Zeta Functions for Images of Graph Coverings by Some Operations

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We consider the zeta functions of the line graph and the middle graph of a regular covering of a graph G, and their related topics. Let M(G) be the middle graph of G and T(G) the total graph of G. We show that the middle graph and the total graph of a regular covering of a graph G with covering transformation group A is a regular covering of M(G) and T(G) with the same covering transformation group A, respectively. For a regular graph G, we express the zeta functions of the line graph and the middle graph of a regular covering of G by using the characteristic polynomial of that regular covering.

KEYWORDS: zeta function, graph covering, complexity, line graph

1 Introduction

In this paper, we discuss the line graph and the middle graph of a regular covering of a graph G, and consider the zeta functions and the complexities of them. In this section, we mention what is known about zeta functions of finite graphs and their related topics.


Graphs and digraphs treated here are finite. Let G be a connected graph with a set V(G) of vertices and a set E(G) of edges, and let D be the symmetric digraph corresponding to G. A path P of length n in D (or G) is a sequence P = (v0, v1, ..., vn) of n + 1 vertices such that consecutive vertices share an arc (or edge) (we do not require that all vertices are distinct). Also, P is called a (v0, vn)-path. We say that a path has a backtracking if a subsequence of the form · · ·, x, y, x, · · · appears. A (v, w)-path is called a cycle (or closed path) if v = w. The inverse cycle of a cycle C = (v, v1, · · · , vn−1, v) is the cycle C−1 = (v, vn−1, · · · , v1, v).

We introduce an equivalence relation between backtracking-less cycles. Such two cycles C1 = (v1, · · · , vm) and C2 = (w1, · · · , wn) are called equivalent if wi = vj+k for all j. The inverse cycle of C is not equivalent to C. Let [C] be the equivalence class which contains a cycle C. Let B be the cycle obtained by going r times around a cycle B. Such a cycle are called a multiple of B. A backtracking-less cycle C is primitive if both C and C2 have no backtracking, and it is not a multiple of a strictly smaller cycle.

The (Ihara) zeta function of a graph G is defined to be a function of u ∈ C with u sufficiently small, by

\[ Z(G, u) = Z_G(u) = \prod_{C \in \mathcal{C}} (1 - u^{\mid C \mid})^{-1}, \]

where \( \mathcal{C} \) runs over all equivalence classes of primitive, backtracking-less cycles of G, and \( \mid C \mid \) is the length of C.

Let G be a connected graph with n vertices \( v_1, \cdots, v_n \). The adjacency matrix \( A(G) = (a_{ij}) \) is the square matrix such that \( a_{ij} = 1 \) if \( v_i \) and \( v_j \) are adjacent, and \( a_{ij} = 0 \) otherwise. Let \( D = (d_{ii}) \) be the diagonal matrix with \( d_{ii} = deg(v_i) \), and \( Q = D - I \).


**Theorem 1.1 (Bass)** The reciprocal of the zeta function of G is given by

\[ Z_G(u)^{-1} = (1 - u^2)^{-n} \det (I - uA(G) + u^2Q), \]

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where \( n = |V(G)| \) and \( l = |E(G)| \).

Stark and Terras [18] gave an elementary proof of Theorem 1.1, and discussed three different zeta functions of any graph. Recently, another proof of Theorem 1.1 is obtained by Kotani and Sunada [13].

The complexity \( \kappa(G) \) (= the number of spanning trees in \( G \)) of a connected graph \( G \) is closely related to the zeta function of \( G \). Hashimoto expressed the complexity of a regular graph as a limit involving its zeta function in [8]. For an irregular graph \( G \), Hashimoto [9] and Northshield [17] gave the value of \((1 - u)^{-\chi(G)} \left| u = 1 \right.\) in term of the complexity of \( G \), where \( \chi(G) \) is the Betti number of \( G \).

**Theorem 1.2 (Hashimoto; Northshield)** For any finite graph \( G \) such that \( r > 1 \), we have

\[
(1 - u)^{-\chi(G)} \left| u = 1 \right. = 2^r \chi(G) \kappa(G),
\]

where \( \chi(G) = 1 - r \) is the Euler number of \( G \).

Kotani and Sunada [13] presented an elementary proof of Theorem 1.2.

Northshield [17] showed that the complexity of \( G \) is given by the derivative of a determinant contained in the reciprocal of its zeta function. For a connected graph \( G \), let \( f_c(u) = \det(1 - u A(G) + u^2 Q) \).

**Theorem 1.3 (Northshield)** The complexity of \( G \) is given as follows:

\[
f_c(1) = 2(l - n) \kappa(G),
\]

where \( n = |V(G)| \) and \( l = |E(G)| \).

Biggs [2], and Kotani and Sunada [12] obtained an excellent result that the reciprocal of the complexity of a graph \( G \) is equal to the square of the volume of the Jacobian torus \( Jac(G) \) of \( G \).

**Theorem 1.4 (Biggs; Kotani and Sunada)** For any graph \( G \), \( \kappa(G)^{-1} = \text{vol}(Jac(G))^2 \).

The complexities for various graphs were given in [4]. Let \( \Phi(G; \lambda) = \det(\lambda I - A(G)) \) be the characteristic polynomial of \( G \).

**Theorem 1.5** The complexity \( \kappa(G) \) of an \( r \)-regular graph \( G \) with \( n \) vertices is

\[
\kappa(G) = \frac{1}{n} \Phi'(G; r),
\]

where \( \Phi'(G; \lambda) \) is the derivative of \( \Phi(G; \lambda) \).

