alpha^*(G)$ are defined as follows:

$$\alpha(G) = \inf \{|E(\partial_e K)|/|\text{Area}(K)| \mid K \text{ is a finite subgraph of } G\},$$  \hspace{2cm} (1.2)

where

$$\text{Area}(K) = \sum_{x \in V(K)} \deg_G(x)$$

and

$$E(\partial_e K) = \{xy \in E(G) \mid x \in V(K), y \in V(G) \setminus V(K)\};$$

$$\alpha^*(G) = \inf \{|E(\partial_e K)|/|F(K)| \mid K \text{ is a finite subgraph of } G\},$$  \hspace{2cm} (1.3)

where $E(\partial_e K) = \{xy \in E(K) \cap E(F') \mid F' \in F(G) \setminus F(K)\}$. Both of them are equal to 0 when $G$ is finite.

Either of them is a discrete analogue of the isoperimetric constant of a manifold: each denominator corresponds to the “area” of $K$, and each numerator to the “length” of the boundary of $K$.

It is known that $\alpha(G)$ is positive if and only if $\lambda_0(G)$ is positive [3–5]; moreover, Fujiwara [5] gives the following estimate:

**Theorem 1.2 ([5]).** For every graph $G$, $1 - \sqrt{1 - \alpha^2(G)} \leq \lambda_0(G) \leq \alpha(G)$.

On the other hand, the following fact which is represented in terms of graph theory is well known as small cancellation theory in combinatorial group theory [7, 8, 10]:

**Theorem 1.3.** Let $G$ be an infinite planar graph such that $3 \leq m_0 \leq \deg_G(x) < \infty$ for every $x \in V(G)$ and $3 \leq l_0 \leq d(R) < \infty$ for every $R \in F(G)$. Then, $\alpha^*(G) > 0$ if $1/m_0 + 1/l_0 < 1/2$.

In addition, the following relationship between $\alpha(\cdot)$ and $\alpha^*(\cdot)$ is known.

**Theorem 1.4 ([11]).** Let $G$ be an infinite planar graph such that $d(R) \text{ and } \deg_G(x) \text{ are bounded in } G$. Then the following four inequalities are mutually equivalent: $\alpha^*(G) > 0$, $\alpha(G) > 0$, $\alpha^*(G^*) > 0$, and $\alpha(G^*) > 0$, where $G^*$ is a dual graph of $G$.

If $\alpha(G) > 0$, then $\lambda_0(G) > 0$, which implies the simple random walk on $G$ is strongly transient. In combinatorial group theory, the group $\Gamma$ is said to be hyperbolic if $\alpha^*(G) > 0$, where $G$ is the graph associated with $\Gamma$ [7, 8]. Thus we may say an infinite planar graph $G$ is “hyperbolic” if $\alpha(G) > 0$ or $\alpha^*(G) > 0$, since $G$ which is embedded into the plane can be considered as the discrete version of a noncompact 2-dimensional surface. In this sense, “$1/m_0 + 1/l_0 = 1/2$” in Theorem 1.3 or $H(d, f)$ may be considered as a kind of “curvature” for planar graphs. More general curvature for planar graphs is discussed in [10].

By Theorems 1.3 and 1.4, it is easy to see that, for a $(d, f)$-regular planar graph such that $H(d, f) < 0$, $\alpha(G)$, $\alpha^*(G)$, $\alpha(G^*)$ and $\alpha^*(G^*)$ are all positive. Remark $G^*$ corresponds to an $(f, d)$-regular planar graph.

Our purpose is to give the following expressions of $\alpha(G)$ and $\alpha^*(G)$ for a $(d, f)$-regular planar graph by purely combinatorial methods:

**Main Theorem.** For a $(d, f)$-regular planar graph $G$ satisfying $H(d, f) = 4 - (d - 2)(f - 2) \leq 0$ and Assumption 1.1, we have

$$\alpha(G) = \frac{d - 2}{d} \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}$$  \hspace{2cm} (1.4)

and

$$\alpha^*(G) = (f - 2) \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}$$  \hspace{2cm} (1.5)

Our strategy to show the above is as follows. In Sect. 2, we introduce notation and terminology used in our proof and give some lemmas including upper estimates of $\alpha(G)$ and $\alpha^*(G)$ (Lemma 2.2 and Lemma 2.3). In Sect. 3, we manage to show a lower estimate of $\alpha^*(G)$, which is the main part of this proof (Proposition 3.1).

