A Diagrammatic Construction of the \((sl(N, \mathbb{C}), \rho)\)-Weight System

Fumikazu NAGASATO

Graduate School of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
E-mail: fukky@math.kyushu-u.ac.jp

Received April 26, 2002; final version accepted August 1, 2002

In this paper, we first give a diagrammatic analogue of the Young symmetrizer. By using this, the \((sl(N, \mathbb{C}), \rho)\)-weight system for an arbitrary finite-dimensional irreducible representation \(\rho\) is formulated in a diagrammatic way. The formula is useful for the calculations of the \((sl(N, \mathbb{C}), \rho)\)-weight system in the sense that we do not need actual constructions of the representations of \(sl(N, \mathbb{C})\) essentially. Hence by using this and the modified Kontsevich integral we can get the quantum \((sl(N, \mathbb{C}), \rho)\)-invariant for any finite-dimensional irreducible representation without actual constructions of the representations of \(sl(N, \mathbb{C})\). The diagrammatic construction is a generalization of the formula given in "Remarks on the \((sl(N, \mathbb{C}), sl)\)-weight system".

KEYWORDS: modified Kontsevich integral, quantum invariant, weight system, Young symmetrizer

1. Introduction

The quantum invariants for framed links have been studied in the low-dimensional topology and have been played important roles. Recently the quantum \((sl(2, \mathbb{C}), \rho)\)-invariant for any finite-dimensional irreducible representation \(\rho\), which is called the colored Jones polynomial, is done intensively in a study of the volume conjecture.

For any pair \((g, \rho)\) of a semi simple Lie algebra \(g\) and its finite-dimensional irreducible representation \(\rho\) of \(g\), the quantum \((g, \rho)\)-invariant \(Q_{g,\rho}\) for framed links is defined by using the universal \(R\)-matrix of the quantum group \(\mathcal{U}_q(g)\) associated with \(g\) and \(\rho\). The original definition is complicated, and so it is omitted in this paper. In the case of \(g = sl(N, \mathbb{C})\), see [14].

On the other hand, we can get the quantum \((g, \rho)\)-invariant by using the modified Kontsevich integral and the \((g, \rho)\)-weight system. The modified Kontsevich integral \(\mathcal{Z}\) is defined as a iterated integral valued in the space \(\mathcal{A}(\text{L}^3)\). This is also a framed link invariant. (Refer [8] and [13] in detail.) The \((g, \rho)\)-weight system is a linear map from \(\mathcal{A}(\text{L}^3)\) to \(\mathbb{C}\), which is associated with the pair \((g, \rho)\). (Refer [1] and [13] in detail.) Indeed, the quantum \((g, \rho)\)-invariant is derived by composing \(\mathcal{Z}\) and the \((g, \rho)\)-weight system \(W_{g,\rho}\) also. The following Theorem 1.1 states this fact.

**Theorem 1.1 (Kassel [7], Le and Murakami [10]).** The quantum \((g, \rho)\)-invariant \(Q_{g,\rho}\) is reconstructed by the composition of the modified Kontsevich integral \(\mathcal{Z}\) and the \((g, \rho)\)-weight system \(W_{g,\rho}\) with degree, that is,

\[
Q_{g,\rho}(L)_{q=\exp(h)} = \tilde{W}_{g,\rho}(\mathcal{Z}(L)),
\]

for an arbitrary oriented framed link \(L\), where \(\tilde{W}_{g,\rho}(D) = W_{g,\rho}(D) \cdot h^{\text{Reg}(D)}\) for any Jacobi diagram \(D\).

Namely \(\mathcal{Z}\) is a lift of \(Q_{g,\rho}\) to \(\mathcal{A}(\text{L}^3)\) and \(\tilde{W}_{g,\rho}\) is the projection of \(\mathcal{Z}\) to \(Q_{g,\rho}\). This construction is somewhat useful in the sense that we do not need the quantum group \(\mathcal{U}_q(g)\) but need \(g\) and its representations.