Mizuno and Sato [15,16] expressed the zeta function and the complexity of a connected regular covering of \( G \) by using that of \( G \), respectively. The Kronecker product \( A \otimes B \) of matrices \( A \) and \( B \) is considered as the matrix \( A \) having the element \( a_{ij} \) replaced by the matrix \( a_{ij} B \). Set \( Q_m = I_m \otimes Q \), where \( I_m \) is the identity matrix of order \( m \).

**Theorem 1.6 (Mizuno and Sato)** Let \( G \) be a connected graph with \( n \) vertices and \( l \) edges, \( A \) a finite group and \( \alpha: D(G) \rightarrow A \) an ordinary voltage assignment. Furthermore, let \( \rho_1 = 1, \rho_2, \cdots, \rho_l \) be the irreducible representations of \( A \), and \( f_i \) the degree of \( \rho_i \), for each \( i \), where \( f_1 = 1 \). For \( g \in A \), the matrix \( A_g = (\alpha_{ij}^{(g)}) \) is defined as follows:

\[
\alpha_{ij}^{(g)} = \begin{cases} 1 & \text{if } (\alpha, v) = g \text{ and } (u, v) \in D(G), \\ 0 & \text{otherwise}. \end{cases}
\]

Suppose that the \( A \)-covering \( G^a \) of \( G \) is connected. Then the reciprocal of the zeta function of \( G^a \) is

\[
Z(G^a, u)^{-1} = Z(G, u)^{-1} \cdot \prod_{i=2}^{l} \left( (1 - u^2)^{l - n_f} \det \left( I_{f_i} - u \sum_{g \in A} \rho_i(g) \otimes A_g + u^2 Q_{f_i} \right) \right)^{f_i}. \tag{4}
\]

Suppose that \( l > n \). Then the complexity of \( G^a \) is

\[
\kappa(G^a) = \frac{1}{|A|} \kappa(G) \cdot \prod_{i=2}^{l} \det \left( I_{f_i} - \sum_{g \in A} \rho_i(g) \otimes A_g + Q_{f_i} \right)^{f_i}. \tag{5}
\]

Mizuno and Sato [14] expressed the characteristic polynomial of a regular covering of \( G \) by using that of \( G \). Let \( \Phi(F; \lambda) = \det(\lambda I - F) \) for any square matrix \( F \).

**Theorem 1.7 (Mizuno and Sato)** Let \( G, A, \alpha, \rho_i \) and \( f_i \) be as in Theorem 1.6. Then the characteristic polynomial of \( G^a \) is

\[
\Phi(G^a; \lambda) = \Phi(G; \lambda) \cdot \prod_{i=2}^{l} \Phi \left( \sum_{g \in A} \rho_i(g) \otimes A_g; \lambda \right)^{f_i}. \tag{6}
\]

Typical operations of graphs are the line graph \( L(G) \), the middle graph \( M(G) \) and the total graph \( T(G) \) of a graph \( G \) (see [4]).

**Theorem 1.8** The adjacency matrices \( A_L = A(L(G)) \) and \( A_M = A(M(G)) \) are given as follows:

\[
A_L = B'B - 2I_n, \quad A_M = \begin{bmatrix} A_L & B \\ B & 0 \end{bmatrix}, \tag{7}
\]
where \( l = |E(G)| \) and \( B = (b_{ij}) \) is the incidence matrix of \( G \): \( b_{ij} = 1 \) if the edge \( e_i \) and the vertex \( v_j \) are incident, and \( b_{ij} = 0 \) otherwise.

In [12], Kotani and Sunada showed that the line graph of a regular covering of a graph \( G \) with covering transformation group \( A \) is a regular covering of \( L(G) \) with the same covering transformation group \( A \), and gave a formula between two subtori of the Jacobian torus of \( G \) associated to \( L(G) \) and \( G \).

**Theorem 1.9 (Kotani and Sunada)** Let \( H \) be a regular covering of a graph \( G \) with covering transformation group \( A \), then \( L(H) \) is a regular covering of \( L(G) \) with the same covering transformation group \( A \).

The characteristic polynomial and the complexity of the line graph \( L(G) \) of an \( r \)-regular graph \( G \) are given as follows (see [4]):

**Theorem 1.10** Let \( G \) be a connected \( r \)-regular graph with \( n \) vertices and \( l \) edges. Then

\[
\phi(L(G); \lambda) = (\lambda + 2)^{l-n} \phi(G; \lambda + 2 - r) \quad \text{and} \quad \kappa(L(G)) = 2^{l-n+1} r^{l-n-1} \kappa(G).
\]

In Section 2, we consider finite regular covering of graphs through ordinary voltage assignments, and present the proofs of Theorems 1.6 and 1.7. In Section 3, we treat line graphs of regular coverings of graphs. Let \( G^* \) be the regular covering of a connected graph \( G \) derived from a voltage assignment \( \alpha \) in a group \( A \). We determine a voltage assignment \( \beta \) in \( A \) such that \( L(G^*) = L(G)^\beta \). The zeta function of the line graph \( L(G^*) \) for a regular graph \( G \) is expressed by using the characteristic polynomial of \( G^* \). In Section 4, we show that the middle graph \( M(G^*) \) of \( G^* \) is a regular covering of the middle graph \( M(G) \) of \( G \). For a regular graph \( G \), we write the zeta function and the complexity of \( M(G^*) \) with the data of \( G^* \), respectively. Furthermore, we discuss the total graph of a regular covering. In Section 5, we present some examples.

For a general theory of the representation of groups and graph coverings, the reader is referred to [3] and [6], respectively.

