Digital Sum Problems for the $p$-adic Expansion of Natural Numbers

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Let $p$ be a positive integer greater than 1 and denote the $p$-adic expansion of $n \in \mathbb{N}$ by $n = \sum_{j=0}^{\infty} a_j(n)p^j$, where $a_j(n) \in \{0, 1, \ldots, p-1\}$. We set $s(n, l) = \sum_{j=0}^{l} a_j(n)p^j$ for $l = 1, 2, \ldots, p-1$, and $s(n) = \sum_{j=0}^{\infty} a_j(n)$. We define the power sum of sum of digits by $S_k(N) = \sum_{n=0}^{N-1} s(n)^k$, $k \in \mathbb{N}$, and the exponential sum of sum of digits by $F(\xi, N) = \sum_{n=0}^{N-1} e^{2\pi i \xi n}, \xi \in \mathbb{R}$ for $N \in \mathbb{N}$. Explicit formulas of these sums have been investigated by many authors for a long time.

In this paper, developing our previous results, we shall give explicit formulas of $F(\xi, N)$ and $S_k(N)$.

KEYWORDS: digital sum problem, singular function, multinomial measure, Bernoulli polynomial

1 Introduction

We first give a short historical sketch for explicit formulas of these sums. Trollope [16] obtained an explicit formula of $S_2(N)$ for $p = 2$ and Delange [4] gave its elegant proof. Coquet [3] studied an explicit formula of $S_k(N)$ for $k \geq 2, p = 2$ and obtained an explicit one. However his formula contains functions defined by recursive equations, which cannot be solved in general. Osbaldestin [11] constructed these functions for $k = 2$. In Okada, Sekiguchi and Shiota [8], we have obtained an explicit formula of $S_k(N)$ for $p = 2$ by use of the binomial measure and the result obtained in Sekiguchi and Shiota [13]. Our formula also contains functions defined by recursive equations. We emphasize that these equations can be solved completely. For an explicit formula of $F(\xi, N)$, Stein [14] gave a one. In Okada, Sekiguchi and Shiota [9], we have given another type of explicit formula of $F(\xi, N)$ and noticed that we can directly get explicit formulas of $S_k(N)$ by use of the equality

$$\frac{\partial^k}{\partial \xi^k} F(\xi, N) |_{\xi = 0} = \sum_{n=0}^{N-1} s(n)^k e^{2\pi i \xi n} |_{\xi = 0} = S_k(N). \quad (1)$$

In this paper, developing this direct method and using the result on multinomial measures obtained in Okada, Sekiguchi and Shiota [10], we give an explicit formula of $F(\xi, N)$ and explicit formulas of $S_k(N)$. We finally remark that the study of explicit formulas of $S_k(N)$ and $F(\xi, N)$ is essentially reduced to that of multinomial measures.

2 Preliminaries

Let $I = I_{0,0} = [0, 1]$ and $I_{n,j} = [j/p^n, (j + 1)/p^n), j = 0, 1, \ldots, p^n - 2, I_{n,p^n-1} = [(p^n - 1)/p^n, 1]$ for $n = 1, 2, 3, \ldots$. Let $r = (r_0, r_1, \ldots, r_{p^n-2})$ be a vector such that $0 < r_l < 1$ for $l = 0, 1, \ldots, p - 1$ and $\sum_{l=0}^{p-1} r_l < 1$ and set $r_{p-1} = 1 - \sum_{l=0}^{p-2} r_l$. The probability measure $\mu_\xi$, on $I$ defined by

$$\mu_\xi(I_{n+1,0}) = r_l \mu_\xi(I_{n,j}) \quad (2)$$

for $n = 0, 1, 2, \ldots, j = 0, 1, \ldots, p^n - 1, l = 0, 1, \ldots, p - 1$, is said to be a multinomial measure. If $r_l = 1/p$ for all $l$, $\mu_\xi$ is the Lebesgue measure. We denote the distribution function of $\mu_\xi$ by $L(r, \cdot)$:

$$L(r, x) = \mu_\xi([0, x]), x \in I \quad (3)$$

In [10], we summarized some fundamental properties of $L(r, \cdot)$ and got an exact form of the $k$-th derivative of $L(r, \cdot)$ with respect to the parameters $r_l$.