**Remark.** Another proof of Main Theorem is independently obtained by Häggström, Jonasson and Lyons [9]. They use a kind of duality relation between some isoperimetric constants, while we construct a sequence of “balls” and directly show that this sequence is optimal for the isoperimetric constant.
2. Lemmas

If \( H(d, f) = 4 - (d - 2)(f - 2) = 0 \), then \( G \) is the hexagonal, the square, or the triangular lattice. In this case it is obvious that the equalities in (1.4) and (1.5) hold since it is well-known \( \lambda_d(G) = \alpha(G) = \alpha^*(G) = 0 \); in order to shed light on the difference between the Euclidean and hyperbolic cases, we give some observation for this case in Remark 2.6. Therefore we may assume \( H(d, f) < 0 \) and Assumption 1.1 here and hereafter. In addition, we write for simplicity \( \mathcal{D} \) and \( \mathcal{F} \) for \( d - 2 \) and \( f - 2 \), respectively.

Let us first introduce a concept of an \( n \)-ball used in our discussion:

**Definition 2.1.** For a \((d, f)\)-regular planar graph \( G \), we pick and fix a face \( R_0 \in F(G) \) and define the \( n \)-ball \( B^*_n = B^*_n(G, R_0) \) in \( G \) as follows: \( B^*_0 = G[R_0] \) and

\[
B^*_n = G\{R \in F(G) \mid V(R) \cap V(B^*_{n-1}) \neq \emptyset\}
\]

for every \( n = 1, 2, \ldots \), where \( G[A] \) is the subgraph of \( G \) induced by a set \( A \) and \( V(R) \) is a set of vertices of a face \( R \).

We set some abbreviations for facilitating our exposition:

1. \( V^*_n = \{ x \in V(B^*_n) \mid x \in V(B^*_{n-1}) \} \) for \( n = 1, 2, \ldots \), \( V^*_0 = V(R_0) \) and \( v^*_n = |V^*_n| \). Thus we have \( v^*_n = \mathcal{F} + 2 = f \);
2. \( S^*_n = \{ x \in V^*_n \mid N_G(x) \cap V(B^*_{n-1}) \neq \emptyset \} \), \( u^*_n = \sum_{x \in S^*_n} |N_G(x) \cap V(B^*_{n-1})| \) for \( n = 1, 2, \ldots \), and \( u^*_0 = 0 \).

**Remark 2.2.** We notice the following:

1. \( u^*_n = ||xy \in E(G) \mid x \in V^*_n, y \in V^*_{n-1}| = |F(B^*_n)\setminus F(B^*_{n-1})| \);
2. If \( \mathcal{F} = 1 \), that is, \( f = 3 \), then \( S^*_n = V^*_n \) and \( |N_G(x) \cap V^*_{n-1}| = 1 \) or \( 2 \) for \( x \in S^*_n \);
3. If \( \mathcal{F} \geq 2 \), that is, \( f \geq 4 \), then \( S^*_n \subseteq V^*_n \) and \( |N_G(x) \cap V^*_{n-1}| = 1 \) for \( x \in S^*_n \).

These are the consequences derived directly from Definition 2.1 and Assumption 1.1.

It is easy to see that

\[
u^*_n = \mathcal{D} \cdot v^*_n - u^*_n - 1
\]

and

\[
u^*_0 = \mathcal{F} \cdot |F(B^*_n) \setminus F(B^*_{n-1})| - v^*_0 = (\mathcal{D}\mathcal{F} - 1)v^*_n - \mathcal{F} \cdot u^*_n
\]

for \( n = 1, 2, \ldots ; v^*_0 = \mathcal{F} + 2 \) and \( u^*_0 = 0 \). Thus we have the expression

\[
\begin{pmatrix}
u^*_n \\ v^*_n
\end{pmatrix} = \begin{pmatrix}-1 & \mathcal{D} \\ -\mathcal{F} & \mathcal{D}\mathcal{F} - 1
\end{pmatrix} \begin{pmatrix}u^*_{n-1} \\ v^*_{n-1}
\end{pmatrix} = \begin{pmatrix}0 \\ \mathcal{F} + 2
\end{pmatrix}.
\]