### 2 Zeta Functions of Regular Coverings

Let \( G \) be a connected graph, and let \( N(v) = \{ w \in V(G) | vw \in E(G) \} \) for any vertex \( v \) in \( G \). A graph \( H \) is called a covering of \( G \) with projection \( \pi: H \rightarrow G \) if there is a surjection \( \pi: H \rightarrow G \) such that \( \pi|_{N(v^\prime)}: N(v^\prime) \rightarrow N(v) \) is a bijection for all vertices \( v^\prime \in V(G) \) and \( \pi(v') = \pi(v) \). When a finite group \( \Pi \) acts on a graph \( G \), the quotient graph \( G/\Pi \) is a simple graph whose vertices are the \( \Pi \)-orbits on \( V(G) \), with two vertices adjacent in \( G/\Pi \) if and only if some two of their representatives are adjacent in \( G \). A covering \( \pi: H \rightarrow G \) is said to be regular if there is a subgroup \( B \) of the automorphism group \( Aut(H) \) of \( H \) acting freely on \( H \) such that the quotient graph \( H/B \) is isomorphic to \( G \).

Let \( G \) be a graph and \( A \) a finite group. Let \( D(G) \) be the arc set of the symmetric digraph corresponding to \( G \). Then a mapping \( \alpha: D(G) \rightarrow A \) is called an ordinary voltage assignment if \( \alpha(u, v) = \alpha(u, v)^{-1} \) for each \( (u, v) \in D(G) \). The pair \( (G, \alpha) \) is called an ordinary voltage graph. The derived graph \( G^* \) of the ordinary voltage graph \( (G, \alpha) \) is defined as follows: \( V(G^*) = V(G) \times A \) and \( (u, h), (v, k) \in D(G^*) \) if and only if \( (u, v) \in D(G) \) and \( k = h\alpha(u, v) \). The natural projection \( \pi: G^* \rightarrow G \) is defined by \( \pi(u, h) = u, (u, h) \in V(G^*) \). The graph \( G^* \) is called a derived graph covering of \( G \) with voltages in \( A \) or an \( A \)-covering of \( G \). The natural projection \( \pi \) commutes with the right multiplication action of the \( \alpha(\cdot, \cdot) \in D(G) \) and the left action of \( A \) on the fibers: \( g(u, h) = (u, gh), g \in A \), which is free and transitive. Thus, the \( A \)-covering \( G^* \) is an \(| A | \)-fold regular covering of \( G \) with covering transformation group \( A \). Furthermore, every regular covering of a graph \( G \) is an \( A \)-covering of \( G \) for some group \( A \) (see [5]).

Let \( M_1 \oplus \cdots \oplus M_r \) be the block diagonal sum of square matrices \( M_1, \cdots, M_r \). If \( M_1 = M_2 = \cdots = M_r = M \), then we write \( s \cdot M = M_1 \oplus \cdots \oplus M_r \).

**Proof of Theorem 1.7.** Let \( V(G) = \{ v_1, \cdots, v_m \} \) and \( A = \{ 1, g_1, g_2, \cdots, g_m \} \). Arrange vertices of \( G^* \) in \( m \) blocks: \( (v_1), (v_2), \cdots, (v_m) \). We consider the adjacency matrix \( A(G^*) \) under this order. For \( h \in A \), the matrix \( P_h = (P^h) \) is defined as follows:

\[
P^h_{i,j} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise}. \end{cases}
\]

Suppose that \( P^h_{i,j} = 1 \), i.e., \( g_i = g_j h \). Then \( ((u, g_i), (v, g_j)) \in D(G^*) \) if and only if \( (u, v) \in D(G) \) and \( g_j = g_i \alpha(u, v) \), i.e., \( \alpha(u, v) = g_i^{-1} g_j = g_i^{-1} g_i h = h \). Thus we have

\[
A(G^*) = \sum_{h \in A} P_h \otimes A_h.
\]

Let \( \rho \) be the right regular representation of \( A \). Furthermore, let \( \rho_1, \rho_2, \cdots, \rho_l \) be the irreducible representations of \( A \), and \( f_i \) the degree of \( \rho_i \) for each \( i \), where \( f_i = 1 \). Then we have \( \rho(h) = P_h \) for \( h \in A \). Furthermore, there exists a regular matrix \( P \) such that \( P^{-1} \rho(h) P = (1) \oplus f_2 \rho_2(h) \oplus \cdots \oplus f_l \rho_l(h) \) for each \( h \in A \) (see [3]). Putting \( B = (P^{-1} \otimes I_s) A(G^*) (P \otimes I_s) \), we have
\[ B = \sum_{h \in A} \{(1 \oplus f_2 \circ \rho_2(h)) \oplus \cdots \oplus f_i \circ \rho_i(h)\} \otimes A_i. \]  \hspace{1cm} (10)

Note that \( A(G) = \sum_{h \in A} A_h \). Therefore it follows that

\[ \Phi(G^a; \lambda) = \Phi(B; \lambda) = \Phi(G; \lambda) \cdot \prod_{j=2}^i \Phi\left(\sum_{h} \rho_j(h) \otimes A_h; \lambda\right)^{f_i}. \]

**Proof of Theorem 1.6.** Let \( n = |V(G)|, i = |E(G)| \) and \( m = |A| \). By (1), (9), (10) and the fact that \( 1 + f_2 + \cdots + f_i = m \), it follows that

\[ Z(G^a, u)^{-1} = (1 - u^{2})^{i-nm} \det (I_{nm} - uA(G^a) + u^{2}Q_{m}) \]
\[ = (1 - u^{2})^{i-nm} \det (I_{i} - uA(G) + u^{2}Q) \]
\[ \times \prod_{j=2}^{i} \left\{ (1 - u^{2})^{i-nj} \det \left( I_{j} - u \sum_{g \in A} \rho_j(g) \otimes A_g + u^{2}Q_{j}\right)\right\}^{f_j}. \]