3 An explicit formula of $F(\xi, N)$

For $N \in \mathbb{N}$, set $t = \log_p N$, and denote its integer part by $[t]$ and its decimal part by $\{t\}$. Evidently $p^{[t]} \leq N < p^{[t]+1}$ and $1/p \leq N/p^{[t]+1} = 1/p^{t-[t]} < 1$.

Theorem 3.1 We have

$$F(\xi, N) = \frac{1}{p^{[t]+1}} L \left( \frac{r}{p^{t-[t]}}, \frac{1}{p^{t-[t]}} \right). \quad (4)$$
where
\[
 r_i = \frac{e^\xi}{1 + e^{\xi} + \cdots + e^{(p-1)\xi}}
\]
for \(l = 0, 1, \ldots, p - 1\).

**Proof.** By definitions of \(L, \mu, \) and \(s(n, l)\), we have
\[
 L\left(r, \frac{1}{p^{1-\xi(n)}}\right) = L\left(r, \frac{N}{p^{1+\xi(n)}}\right) = \sum_{n=0}^{N-1} \left( L\left(r, \frac{n+1}{p^{1+\xi(n)}}\right) - L\left(r, \frac{n}{p^{1+\xi(n)}}\right) \right)
\]
\[
 = \sum_{n=0}^{N-1} \mu_n (I_{[0, 1]} + L) = \sum_{n=0}^{N-1} \frac{r_0^{[0, 1]} + 1 - \sum_{e^{(p-1)\xi(n)}} r_0^{(n, l)} \cdot r_0^{(n, p-1)}}{r_0^{[0, 1]} + 1 - \sum_{e^{(p-1)\xi(n)}} r_0^{(n, l)} \cdot r_0^{(n, p-1)}}
\]
\[
 = \sum_{n=0}^{N-1} \frac{r_0^{[0, 1]} + 1 - \sum_{e^{(p-1)\xi(n)}} r_0^{(n, l)} \cdot r_0^{(n, p-1)}}{r_0^{[0, 1]} + 1 - \sum_{e^{(p-1)\xi(n)}} r_0^{(n, l)} \cdot r_0^{(n, p-1)}}
\]
\[
 = r_0^{[0, 1]} + 1 \sum_{n=0}^{N-1} e^{\xi(n)} = r_0^{[0, 1]} + 1 F(\xi, N).
\]

4 An explicit formula of \(S_k(N)\)

To obtain an explicit formula of \(S_k(N)\), we have only to calculate the left-hand side of (1). We now set
\[
a(m, x, \xi) = \frac{\partial^m}{\partial x^m} \left[ \frac{1 + e^x + \cdots + e^{(p-1)\xi}}{1 + e^x + \cdots + e^{(p-1)\xi}} \right]_{x = \xi}
\]
for \(x \in \mathbb{R}\) and denote \((\partial^j / \partial x^j)a(m, x, \xi_0)\) by \(a^{(j)}(m, x, \xi_0)\).

**Theorem 4.1.** We have
\[
\frac{\partial^k}{\partial \xi^k} F(\xi, N) \bigg|_{\xi = \xi_0} = (1 + e^\xi + \cdots + e^{(p-1)\xi_0}) \sum_{j=0}^{k-1} \frac{1}{j!} a^{(j)}(m - j, 1 - (x), \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]

for \(k \geq 0\), where \(H_k(x, \xi)\) is a continuous periodic function of period \(1\) with respect to \(x\) defined by
\[
H_k(x, \xi) = (1 + e^\xi + \cdots + e^{(p-1)\xi_0}) \sum_{j=0}^{k-1} \frac{1}{j!} a^{(j)}(m - j, 1 - (x), \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]