The weight system is originally defined as a state sum, which is equivalent to the definition by using representation theory. (Refer [1], [12] and [13].) These two definitions are algebraic. However by using a diagrammatic realization of the Young symmetrizer, which is given in Sect. 2.2, the \((sl(N, \mathbb{C}), \rho)\)-weight system \(W_{sl(N,\mathbb{C}),\rho}(D)\) of a Jacobi diagram \(D\) can be formulated diagrammatically such that it is a linear sum of some diagrams without dashed edge. (Some Jacobi diagrams are shown in Fig. 1. See [11] in detail.) The diagrammatic construction is more manageable than the algebraic one in the sense that essentially it suffices to do everything on the diagrammatic level, that is, we do not need to use the representations of \(sl(N, \mathbb{C})\) essentially. (This will make the calculations of the weight system easier.) Hence we can essentially get the quantum \((sl(N, \mathbb{C}), \rho)\)-invariant without actual construction of the representation \(\rho\) of \(sl(N, \mathbb{C})\).

![Fig. 1. Jacobi diagrams based on \(S^1\)](image-url)
In this paper, a diagrammatic analogue of the Young symmetrizer for $sl(N, \mathbb{C})$ and a formula of the $(sl(N, \mathbb{C}), \rho)$-weight system through a diagrammatic analogue of the Young symmetrizer are given. The formula is a generalization of formula given by Takamuki and the author in [11].

This paper is organized as follows. In Sect. 2 we make a diagrammatic realization of the Young symmetrizer for $sl(N, \mathbb{C})$. In Sect. 3 by using this a diagrammatic construction of the $(sl(N, \mathbb{C}), \rho)$-weight system is given. In Sect. 4 we reconstruct the formula of the $(sl(N, \mathbb{C}), \rho)$- and $(sl(N, \mathbb{C}), \text{ad})$-weight system given in [8] and [11] respectively by using the diagrammatic construction. Moreover we mention a concrete observation such that the quantum $(sl(2, \mathbb{C}), \text{ad})$-invariant is the Kauffman polynomial, essentially in one variable.

2. Diagrammatic Analogue of the Young Symmetrizer

In this section, we first discuss how to construct the Young symmetrizer in the algebraic method roughly. Every topic in this section is stated in [5] and [15] in detail, so we do not state about it in detail.

2.1 Young symmetrizer $\hat{E}_{YD}$ for Young tableau $Y_D$

The Young symmetrizers associated with the Young tableaux were defined by Weyl in the algebraic method. (Refer [15] in detail.) In this subsection, we first discuss the Young tableaux.

A Young tableau is a Young diagram with some entries in it. Every Young diagram is presented by an $n$-vector $(m_n, \ldots, m_2, m_1) \in \mathbb{Z}^n$ for $n \in \mathbb{Z}_{>0}$. The Young diagram $D = D(m_n, \ldots, m_2, m_1)$ is defined as in Fig. 2.

![Fig. 2. The Young diagram $D(m_n, \ldots, m_2, m_1)$](image)

In this paper, we assume that every Young tableau on $D(m_n, \ldots, m_2, m_1)$ has entries $1, 2, \ldots, |D| = \sum_{i=1}^{n} im_i$. A sample on $D(1, 1, 1)$ is shown in Fig. 3.

![Fig. 3. A sample of Young tableau](image)

Now let us define the Young symmetrizer associated with a Young tableau. In this paper, we just define it by using the Young tableau $Y_{D(1,1,1)}$ in Fig. 3. With respect to the general case, we can define similarly. (See [15] in detail.)

We first define the elements $h_i$ and $v_i$ of group ring $\mathbb{C}[\mathcal{S}_6]$, where $\mathcal{S}_6$ is the symmetric group. Let $H_i$ be the subgroup of $\mathcal{S}_6$ such that $H_i$ consists of the permutations of the entries in the $i$-th row of $Y_{D(1,1,1)}$. Therefore,

- $H_1 = \{e, (35), (36), (56), (356), (365)\}$,
- $H_2 = \{e, (24)\}$,
- $H_3 = \{e\}$.