By (1) and (4), we have \( f_{c}(u) = f_{c}(u) \prod_{j=2}^{i} \det \left( I_{j} - u \sum_{g \in A} \rho_j(g) \otimes A_g + u^{2}Q_{j}\right) \). Since \( f_{c}(1) = 0, f_{c}^{*}(1) = f_{c}(1) \prod_{j=2}^{i} \det \left( I_{j} - u \sum_{g \in A} \rho_j(g) \otimes A_g + u^{2}Q_{j}\right)^{f_j}. \) By (2), it follows that

\[ 2(l - n)\kappa(G^a) = 2(l - n)\kappa(G) \prod_{j=2}^{i} \det \left( I_{j} - u \sum_{g \in A} \rho_j(g) \otimes A_g + u^{2}Q_{j}\right)^{f_j}. \]

Therefore, the result follows. \( \square \)

### 3 Line Graphs of Regular Coverings

Let \( G \) be a graph. Then the **line graph** \( L(G) \) is the graph whose vertex set is the edge set \( E(G) \) of \( G \), with two vertices of \( L(G) \) being adjacent if and only if the corresponding edges in \( G \) have a vertex in common.

Let \( G \) be a connected graph, \( A \) a finite group and \( \alpha: D(G) \rightarrow A \) an ordinary voltage assignment. For \( e = (u, v) \in D(G) \), let \( o(e) = u \) and \( t(e) = v \). The inverse arc of \( e \) is denoted by \( e^{-1} \). In the \( A \)-covering \( G^a \), set \( v_{y} = (v, g), e_{y} = (e, g) \) and \( [e] = \{e, g\}, \) where \( v \in V(G), e \in D(G), g \in A, \) and \( [e] \) denotes the edge obtained from \( e \) by deleting its direction. For \( e = (u, v) \in D(G) \), the arc \( e \) emanates from \( u \) and terminates at \( v \). Note that \( (e_{y})^{-1} = (e_{y})_{(e, g)} \).

The set \( D(L(G)) \) of arcs in the line graph \( L(G) \) is given by

\[ \{(e, f) | e \neq f^{-1}, t(e) = o(f)\}. \]

Then we have \( o((e, f)) = [e] \) and \( t((e, f)) = [f] \). Furthermore, \( (e, f)^{-1} = (f^{-1}, e^{-1}) \). By Theorem 1.9, the line graph of the \( A \)-covering \( G^a \) of \( G \) is an \( A \)-covering of \( L(G) \).

Now, we determine a voltage assignment \( \beta: D(L(G)) \rightarrow A \) such that \( L(G^a) = L(G)^{\beta} \).

**Lemma 3.1** Let \( G \) be a connected graph with \( n \) vertices \( v_1, \cdots, v_n \), \( A \) a finite group and \( \alpha: D(G) \rightarrow A \) an ordinary voltage assignment. For each edge \( v_i, v_j \in E(G) \), let \( e_{ij} = (v_i, v_j) \). Furthermore, let \( \alpha_L: D(L(G)) \rightarrow A \) be defined by

\[ \alpha_L([e_{ij}], [e_{jk}]) = \begin{cases} 
\alpha(e_{ij}) & \text{if } i < j < k, \\
\alpha(e_{ij})\alpha(e_{jk}) & \text{if } i < j \text{ and } j > k, \\
\alpha(e_{jk}) & \text{if } i > j > k, \\
1 & \text{if } i > j \text{ and } j < k.
\end{cases} \]

**Then** \( L(G^a) = L(G)^{\alpha_L} \).

**Proof.** At first, note that \( ([e_{ij}], [e_{jk}])^{-1} = ([e_{ij}], [e_{jk}]) \) for any \( g \in A \). Let \( [e_{ij}] = ([e_{ij}]_{\rho_1(\rho_{1(i)})}) = [e_{ij}]_{\rho_1} \) if \( i < j \).

Let \( [e_{ij}], [e_{ik}] \in E(L(G)) \) be any edge of \( L(G) \) and \( g \in A \). Set \( v = v_i, w = v_j, z = v_k, x = e_{ij} \) and \( y = e_{jk} \).

Then we have \( y_{y} = (w, \rho_{1(\rho_{1(i)})}), (x_{y})^{-1} = (x_{1})_{\rho_{1(\rho_{1(i)})}}, y_{y_1} = (w_{\rho_{1(\rho_{1(i)})}}, z_{\rho_{1(\rho_{1(i)})}}) \) and \( (y_{y_1})^{-1} = (y_{y_1})_{\rho_{1(\rho_{1(i)})}} \). We consider four cases.

- **Case 1.** \( i < j < k \).
- **Then** we have \( ([x_{y_1}], [y_{y_1}]) \in D(L(G^{a})) \). Since \( \alpha_L([x_{y_1}], [y_{y_1}]) = \alpha(x_{y_1}), ([x_{y_1}], [y_{y_1}]) \in D(L(G))^{\alpha_L} \).

- **Case 2.** \( i < j \) and \( j > k \).
- **Then** we have \( ([x_{y_1}], [y_{y_1}]) \in D(L(G^{a})) \). Since \( \alpha_L([x_{y_1}], [y_{y_1}]) = \alpha(x_{y_1})\alpha(y_{y_1}), ([x_{y_1}], [y_{y_1}]) \in D(L(G))^{\alpha_L} \).

- **Case 3.** \( i > j > k \).
- **Then** we have \( ([x_{y_1}], [y_{y_1}]) \in D(L(G^{a})) \). Since \( \alpha_L([x_{y_1}], [y_{y_1}]) = \alpha(y_{y_1}), ([x_{y_1}], [y_{y_1}]) \in D(L(G))^{\alpha_L} \).