Here, setting

\[
\mu_1 = \left(\mathcal{D}\mathcal{F} - 2 + \sqrt{\mathcal{D}\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}\right) / 2 \quad \text{and} \quad \mu_2 = 1 / \mu_1,
\]

we find \( \mu_1 \) and \( \mu_2 \) are eigenvalues of the matrix in (2.3); it is obvious that \( \mu_1 > 1 > \mu_2 > 0 \),

\[
\mu_1 + \frac{1}{\mu_1} = \mathcal{D}\mathcal{F} - 2 \quad \text{and} \quad \mu_1 - \frac{1}{\mu_1} = \sqrt{\mathcal{D}\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}.
\]

By straightforward computation, we get

\[
u^*_n = \frac{\mathcal{D}(\mathcal{F} + 2)}{\sqrt{\mathcal{D}\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}} \left(\mu_1^n - \mu_1^{-n}\right), \quad v^*_n = \frac{\mathcal{F} + 2}{\sqrt{\mathcal{D}\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}} \left((\mu_1^{n+1} - \mu_1^{-(n+1)}) + (\mu_1^n - \mu_1^{-n})\right)
\]

and

\[
|V(B^*_n)| = \sum_{k=0}^{n} v^*_k = \frac{\mathcal{F} + 2}{\mathcal{D}\mathcal{F} - 4} \left(\mu_1^{n+1} - 2 + \mu_1^{-(n+1)}\right)
\]

for every non-negative integer \( n \). Hence we have

\[
\frac{|E(\partial B^*_n)|}{|V(B^*_n)|} = \frac{u^*_{n+1}}{\sum_{k=0}^{n} v^*_k} = \sqrt{\frac{\mathcal{D}(\mathcal{D}\mathcal{F} - 4)}{\mathcal{F}}} \left(1 + \frac{2}{\mu_1^{n+1} - 1}\right),
\]

which decreases monotonously as \( n \) increases. Consequently, we obtain

\[
\lim_{n \to \infty} \frac{|E(\partial B^*_n)|}{\text{Area}(B^*_n)} = \lim_{n \to \infty} \frac{|E(\partial B^*_n)|}{d(V(B^*_n))} = \frac{\mathcal{D}}{\mathcal{D} + 2 \sqrt{1 - \frac{4}{\mathcal{D}\mathcal{F}}}}.
\]
This implies the following:

**Lemma 2.3.** For a \((d,f)\)-regular planar graph \(G\) satisfying \(H(d,f) < 0\) and Assumption 1.1, we have

\[
\alpha(G) \leq \frac{d - 2}{d} \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}.
\]

(2.10)

Let us secondly introduce another \(n\)-ball for simplifying our discussion.

**Definition 2.4.** For a \((d,f)\)-regular planar graph \(G\), we pick and fix a vertex \(x_0 \in V(G)\) and define the \(n\)-ball \(B_n = B_n(G,x_0)\) in \(G\) as follows: \(B_0 = G[x_0]\) and

\[
B_n = G\{R \in F(G) \mid V(R) \cap V(B_{n-1}) \neq \emptyset\}
\]

for every \(n = 1, 2, \ldots\)

For this \(n\)-ball \(B_n\), we also set

1. \(V_n = \{x \in V(B_n) \mid V(B_{n-1})\} \) for \(n = 1, 2, \ldots\), \(V_0 = x_0\) and \(v_n = |V_n|\).
2. \(u_n = \sum_{x \in V_n} |N_G(x) \cap V(B_{n-1})|\) for \(n = 1, 2, \ldots\), and \(u_0 = 0\).