Similarly let $V_i$ be the subgroup of $\mathcal{S}_6$ such that $V_i$ consists of the permutations of the entries in the $i$-th column of $Y_{D(1,1,1)}$. Hence,
By using $H_i$ and $V_i$, the elements $h_i$ and $v_i$ are defined as follows:

$$h_i = \frac{1}{|H_i|} \sum_{\sigma \in H_i} \sigma,$$
$$v_i = \frac{1}{|V_i|} \sum_{\rho \in V_i} (-1)^{|\text{sign}(\rho)} \rho,$$

where $|H_i|$ and $|V_i|$ are the cardinality of $H_i$ and $V_i$ respectively. Therefore,

$$h_1 = \frac{1}{6} \{ e + (35) + (36) + (56) + (356) + (365) \},$$
$$h_2 = \frac{1}{2} \{ e + (24) \},$$
$$h_3 = e,$$
$$v_1 = \frac{1}{6} \{ e - (12) - (13) - (23) + (123) + (132) \},$$
$$v_2 = \frac{1}{2} \{ e - (45) \},$$
$$v_3 = e.$$

Then the Young symmetrizer for $Y_{(1,1,1)}$ is defined as the element $(\prod_{i=1}^{3} v_i)(\prod_{j=1}^{3} h_i)$ of $\mathbb{C}[\mathcal{S}_3]$. The Young symmetrizer plays an important role with respect to the representation of $sl(N, \mathbb{C})$. It is well known that every finite-dimensional irreducible representation of $sl(N, \mathbb{C})$ corresponds to the Young diagram through the highest weight of the representation as follows. (With respect to the highest weight, for example, see [3] and [5].) The highest weight of each finite-dimensional irreducible representation of $sl(N, \mathbb{C})$ is presented by an $(N-1)$-vector $(m_{N-1}, \ldots, m_2, m_1)$. Therefore we can make an explicit correspondence between the representation with the highest weight $(m_{N-1}, \ldots, m_2, m_1)$ and the Young diagram $D = D(m_{N-1}, \ldots, m_2, m_1)$. Let $V_N$ be the vector space $\mathbb{C}^N$, which is a representation space of the fundamental representation $\rho_0$ of $sl(N, \mathbb{C})$, and let $\rho$ be the representation with the highest weight $(m_{N-1}, \ldots, m_2, m_1)$. Let $Y_D$ be a Young tableau on $D$ with entries $1, 2, \ldots, |D| = \sum_{i=1}^{N-1} m_i$ which is associated with $\rho$. Then the Young symmetrizer $\mathcal{E}_{Y_D}$ acts on $V_N^{|D|}$ as a permutation of the tensor product. In fact, the Young symmetrizer $\mathcal{E}_{Y_D}$ is a projection from $V_N^{|D|}$ to an irreducible representation space $V_{\rho} \subset V_N^{|D|}$ of $\rho$. So any representation can be reduced to the fundamental representation $\rho_0$ by using the Young symmetrizer. This is the important role which the Young symmetrizer plays with respect to the representation of $sl(N, \mathbb{C})$. (Refer [3] and [5].)

### 2.2 Diagrammatic realization of $\mathcal{E}_\rho$

From now on, $\mathcal{E}_\rho$ and $\mathcal{E}_{Y_D}$ are used without distinctions. We first discuss the diagrammatic realization of the Young symmetrizer $\mathcal{E}_\rho$. Let $\rho$ be the finite-dimensional irreducible representation of $sl(N, \mathbb{C})$ with the highest weight $(m_{N-1}, \ldots, m_2, m_1)$. We may fix the Young tableau $Y_D$ of $D = D(m_{N-1}, \ldots, m_2, m_1)$ as follows:

$$Y_D = \begin{array}{cccccc}
N-1 & 12N-2 & \cdots & \cdots & \cdots & |D| \\
\vdots & \vdots & \ddots & \vdots & \vdots & \\\n2 & N+1 & \cdots & \cdots & \\\n1 & N & \cdots & \end{array}$$

because two Young tableaux $Y_D$ and $Y_D'$ having the same Young diagram $D$ project $V_N^{|D|}$ to two isomorphic subspaces $\mathcal{E}_{Y_D}(V_N^{|D|})$ and $\mathcal{E}_{Y_D'}(V_N^{|D|})$ as the representation space, where $|D|$ is the number of boxes of $D$. For example, the $Y_{D(1,1,1)}$ is as in Fig. 3.