- **Case 4.** \( i > j \) and \( j < k \).
Then we have \((x)_{\partial(G)} \cdot \gamma(\gamma)_{\partial(G)} \in D(L(G^c))\). Since \(\alpha_L([x], [\gamma]) = 1\), \((x)_{\partial(G)} \cdot \gamma(\gamma)_{\partial(G)} \in D(L(G)^c)\). 

Let \(G\) be a connected graph with \(n\) vertices \(v_1, \cdots, v_n\) and \(l\) edges \(e_1, \cdots, e_l\). For \(g \in A\), the matrix \((A_L)_g = (a_{ij}^g)\) is defined as follows:

\[
\alpha^g_{ij} = \begin{cases} 1 & \text{if } \alpha_L(e, f) = g \text{ and } (e, f) \in D(L(G)), \\ 0 & \text{otherwise}. \end{cases}
\]

Furthermore, let \(D_r = (d_{ij})\) be the diagonal matrix with \(d_{ii} = \deg \gamma_i\), and \(Q_L = D_r - I_l\). By Theorem 1.6, the decomposition formulas for the zeta function and the complexity of the line graph \(L(G^c)\) of a regular covering \(G^c\) of a graph \(G\) are obtained.

**Corollary 3.1** Let \(G\) be a connected graph with \(n\) vertices and \(l\) edges, \(A\) a finite group and \(\alpha: D(G) \to A\) an ordinary voltage assignment. Furthermore, let \(\rho_1, \rho_2, \cdots, \rho_l\) be the irreducible representations of \(A\), and \(f\) the degree of \(\rho_i\) for each \(i\), where \(f_i = 1\). Suppose that the \(A\)-covering \(G^c\) of \(G\) is connected. Then the reciprocal of the zeta function of \(L(G^c)\) is

\[
Z(L(G^c), u)^{-1} = Z(L(G), u)^{-1} \cdot \prod_{i=2}^l \left(1 - u^{2l} - 2u \sum_{g \in A} \rho_i(g) \otimes (A_L)_g + u^2 (Q_L)_i^l\right)^{f_i},
\]

where \((Q_L)_i = I_l \otimes Q_L\) and \(l_c = |E(L(G))|\). Suppose that \(l > n\). Then the complexity of \(L(G^c)\) is

\[
\kappa(L(G^c)) = \frac{1}{|A|} \kappa(L(G)) \prod_{i=2}^l \det \left(I_l \otimes D_r - \sum_{g \in A} \rho_i(g) \otimes (A_L)_g\right)^{f_i}.
\]

We express the zeta function of the line graph \(L(G^c)\) for a regular graph \(G\) by using the characteristic polynomial of \(G^c\).

**Theorem 3.1** Let \(G\) be a connected r-regular graph with \(n\) vertices and \(l\) edges, \(A\) a finite group and \(\alpha: D(G) \to A\) an ordinary voltage assignment. Set \(|A| = m\). Suppose that the \(A\)-covering \(G^c\) of \(G\) is connected. Then

\[
Z(L(G^c), u)^{-1} = (1 - u^{2l})^{r-2m} u^m \cdot \left(1 + 2u + (2r - 3)u^2\right)^{r-2m} \cdot \Phi \left(\frac{1 + (2 - r)u + (2r - 3)u^2}{u}\right).
\]

**Proof.** At first, both \(L(G)\) and \(L(G^c)\) are \((2r - 2)\)-regular graphs. By (1), we have

\[
Z(L(G), u)^{-1} = (1 - u^{2l})^{r-2l} \det (I_l - uA_L + au^2 I_l) = (1 - u^{2l})^{r-2l} u^l \cdot \Phi \left(\frac{1 + au^2}{u}\right),
\]

where \(a = 2r - 3\). By (8), \(Z(L(G), u)^{-1} = (1 - u^{2l})^{r-2l} u^l \cdot \Phi \left(\frac{1 + 2au^2}{u}\right)\). Therefore, (11) follows.

4 Middle Graphs of Regular Coverings

Let \(G\) be a graph. The middle graph \(M(G)\) is the graph obtained from \(G\) inserting a new vertex into every edge of \(G\), and by joining each pair of these new vertices which lie on adjacent edges of \(G\). Furthermore, the total graph \(T(G)\) is the graph whose vertex set is the union of the vertex set \(V(G)\) and the edge set \(E(G)\) of \(G\), with two vertices of \(T(G)\) being adjacent if and only if the corresponding elements of \(G\) are adjacent or incident.

Let \(G\) be a connected graph with \(n\) vertices \(v_1, \cdots, v_n\). Then the middle graph \(M(G)\) of \(G\) is the graph with \(V(M(G)) = V(G) \cup E(G)\) and \(E(M(G)) = E(L(G)) \cup \{ue \in E(G), u \in V(G) \text{ are incident in } G\}\). The endline graph \(G^*\) of \(G\) is given as follows: \(V(G^*) = \{v_1, \cdots, v_n, v_1', \cdots, v_n', v_{12}', \cdots, v_{nn}'\}\) and \(E(G^*) = E(G) \cup \{v_iv_i', \cdots, v_nv_n'\}\). Hamada and Yoshimura [7] showed that \(M(G) = L(G^*)\).

For a finite group \(A\) and an ordinary voltage assignment \(\alpha: D(G) \to A\), we show that the middle graph of the \(A\)-covering \(G^c\) of \(G\) is an \(A\)-covering of the middle graph \(M(G)\) of \(G\).

**Theorem 4.1** Let \(G\) be a connected graph with \(n\) vertices \(v_1, \cdots, v_n\), \(A\) a finite group and \(\alpha: D(G) \to A\) an ordinary voltage assignment. Then \(M(G^c)\) is an \(A\)-covering of \(M(G)\).