Let \(G^*\) be the dual graph of an \((f,d)\)-regular planar graph \(G\); thus we find \(G^*\) is a \((d,f)\)-regular planar graph. Here, the dual \(G^*\) of \(G\) satisfies the following conditions: there exist bijections

\[
* : F(G) \rightarrow V(G^*), \quad * : E(G) \rightarrow E(G^*), \quad * : V(G) \rightarrow F(G^*)
\]

(2.11)

such that vertices \(R^*_n\) and \(R^*_m\) in \(G^*\) are joined by the edge \(e^*\) if and only if faces \(R_1\) and \(R_2\) in \(G\) have the common edge \(e\).

Setting \(s(R_0) = x_0\) for \(R_0 \subseteq F(G)\) and \(x_0 \in V(G^*)\), we observe that

\[
G^*\{s(F(B_n^*(G,R_0)))\} = B_n(G^*, x_0)
\]

(2.12)

and that

\[
|V(B_n^*(G,R_0))| = |F(B_{n+1}(G^*,x_0))|, \quad |E(\partial_1 B_n^*(G,R_0))| = |E(\partial_1 B_{n+1}(G^*,x_0))|.
\]

(2.13)

It follows from (2.8) that, for \(n \geq 1\),

\[
\frac{|E(\partial_1 B_n^*(G^*,x_0))|}{|F(B_n(G^*,x_0))|} = \frac{\mathcal{F}(\mathcal{D}^2 - 4)}{2d} \left(1 + \frac{2}{\mu_1^* - 1}\right),
\]

(2.14)

which decreases monotonously as \(n\) increases. Then we have the following:

**Lemma 2.5.** For a \((d,f)\)-regular planar graph \(G\) satisfying \(H(d,f) < 0\) and Assumption 1.1, we have

\[
\alpha^*(G) \leq (f - 2) \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}.
\]

(2.15)

**Remark 2.6.** If \(H(d,f) = 4 - \mathcal{D}^2 = 0\), then behaviour of \(u_n^*\) and \(v_n^*\) are also given by (2.3). Now we have \(u_0^* = n\mathcal{D}(\mathcal{F} + 2)\) and \(v_0^* = (1 + 2n)(\mathcal{F} + 2)\) instead of (2.6). Then \(|E(\partial_1 B_n^*)) = u_n^* = (n + 1)\mathcal{D}(\mathcal{F} + 2)\) and \(|V(B_n^*)) = \sum_{x \in V_n} v_n^* = (n + 1)(\mathcal{F} + 2)\), thus the ratio similar to the one given in (2.8) tends to 0 and \(\alpha(G) = 0\), which gives \(\lambda_0(G) = 0\) by Theorem 1.2. Also we can easily obtain \(\alpha^*(G) = 0\) by Theorem 1.4 or the corresponding modified formula (2.14).

In order to show \(\alpha^*(G) = \mathcal{F}(1 - 4/\mathcal{D}^2)\) for a \((d,f)\)-regular planar graph, we only have to show the following proposition, whose proof is given in the next section:

**Proposition 2.7.** For a \((d,f)\)-regular planar graph \(G\) satisfying \(H(d,f) < 0\) and Assumption 1.1, we have

\[
\alpha^*(G) \geq (f - 2) \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}.
\]

(2.16)

In the rest of this section, let us observe that Proposition 2.7 brings the rest part of Main Theorem to us:

**Lemma 2.8.** For a \((d,f)\)-regular planar graph \(G\) satisfying \(H(d,f) < 0\) and Assumption 1.1, it holds that

\[
\alpha(G) \geq \frac{d - 2}{d} \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}.
\]

(2.17)

Therefore we find the equality holds in (2.17) by Lemma 2.3.
Proposition 3.1. Let \( G \) be a \((d,f)\)-regular planar graph satisfying \( H(d,f) < 0 \) and Assumption 1.1. For any finite subgraph \( K \) of \( G \), we have

\[
\frac{|E(\partial K)|}{\text{Area}(K)} \geq \inf_{K'} \left\{ \frac{1}{\mathcal{D} + 2} \frac{|E(\partial K')|}{|F(K')|} \right\} \geq \frac{1}{\mathcal{D} + 2} \alpha^*(G^*). 
\]

Since \( G^* \) is an \((f,d)\)-regular planar graph with \( H(f,d) = H(d,f) < 0 \) and the inequality (2.16) holds, we get the inequality (2.17).