**Proof.** The assertion for \(k = 0\) is that of Theorem 3.1. We show for \(k > 0\). We have, by Leibniz's formula,
\[
\frac{\partial^k}{\partial \xi^k} F(\xi, N) \bigg|_{\xi = \xi_0} = (1 + e^\xi + \cdots + e^{(p-1)\xi_0}) \sum_{j=0}^{k} \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]
\[
\times a(k - j, 1 - (x), \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]
\[
= (1 + e^\xi + \cdots + e^{(p-1)\xi_0}) \sum_{j=0}^{k} \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]
\[
\times a(k - j, t + 1 - (t), \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]

Since \(a(m, x, \xi_0)\) is a polynomial in \(x\) of degree \(m\), we obtain by Taylor's expansion
\[
\frac{\partial^j}{\partial x^j} F(\xi, N) \bigg|_{\xi = \xi_0} = (1 + e^\xi + \cdots + e^{(p-1)\xi_0}) \sum_{j=0}^{k} \frac{\partial^j}{\partial x^j} L\left(r, \frac{1}{p^{1-(x)}}\right)_{\xi = \xi_0}
\]
\[ x = \sum_{i=0}^{\infty} \frac{u^i}{i!} a(i)(k - j, 1 - (t), \xi_0) t^i \left( \frac{1}{p^{1-t}} \right) \left( \frac{1}{\xi_0} \right) \]

\[ = (1 + e^{\xi_0} + \cdots + e^{(p^{-1})}) \sum_{i=0}^{\infty} \frac{u^i}{i!} a(i)(k - j, 1 - (t), \xi_0) \]

\[ = \frac{1}{l!} \frac{\partial^j}{\partial x^j} \left( \frac{1}{e^{x} + \cdots + e^{(p^{-1})}} \right) \]

The periodicity of \( H_{k,i}(x, \xi_0) \) is evident. It suffices to show the continuity of \( H_{k,i}(x, \xi_0) \) at \( x = 0 \). As \( L(1, 1/p) = 1/(1 + e^x + \cdots + e^{(p^{-1})}) \),

\[ H_{k,i}(1, \xi_0) = (1 + e^{\xi_0} + \cdots + e^{(p^{-1})}) \sum_{i=0}^{\infty} \frac{1}{i!} a(i)(k - j, 1 - (t), \xi_0) \frac{\partial^j}{\partial x^j} L(r, 1/p) \]

\[ = \frac{1}{l!} \frac{\partial^j}{\partial x^j} \left( \frac{1}{e^{x} + \cdots + e^{(p^{-1})}} \right) \]

for \( k \geq l \geq 0 \). Since \( d^{P}(k - j, 1, \xi_0) = 0 \) for \( k - l < j \) and \( L(r, 1/p) = 1/(1 + e^X + \cdots + e^{(p^{-1})}) \),

\[ H_{k,i}(0, \xi_0) = (1 + e^{\xi_0} + \cdots + e^{(p^{-1})}) \sum_{i=0}^{\infty} \frac{1}{i!} a(i)(k - j, 1, \xi_0) \frac{\partial^j}{\partial x^j} L(r, 1/p) \]

\[ = \frac{1}{l!} \frac{\partial^j}{\partial x^j} \left( \frac{1}{e^{x} + \cdots + e^{(p^{-1})}} \right) \]

for \( k \geq l \geq 0 \). Hence \( H_{k,i}(x, \xi_0) \) is continuous at \( x \in \mathbb{Z} \).

To calculate the function \( a(m, x, \xi_0) \), we use the following well-known lemma in calculus.

**Lemma 4.1** Let \( f, g \) be infinitely differentiable functions. Then we have

\[ (f \cdot g)^{(m)} = \sum_{r=1}^{m} f^{(r)} \cdot g \sum_{r=1}^{m} \frac{m!}{v_1 \cdots v_m} \frac{\prod_{i=1}^{m} \left( g^{(r)} \right)}{i!} \]

where the summation \( \Sigma \) is taken over \( v_1, v_2, \ldots, v_m \geq 0, v_1 + v_2 + \cdots + v_m = v \) and \( v_1 + 2v_2 + \cdots + mv_m = m \).