**Proof.** At first, note that \(M(G) = L(G^*)\).

Now, let \(V(G) = \{v_1, \cdots, v_n\}\) and \(V(G^c) = V(G) \cup \{v_1', \cdots, v_n'\}\). Furthermore, let \(\alpha^* : D(G^c) \to A\) be the function defined as follows:

\[
\alpha^*(u, v) = \begin{cases} \alpha(u, v) & \text{if } (u, v) \in D(G), \\ 1 & \text{if } uv = v_iv_i'. \end{cases}
\]

Then \(\alpha^*\) is an ordinary voltage assignment. It is clear that \((G^c)^+ = (G^+)\), i.e., \(M(G^c) = L((G^c)^+) = L(G^+)^c\) and \(M(G^c) = L((G^+)\). By Theorem 1.9, \(L((G^c)^c)\) is an \(A\)-covering of \(L(G^c)\). Therefore, \(M(G^c)\) is an \(A\)-covering of \(M(G)\). 

**Corollary 4.1** \(M(G^c) = M(G)^{c_2}\).
Proof. Since $L(G^a) = L(G)^{va}, M(G^{a}) = L((G^a)^{v_1}) = L(G^{v_1})^{v_2} = M(G)^{v_1}$. □

We show that the total graph $T(G^a)$ of an $A$-covering $G^a$ of $G$ is an $A$-covering of the total graph $T(G)$ of $G$. For two graphs $G$ and $H$, let $G \cup H$ be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$.

Theorem 4.2 Let $G$ be a connected graph, $A$ a finite group and $\alpha: D(G) \to A$ an ordinary voltage assignment. Then the total graph $T(G^a)$ of $G^a$ is an $A$-covering of the total graph $T(G)$ of $G$.

Proof. At first, note that $T(G) = M(G) \cup G$. Thus, we have $T(G^a) = M(G^{va}) \cup G^a = M(G)^{va} \cup G^a$. We define the function $\alpha_T: D(T(G)) \to A$ as follows:

$$\alpha_T(u, v) = \begin{cases} 
\alpha_M(u, v) & \text{if } (u, v) \in D(M(G)), \\
\alpha(u, v) & \text{if } (u, v) \in D(G), 
\end{cases}$$

Then it follows that $T(G^a) = M(G^{va}) \cup G^a = (M(G))^{va} \cup G^a = T(G^a)$. □

We consider the ordinary voltage assignment $\alpha_T: D(T(G)) \to A$. Set $\alpha_M = \alpha^g$. We introduce the following order in $V(G^a)$: $v_1, \ldots, v_n, v'_1, \ldots, v'_n$. By the definition of $\alpha^g$, $\alpha_M$ is given as follows:

$$\alpha_M(u, v) = \begin{cases} 
\alpha_L(u, v) & \text{if } (u, v) \in D(L(G)), \\
1 & \text{if } u = [e_i], v = v_ji' \text{ and } i > j, \\
\alpha(e_i) & \text{if } u = [e_i], v = v_ji' \text{ and } i < j, 
\end{cases}$$

where $e_i = (v_i, v_j)$.

For $g \in A$, the matrix $(A_M)_{ij} = (a_{ij}^g)$ is defined as follows: $a_{ij}^g = 1$ if $\alpha_M(u, v) = g$ and $(u, v) \in D(M(G))$, and $a_{ij}^g = 0$ otherwise. Furthermore, let $D_M = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_{M(G)}(e_i) (1 \leq i \leq l)$; $d_{ii} = 
\deg_{M(G)}(e_i) (1 \leq i \leq l)$; and $Q_M = D_M - I_{l,n}$, where $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_l\}$. By Theorem 1.6, the decomposition formulas for the zeta function and the complexity of the middle graph $M(G^a)$ of a regular covering $G^a$ of a graph $G$ are obtained. Note that $|E(M(G))| > |V(M(G))|$ if $l \geq n$.

Corollary 4.2 Let $G$ be a connected graph with $n$ vertices and $l$ edges, a finite group and $\alpha: D(G) \to A$ an ordinary voltage assignment. Furthermore, let $\rho_1, \rho_2, \ldots, \rho_l$ be the irreducible representations of $A$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_i = 1$. Suppose that the $A$-covering $G^a$ of $G$ is connected. Then the reciprocal of the zeta function of $M(G^a)$ is

$$Z(M(G^a), u)^{-1} = Z(M(G), u)^{-1} \prod_{i=1}^{l} \left\{ 1 - u^{-2}(\rho_i(g) \otimes (A_M)) + u^{-2}(Q_M) f_i \right\},$$

where $(Q_M) f_i = I_{f_i} \otimes Q_M$ and $l_M = |E(M(G))|$. Suppose that $l \geq n$. Then the complexity of $M(G^a)$ is

$$\kappa(M(G^a)) = \frac{1}{|A|} \kappa(M(G)) \prod_{i=1}^{l} \det \left( I_{f_i} \otimes D_M - \sum_{g \in A} \rho(g) \otimes (A_M) f_i \right).$$

We express the zeta function of the middle graph $M(G^a)$ for a regular graph $G$ by using the characteristic polynomial of $G^a$.