3. Main Proof

Our aim in this section is to give the proof of Proposition 2.7 in the previous section, which is equivalent to the following:

**Proposition 3.2.** Let \( G \) be a \((d,f)\)-regular planar graph satisfying \( H(d,f) < 0 \) and Assumption 1.1. For any finite subgraph \( K \) of \( G \), we have

\[
\frac{\sqrt{\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}}{\mathcal{D}} \cdot |F(K)| \leq |E(\partial K)|, 
\]

where \( \mathcal{D} = d - 2 \geq 1 \) and \( \mathcal{F} = f - 2 \geq 1 \).

In order to show this, we may assume any finite subgraph \( K \) of \( G \) satisfies that

(a) for every \( e \in E(K) \), there exists a face \( R \in F(K) \) such that \( e \in E(R) \);

(b) \( K \) is connected and has no cut-vertex.

For (a), if there exists an edge \( e \) which belongs to \( E(K) \) and does not lie on any face \( R \in F(K) \), then it is sufficient to show the inequality (3.1) for \( K - \{e\} \). For (b), if \( K \) is disconnected, then it is sufficient to show the inequality (3.1) for each component of \( K \). Moreover, if \( K \) has a cut-vertex \( x \), \( K - \{x\} \) consists of some components \( K_1, K_2, \ldots, K_l \). Then it is sufficient to show (3.1) for each finite subgraph of \( G \) induced by \( V(K) \cup \{x\} \) for \( i = 1, 2, \ldots, l \).

Let us recall \( G \) satisfies Assumption 1.1. Namely, \( G \) can be realized as the canonical tessellation of the Poincaré disc \( \mathbb{H}^2 \) by the triangle group \( G(2,f,d) \) (cf. [13]). Thus, no accumulation point is on \( \mathbb{H}^2 \setminus \partial \mathbb{H}^2 \) by discreteness of this group. Then, we may assume also

(c) \( \partial K \), the subgraph of \( K \) induced by \( E(\partial K) \), is connected and a cycle.

If \( \partial K \) is not, we can choose a finite graph \( K' \) such that \( \partial K \subset \partial K' \) and \( K \) is a subgraph of \( K' \); \( |F(K')| > |F(K)| \) and \( |E(\partial K')| < |E(\partial K)| \). Then it is sufficient to show the inequality (3.1) for such \( K' \).

For any finite subgraph \( K \) of \( G \), we pick and fix a vertex \( x_0 \in V(K) \); we set \( B_n = B_n(G, x_0) \) in \( G \), which is defined in Definition 2.4. We use some notation around Definition 2.4 and set

\[
N = N(K, x_0) = \max \{n \mid V(K) \cap V_n \neq \emptyset\}. 
\]

If \( N = 1 \), then

\[
\frac{|E(\partial K)|}{|F(K)|} \geq \frac{(\mathcal{D} + 2)\mathcal{F}}{\mathcal{D} + 2} = \mathcal{F} > \frac{\sqrt{\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}}{\mathcal{D}} \cdot \frac{1}{\mathcal{D} + 2} = \frac{\sqrt{\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}}{\mathcal{D}}. 
\]

In the case where \( N \geq 2 \), we denote by \( W_n \) the subgraph of \( K \) induced by \( F(K) \cap F(B_n) \) for \( n = 1, 2, \ldots, N \).

**Lemma 3.2.** For every \( n (n = 2, 3, \ldots, N) \) such that \( W_n \neq B_n \), we have

\[
|E(\partial W_n)| - |E(\partial W_{n-1})| > \frac{\sqrt{\mathcal{F}(\mathcal{D}\mathcal{F} - 4)}}{\mathcal{D}} (|F(W_n)| - |F(W_{n-1})|). 
\]