**Proposition 4.1** We have

\[ a(m, x, \xi_0) = \sum_{v=1}^{m} \frac{1}{v_1 \cdots v_m} \frac{\prod_{i=1}^{m} \phi_i(x)}{i!} \]

for \( m > 0 \). Here \( \phi_i \) is Bernoulli's polynomial of degree \( i \), that is,

\[ t(e^x - 1) = \sum_{i=0}^{\infty} \frac{\phi_i(x)}{i!} t^i \]

and the summation \( \Sigma \) is taken over \( v_1, v_2, \ldots, v_m \geq 0, v_1 + v_2 + \cdots + v_m = v \) and \( v_1 + 2v_2 + \cdots + mv_m = m \).

Taking \( \xi_0 = 0 \), we have the following corollaries.

**Corollary 4.1** We have

\[ S_k(N) = N \sum_{i=0}^{k} H_{k,i}(t, 0) \left( \frac{1}{p} \right)^i \]

**Corollary 4.2** We have

\[ a(m, x, 0) = \sum_{v=1}^{m} \frac{1}{v_1 \cdots v_m} x(x-1) \cdots (x-v) \prod_{i=0}^{m} \left( \frac{\phi_i(x)}{i!} \right)^{v_i} \]

for \( m > 0 \).
5 Remarks

5.1 Asymptotic behaviors

We can investigate various types of asymptotic behavior of \( \frac{\partial^k}{\partial \xi^k} F(\xi, N) \) by use of Theorem 4.1. For example, since

\[
\frac{\partial^k}{\partial \xi^k} F(\xi, N) \sim \frac{H_{k,k}(\log_N N, \xi)}{(1 + e^\xi + \cdots + e^{(\rho - 1)\xi})^k}
\]

as \( N \to \infty \), it follows that

\[
\limsup_{N \to \infty} \frac{\partial^k}{\partial \xi^k} F(\xi, N) = \left( \sum_{l=0}^\infty (l+1) \frac{\phi_l + \gamma(p)}{(l+2)!} \right) \frac{x^{\log_{\rho} L(r, x)}}{1 + e^\xi + \cdots + e^{(\rho - 1)\xi}}, \quad \text{max}_{1/p \leq x \leq 1} x^{\log_{\rho} L(r, x)}, \tag{14}
\]

\[
\liminf_{N \to \infty} \frac{\partial^k}{\partial \xi^k} F(\xi, N) = \left( \sum_{l=0}^\infty (l+1) \frac{\phi_l + \gamma(p)}{(l+2)!} \right) \frac{x^{\log_{\rho} L(r, x)}}{1 + e^\xi + \cdots + e^{(\rho - 1)\xi}}, \quad \text{min}_{1/p \leq x \leq 1} x^{\log_{\rho} L(r, x)}. \tag{15}
\]

To obtain the precise values of the left-hand sides, it suffices to estimate the function \( x^{\log_{\rho} L(r, x)} \). However it is hard, because \( L(r, x) \) is a singular function for \( \xi \neq 0 \).

5.2 Inductive formula of \( H_{k,l} \)

We notice that \( H_{k,l} \) satisfies the next inductive formula.

\[
H_{0,0}(x, \xi) = (1 + e^\xi + \cdots + e^{(\rho - 1)\xi})^{-1} L(r, \frac{1}{p^{1-\omega}}), \tag{16}
\]

\[
H_{k,0}(x, \xi) = (1 + e^\xi + \cdots + e^{(\rho - 1)\xi})^{-1} \frac{\partial^k}{\partial \xi^k} L(r, \frac{1}{p^{1-\omega}}) - \sum_{j=0}^{k-1} \binom{k}{j} a(k-j, \{x\}-1, \xi) H_{j,0}(x, \xi), \tag{17}
\]

\[
H_{k,l}(x, \xi) = -\sum_{j=0}^{k-1} \sum_{q=0}^{k-j} (-1)^j \binom{k}{j} \left( (1 + e^\xi + \cdots + e^{(\rho - 1)\xi})^q \right) \times a^{(q)}(k-j, \{x\}-1, \xi) H_{j-\ell, \ell}(x, \xi), \tag{18}
\]

for \( k \geq 1 \) and \( 1 \leq l \leq k \). This kind of formula has been treated in \([3]\) and \([8]\).

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