Theorem 4.3 Let $G$ be a connected $r$-regular graph with $n$ vertices and $l$ edges, a finite group and $\alpha: D(G) \to A$ an ordinary voltage assignment. Set $|A| = m$. Suppose that the $A$-covering $G^a$ of $G$ is connected. Then

$$Z(M(G^a), u)^{-1} = (1 - u^{-2}(\rho_i(g) \otimes (A_M)) + u^{-2}(Q_M)) f_i \left\{ 1 + (2-r)u + (2-r-2)u^2 + (r-1)(2-r)u^3 + (2-r)(r-1)u^4 \right\},$$

for $1 \leq n \leq l$. Suppose that $l \geq n$. Then the complexity of $M(G^a)$ is

$$\kappa(M(G^a)) = \frac{1}{|A|} \kappa(M(G)) \prod_{i=1}^{l} \det \left( I_{f_i} \otimes D_M - \sum_{g \in A} \rho(g) \otimes (A_M) f_i \right).$$

Proof. For a vertex $w$ of $M(G)(M(G^a))$, we have

$$\deg w = \begin{cases} 
r & \text{if } w \in V(G)(V(G^a)), \\
2r & \text{if } w \in E(G)(E(G^a)). 
\end{cases}$$

Set $\alpha = 2r - 1$ and $b = r - 1$. By (1), we have

$$Z(M(G), u)^{-1} = (1 - u^{-2}(\rho_i(g) \otimes (A_M)) + u^{-2}(Q_M)) f_i \left\{ 1 + (2-r)u + (2-r-2)u^2 + (r-1)(2-r)u^3 + (2-r)(r-1)u^4 \right\}.$$
By (7) and (8),
\[
\det (I + uA_M + u^2Q_M) = u^2(1 + bu^2)^n(1 + u + bu^2)\Phi \left( \frac{1 + (a + b - 2)u^2 + abu^4}{u(1 + u + bu^2)} \right)
\]
\[
\Phi \left( \frac{1 + (2 - r)u + (a + b - r)u^2 + b(2 - r)u^3 + abu^4}{u(1 + u + bu^2)} \right).
\]
Thus, we have
\[
Z(M(G), u)^{-1} = (1 - u^2)^{\nu(G)} - u(1 + u + bu^2)^n(1 + 2u + au^2)^{\nu(G)} \Phi(G; h(u)),
\]
where \(h(u) = (1 + (2 - r)u + (a + b - r)u^2 + b(2 - r)u^3 + abu^4)/u(1 + u + bu^2)\). Therefore, the result follows.
\]
\[\square\]

**Corollary 4.3** Let \(G\) be a connected regular graph with \(n\) vertices and \(l\) edges. Suppose that \(l \geq n\). Then
\[
\kappa(M(G)) = 2^{l-n+1}(r + 1)^l \kappa(G).
\]

**Proof.** Set \(a = 2r - 1\) and \(b = r - 1\). The equality (2) implies that
\[
\kappa(M(G)) = \frac{f_{M(G)}(1)}{2(|E(M(G))| - |V(M(G))|)} = \frac{f_{M(G)}(1)}{n(r^2 - 2)}.
\]
By (13), we have \(f_{M(G)}(u) = u^n(1 + u + bu^2)^n(1 + 2u + au^2)^{-n}\Phi(G; h(u))\), where \(h(u) = (1 + (2 - r)u + (a + b - r)u^2 + b(2 - r)u^3 + abu^4)/u(1 + u + bu^2)\). Then we have \(\Phi(G; h(1)) = \Phi(G; r) = 0\) and \(h'(1) = 2(r^2 - 2)/(r + 1)\). By (3), \(\Phi'(G; h(1)) = \Phi'(G; r) = n\kappa(G)\). Hence (14) is obtained.
\]
\[\square\]

By Corollary 4.3, we obtain the following result.

**Corollary 4.4** Let \(G\) be a connected r-regular graph with \(n\) vertices and \(l\) edges, \(A\) a finite group and \(\alpha: D(G) \to A\) an ordinary voltage assignment. Suppose that the A-covering \(G^a\) of \(G\) is connected and \(l \geq n\). Set \(|A| = m\). Then the complexity of \(M(G^a)\) is
\[
\kappa(M(G^a)) = 2^{l-n+m+1}(r + 1)^{m-1} \kappa(G^a).
\]

Next, we state an alternative form of the formula on the complexity \(\kappa(M(G^a))\).

**Corollary 4.5** Let \(G\) be a connected r-regular graph with \(n\) vertices and \(l\) edges, \(A\) a finite group and \(\alpha: D(G) \to A\) an ordinary voltage assignment. Furthermore, let \(\rho_1 = 1, \rho_2, \ldots, \rho_i\) be the irreducible representations of \(A\), and \(f_i\) the degree of \(\rho_i\) for each \(i\), where \(f_i = 1\). Suppose that the A-covering \(G^a\) of \(G\) is connected and \(l \geq n\). Set \(|A| = m\). Then the complexity of \(M(G^a)\) is
\[
\kappa(M(G^a)) = \frac{1}{|A|} 2^{l-n+m-1}(r + 1)^{m-1} \kappa(M(G)) \prod_{j=1}^{m} \Phi \left( \sum_{j=1}^{n} \rho_j(g) \otimes A_j; r \right)^{f_i}.
\]

**Proof.** By (15) and (3), we have \(\kappa(M(G^a)) = 2^{l-n+m+1}(r + 1)^{m-1}(1/nm)\Phi'(G^a; r)\). The equality (6) implies that \(\Phi'(G^a; r) = \Phi'(G; r)^{1/2} \Phi(\sum_{j=1}^{n} \rho_j(g) \otimes A_j; r)^{f_i}\). By (3) and (14), \(\Phi'(G; r) = n\kappa(G) = n2^{-l-n+1}(r + 1)^{-l-1}\kappa(M(G))\). Therefore, the result follows.
\]
\[\square\]