**Proof.** Take such \( n \) that \( 2 \leq n \leq N \) and \( W_n \neq B_n \). We denote \( V(W_n) \cap V_n \) by \( A_n \) and set \( a_n = |A_n| \). Then \( K[A_n] \), which is a subgraph of \( K \) induced by \( A_n \), consists of some disjoint paths \( P_1, P_2, \ldots, P_l \) in general. Let us first assume \( l = 1 \), that is, \( K[A_n] \) is a path. At the end of this proof, we observe that it is essential to give a proof when \( l = 1 \). Let

\[
F_n = F(W_n) \setminus F(W_{n-1}) \quad \text{and} \quad f_n = |F_n|. 
\]

In addition, we set

\[
A^+_n = V(G[F_n]) \cap V_{n-1}, \\
A^-_n = \{x \in A^+_n \mid xy \in Z_n \} \text{ for any } y \in N_G(x) \cap V_n, \\
a_{n-1} = |A_{n-1}|. 
\]
where $Z_n$ is the set of edges $xy$ satisfying there exist two faces $R_1, R_2 \in F_n$ such that $xy \in E(R_1) \cap E(R_2)$. It is easy to see the following:

$$|E(\partial_y W_n)| - |E(\partial_y W_{n-1})| = (a_n + 1) - (a_{n-1} + 1) = a_n - a_{n-1},$$  \hspace{1cm} (3.7)

and

$$|F(W_n)| - |F(W_{n-1})| = f_n \leq \frac{a_n + a_{n-1}}{\mathcal{F}}. \hspace{1cm} (3.8)$$

If $a_{n-1} = 0$, then $f_n \leq a_n/\mathcal{F}$ and $|E(\partial_y W_n)| - |E(\partial_y W_{n-1})| \geq a_n$. Thus it holds that

$$|E(\partial_y W_n)| - |E(\partial_y W_{n-1})| > \sqrt{\frac{\mathcal{F}(\mathcal{D} \mathcal{F} - 4)}{\mathcal{D}}}(|F(W_n)| - |F(W_{n-1})|),$$

since $0 < \sqrt{(\mathcal{D} \mathcal{F} - 4)/(\mathcal{D} \mathcal{F})} < 1$.

If $a_{n-1} \neq 0$, then we define an auxiliary sequence $\{a_{n-1}\}$ by the following procedure. Let us set

$$F_{n-1} = \{R \in F(B_{n-1}) \mid V(R) \cap A_{n-1} = V(R) \cap V_{n-1}\} \quad \text{and} \quad f_{n-1} = |F_{n-1}|. \hspace{1cm} (3.9)$$

For $G[F_{n-1}]$, we set also

$$A_{n-2}^+ = V(G[F_{n-1}]) \cap V_{n-2}, \hspace{1cm} A_{n-2} = \{x \in A_{n-2}^+ \mid xy \in Z_{n-1} \} \quad \text{for any} \ y \in N_G(x) \cap V_{n-1},$$

\hspace{1cm} (3.10)

where $Z_{n-1}$ is the set of edges $xy$ satisfying there exist two faces $R_1, R_2 \in F_{n-1}$ such that $xy \in E(R_1) \cap E(R_2)$. We inductively define $F_k, f_k, A_{k-1}^+, A_{k-1}, X_k, A_{k-1}^+$ and $a_{k-1}$ for $1 \leq k \leq n - 2$. We remark $a_0 = 0$. Otherwise, namely if $a_0 = 1$, $W_n$ must coincide with $B_n$; this contradicts the assumption $W_n \neq B_n$.

We now set

$$m = m(n, K) = \max\{0 \leq k \leq n - 2 \mid a_k = 0\}. \hspace{1cm} (3.11)$$

Let us see the following claim:

**Claim 3.3.** If $0 \leq m \leq n - 2$, then

$$a_k \geq (\mathcal{D} \mathcal{F} - 2)a_{k-1} - a_{k-2}, \hspace{1cm} (3.12)$$

for $m + 2 \leq k \leq n$. In particular, $a_{m+2} \geq (\mathcal{D} \mathcal{F} - 2)a_{m+1}$. Furthermore, we have

$$\frac{a_n}{a_{n-1}} > \mu_1, \hspace{1cm} (3.13)$$

where $\mu_1$ is the same as in (2.4).