### 5 Examples

Now, we give an example. Let \(G = K_3\) be the complete graph with three vertices \(v_1, v_2, v_3\) and \(A = Z_3 = (1, \tau, \tau^2) (\tau^3 = 1)\) the cyclic group of order 3. Furthermore, let \(\alpha: D(K_3) \to Z_3\) be the ordinary voltage assignment such that \(\alpha(v_1, v_2) = \tau\) and \(\alpha(v_1, v_3) = \alpha(v_2, v_3) = 1\). Then, the A-covering \(K_3^a\) is the cycle graph \(C_3\) with nine vertices. Furthermore, we have \(\alpha_1([e_{23}]) = \alpha_2([e_{23}]) = \alpha_3([e_{23}]) = \alpha([v_1, v_2, v_3]) = 1\); \(\alpha_1([e_{23}]) = \alpha_2([e_{23}]) = 1\); \(\alpha_1([e_{23}]) = \alpha_2([e_{23}]) = 1\). Since \(L(K_3^a) = L(C_3) = C_3\) and \(L(K_3) = K_3\), it is certain that \(L(K_3^a) = L(K_3)\).

All primitive, backtracking-less cycles of \(K_3^a\) are \(C, C^{-1}\), where \(C = (v_1, v_2, v_3, v_3)\). Thus, \(Z(K_3, u)^{-1} = (1 - u^3)^2\). Similarly, we have \(Z(L(K_3)), u)^{-1} = (1 - u^3)^2\).

By (6), we have
\[
\Phi(K_3^a; \lambda) = \Phi(K_3; \lambda)^2 \prod_{j=1}^{2} \Phi \left( \sum_{i=0}^{1} \chi_i(\tau^j)A_{\lambda}; \lambda \right)
\]
\[
= \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \det \begin{bmatrix} \lambda & -\zeta & -1 \\ -\zeta^2 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \det \begin{bmatrix} \lambda & -\zeta^2 & -1 \\ -\zeta & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}
\]
\[
= (\lambda^3 - 3\lambda - 2)(\lambda^3 - 3\lambda + 1)^2,
\]
where \(\chi_i(\tau^j) = (\zeta^j)^i\) and \(\zeta = (-1 + \sqrt{3} i)/2\). By (11), we have \(Z(L(K_3)), u)^{-1} = u^3\Phi(K_3^a; (1 + u^3)/u) = (1 - u^3)^2\).
\[ -u^3(1 + u^1 + u^3)^2 = (1 - u^2)^2. \]

Let \( V(M(K_3)) = \{ [v_1], [v_2], [v_3], [e_{12}], [e_{23}], [e_{13}] \} \), where we set \([v_i] = v_i v_i', i = 1, 2, 3\). Then, the ordinary voltage assignment \( \alpha_M : D(M(K_3)) \rightarrow Z_3 \) is given as follows:

\[
\alpha_M(u, v) = \alpha_L(u, v), (u, v) \in D(L(K_3)); \quad \alpha_M([v_1], [e_{12}]) = \alpha([v_1], v_1) = \tau^2;
\]

\[
\alpha_M([v_2], [v_3]) = \alpha_M([v_1], [e_{13}]) = \alpha_M([v_1], [e_{23}]) = \alpha([v_1], [v_1]) = \alpha([v_2], [v_2]) = 1.
\]

It is certain that \( M(K_3) = M(C_3) = M(K_3)^{\nu} \). By (12), we have

\[
Z(M(K_3)), u^{-1} = (1 - u^4)^3u^2(1 + u + u^2)^3 \Phi \left( K_3^\nu, \frac{1 + 2u^2 + 3u^4}{u + u^2 + u^3} \right)
\]

\[
= (1 - u^4)^3(1 + u + 3u^2 + u^3 + 3u^4)(1 - u)(1 + u^2 - 6u^3 - 4u^4 - 13u^5 - 6u^6 - 9u^7)
\]

\[
\times (1 - u + 3u^2 - 8u^3 + 10u^4 - 20u^5 + 21u^6 - 36u^7 + 22u^8 - 32u^9 + 39u^{10} - 9u^{11} + 27u^{12})^2.
\]

Finally, we give another example. Let \( G = K_4 \) be the complete graph with four vertices \( v_1, v_2, v_3, \) and \( A = Z_3 = \{ 1, \lambda, \lambda^2 \} \) the cyclic group of order 3. Furthermore, let \( \alpha : D(K_4) \rightarrow Z_3 \) be the ordinary voltage assignment such that \( \alpha(v_1, v_2) = \lambda, \alpha(v_1, v_3) = \lambda^2 \) and \( \alpha(v_1, v_4) = \alpha(v_2, v_3) = \alpha(v_2, v_4) = \alpha(v_3, v_4) = 1 \). The complexity of \( K_4 \) is \( 4^{4} = 16 \) (see [4]). By (5), we have

\[
\kappa(K_4) = \frac{1}{3} \kappa(G) \det \left[ \begin{array}{ccc}
3 & -\zeta & -\zeta^2 \\
-\zeta & 3 & -1 \\
-\zeta^2 & 3 & -1 \\
-1 & -1 & 3 \\
\end{array} \right] \cdot \det \left[ \begin{array}{ccc}
3 & -\zeta^2 & -\zeta \\
-\zeta & 3 & -1 \\
-\zeta^2 & 3 & -1 \\
-1 & -1 & 3 \\
\end{array} \right] = \frac{1}{3} \cdot 4^3 \cdot 1296 = 6912,
\]

where \( \zeta = (-1 + \sqrt{3}i)/2 \). By (15), we have \( \kappa(M(K_3)) = 2^1 4^{17} \kappa(K_3) = 2^1 4^{17} \cdot 6912 \). Furthermore, the equality (14) implies that \( \kappa(M(K_4)) = 2^3 4^{4} \kappa(K_4) = 2^3 4^{4} \cdot 131072 \).}

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