**Proof of Claim 3.3.** By the definition of $\{a_k\}$, we have

$$\mathcal{F} \cdot f_k + 2 = |E(\partial_y G[F_k])| \leq a_k + a_{k-1} + 2, \hspace{1cm} (3.14)$$

and

$$\mathcal{D} \cdot a_{k-1} = |\{xy \in E(G) \mid x \in A_{k-1}, y \in A_k \cup A_{k-2}^+\}| \leq (f_k - 1) + (f_{k-1} + 1) = f_k + f_{k-1}, \hspace{1cm} (3.15)$$

for $m + 1 \leq k \leq n$. Combining (3.14) and (3.15), we have

$$\mathcal{D} \mathcal{F} \cdot a_{k-1} \leq \mathcal{F} \cdot (f_k + f_{k-1}) \leq (a_k + a_{k-1}) + (a_{k-1} + a_{k-2}), \hspace{1cm} (3.16)$$

for $m + 2 \leq k \leq n$; thus we obtain the inequalities (3.12). Now let us consider a sequence $\{b_k\}$ as follows: $b_{m+2} = \mathcal{D} \mathcal{F} - 2$ and

$$b_k = (\mathcal{D} \mathcal{F} - 2) - \frac{1}{a_{k-1}}. \hspace{1cm} (3.17)$$

for $k \geq m + 3$. Since $b_k \geq \mathcal{D} \mathcal{F} - 3 > 1$ for every $k$ and $\{b_k\}$ is strictly decreasing, it converges to $\mu_1$. Comparing (3.12) with (3.17), we obtain $a_n/a_{n-1} \geq b_n > \mu_1$.

We note that, if $x > \mu_1$,

$$x - 1 > \sqrt{(\mathcal{D} \mathcal{F} - 4)/(\mathcal{D} \mathcal{F})} \cdot (x + 1). \hspace{1cm} (3.18)$$

It follows from (3.12), (3.13) and (3.18) that
\[ \frac{a_n}{a_{n-1}} - 1 > \sqrt{\frac{D F - 4}{D F} \left( \frac{a_n}{a_{n-1}} + 1 \right)}. \]

Thus it holds that
\[ a_n - a_{n-1} > \sqrt{\frac{F (D F - 4)}{D} a_n + a_{n-1}}, \]

which implies the inequality (3.4) by (3.7) and (3.8).

At the beginning of this proof, we assume \( K[A_n] \) is a path. In general, \( K[A_n] \) consists of some disjoint paths \( P_1, P_2, \ldots, P_l \) as stated before. Let
\[ F_n^j = \{ R \in F(B_n) \mid V(R) \cap V(P_j) = V(R) \cap V_n \} \quad \text{and} \quad f_n^j = |F_n^j| \]
for \( j = 1, 2, \ldots, l \). As in (3.6), we moreover set
\[ A_n^{j,+} = V(G[F_n^j]) \cap V_{n-1}, \]
\[ A_n^{j,-} = \{ x \in A_n^{j,+} \mid xy \in Z_n^j \} \text{ for any } y \in N_G(x) \cap V_n, \]
\[ a_n^{j,-} = |A_n^{j,-}|, \]
(3.22)

where \( Z_n^j \) is the set of edges \( xy \) satisfying there exist two faces \( R_1, R_2 \in F_n^j \) such that \( xy \in E(R_1) \cap E(R_2) \). It is obvious that the estimate (3.20) holds for each \( j \) and that \( A_n^{j,-} \cap A_n^{k,-} = \emptyset \) if \( j \neq k \). Hence we have
\[ |E(\partial_j W_n)| - |E(\partial_j W_{n-1})| = \sum_{j=1}^l a_n^j - a_{n-1}^j \quad \text{and} \quad |F(W_n)| - |F(W_{n-1})| = \sum_{j=1}^l f_n^j, \]
then this completes the proof of Lemma 3.2.

We are now in a position to give the proof of Proposition 3.1.

**Proof of Proposition 3.1.** For any given finite subgraph \( K \) of a \((d,f)\)-regular planar graph \( G \), \( N = N(K, x_0) \) is a value defined in (3.2). Let us first assume that \( W_n \neq B_n \) for \( n = 1, 2, \ldots, N \). It follows from (3.3) and Lemma 3.2 that
\[ |E(\partial_j K)| = \sum_{n=2}^N \left( |E(\partial_j W_n)| - |E(\partial_j W_{n-1})| \right) + |E(\partial_j W_1)| \]
\[ > \sqrt{\frac{F (D F - 4)}{D} \left( \sum_{n=2}^N \left( |F(W_n)| - |F(W_{n-1})| \right) + |F(W_1)| \right) \}
\[ = \sqrt{\frac{F (D F - 4)}{D} |F(W_N)|} = \sqrt{\frac{F (D F - 4)}{D} |F(K)|}. \]

Let us next assume that there exists \( n \) such that \( 1 \leq n \leq N \) and \( W_n = B_n \); we set \( n_0 = \max \{ n \mid W_n = B_n \} \).

We remark that \( W_n = B_n \) for every \( n \leq n_0 \). If \( n_0 = N \), then it follows from (2.14) that
\[ |E(\partial_j K)| = |E(\partial_j B_N(x_0))| = \sqrt{\frac{F (D F - 4)}{D} \left( 1 + \frac{2}{\mu_1^{n_0} - 1} \right) |F(B_N(x_0))|} \]
\[ > \sqrt{\frac{F (D F - 4)}{D} |F(K)|}. \]

If \( n_0 \leq N - 1 \), then it follows from (2.14) and Lemma 3.2 that
\[ |E(\partial_j K)| = \sum_{n=n_0+1}^N \left( |E(\partial_j W_n)| - |E(\partial_j W_{n-1})| \right) + |E(\partial_j W_{n_0})| \]
\[ > \sqrt{\frac{F (D F - 4)}{D} \sum_{n=n_0+1}^N \left( |F(W_n)| - |F(W_{n-1})| \right) + \sqrt{\frac{F (D F - 4)}{D} \left( 1 + \frac{2}{\mu_1^{n_0} - 1} \right) |F(W_{n_0})|} \]
\[ > \sqrt{\frac{F (D F - 4)}{D} |F(W_N)|} = \sqrt{\frac{F (D F - 4)}{D} |F(K)|}. \]

This completes the proof of Proposition 3.1; thus we obtain Main Theorem.

**Remark 3.4.** Let us give some remarks on \( \lambda_0(G) \) for a \((d,f)\)-regular planar graph \( G \) with \( H(d,f) < 0 \). Applying (1.4) and Theorem 1.2, we have
\[ 1 - \frac{2}{d} \sqrt{(d - 2)\left(1 + \frac{1}{f - 2}\right) + 1} \leq \lambda_0(G); \]  
(3.27)

If \( f \geq 4 \), we know also another estimate [17]:

\[ \frac{d - 2}{d} \left(1 - \frac{2\sqrt{d - 3}}{d - 2}\right) \leq \lambda_0(G). \]  
(3.28)

For the \( d \)-regular tree \( T_d \), it is easy to calculate \( \alpha(T_d) = 1 - 2/d \); this quantity is equal to the right hand side of (1.4) when we formally set \( f = \infty \). Referring to a general fact \( \text{Spec}(-\Delta_{T_d}) = \{1 - 2\sqrt{d - 1}/d, 1 + 2\sqrt{d - 1}/d\} \), we immediately see the equality in (3.27) holds for \( T_d \). For \((d, f)\)-regular planar graphs, we find the equality in (3.27) does not hold in general by observing the estimate (3.28). On the other hand, applying results in, [6, 12] we have

\[ \lambda_0(G) \leq \min \left\{1 - \frac{2\sqrt{d - 1}}{d}, \frac{d - 2}{d} \left(1 - \frac{2}{\sqrt{(d - 2)(f - 2)}}\right)\right\}. \]  
(3.29)

In particular, when \( f = 3 \), that is, when \( G \) is a “hyperbolic” triangulation, the estimates (3.27) and (3.29) gives us

\[ 1 - \frac{2\sqrt{2d - 3}}{d} \leq \lambda_0(G) \leq 1 - \frac{2 + 2\sqrt{d - 2}}{d}. \]  
(3.30)

We should be interested to know which equality in (3.30) holds, or neither of them holds.

REFERENCES