Now let us define some diagrams. First, $(i \ i+1) \in \mathcal{S}_n$ and the product on $\mathcal{S}_n$ can be depicted as follows:
For a positive integer $n$ and an arbitrary element $\gamma \in S_n$, let $\sigma \cdot \tau$ be a depiction of $\gamma$ by using the above depictions of $(i \ i + 1)$ and the product on $S_n$. For example, $\gamma = (1 \ 3 \ 2) = (1 \ 3) \cdot (1 \ 2) \in S_4$ is depicted as follows:

Then the following lemma gives a depiction of $\hat{E}_\rho = \hat{E}_{YD}$.

**Lemma 2.1.** The Young symmetrizer $\hat{E}_\rho$ is depicted as follows,

\[
\hat{E}_\rho =
\]

where the diagrams $\frac{\ldots}{n}$ and $\frac{\ldots}{n}$ are depictions of $h_i$ and $v_j$ in the previous subsection for some $i$ and $j$ respectively as follows:

\[
\frac{\ldots}{n} = \frac{1}{n} \sum_{\sigma \in S_n} \sigma
\]

\[
\frac{\ldots}{n} = \frac{1}{n} \sum_{\tau \in S_n} (-1)^{\text{sign}(\tau)} \tau
\]

**Proof.** As in the previous subsection, the Young symmetrizer $\hat{E}_{YD}$ is defined by $\hat{E}_{YD} = (\prod_{i=1}^{m_1} v_i)(\prod_{j=1}^{N-1} h_j)$. By using this, the Young symmetrizer $\hat{E}_{YD}$ is depicted as follows:

This completes the proof.

In the next section, we use the modified Young symmetrizer $E_\rho$, which is an idempotent in $\mathbb{C}[S_n]$. (The reason will be mentioned in the following section.) In the rest of this subsection, we define $E_\rho$.

**Lemma 2.2.** Let $\rho$ be the representation as in Lemma 2.1. Then the following diagram $E_\rho = E_{YD}$ is a depiction of an idempotent in $\mathbb{C}[S_n]$:
where
\[ m_\rho = \prod_{n=1}^{N-1} \frac{m_n}{n!} \left( \frac{m_n + \cdots + m_n}{N-n} \right) \prod_{i=1}^{n} \frac{n-i+1}{m_n + \cdots + m_i + n-i+1}, \]
which is the non-zero rational number associated with \( \rho \).

**Proof.** In fact, Yokota constructed the diagrammatic realization of the quantum modified Young symmetrizer in [16]. Hence it suffices to do along [16].

The idempotent \( E_\rho \) is called the modified Young symmetrizer. We find that \( E_\rho \) is a non-zero element in \( C \frac{1}{\mathcal{S} \mathcal{D}} \) by using the result in [16]. Therefore it is not difficult to check that the idempotent \( E_\rho \) is also a projection which is similar to \( E_\rho \).

3. **Formulation of the (sl(\( N, \mathbb{C} \)), \( \rho \))-Weight System**

In Lemma 2.2, we gave a depiction of the modified Young symmetrizer \( E_\rho \). By using that, a diagrammatic construction of the (sl(\( N, \mathbb{C} \)), \( \rho \))-weight system is given as in Theorem 3.1.

**Theorem 3.1.** Let \( \rho \) be the representation as in Lemma 2.1 and let \( E_\rho \) be the modified Young symmetrizer associated with \( \rho \). Then the (sl(\( N, \mathbb{C} \)), \( \rho \))-weight system satisfies the following formula:

\[
W_{\text{sl}(N, \mathbb{C}), \rho}(\ldots) = \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} W_{\text{sl}(N, \mathbb{C}), \rho_0}(\ldots) - \frac{\lfloor \frac{N-1}{2} \rfloor}{N} W_{\text{sl}(N, \mathbb{C}), \rho_0}(\ldots).
\]

For convenience, \( W_{\text{sl}(N, \mathbb{C}), \rho} \) is denoted by \( W_\rho \) from now on.

**Proof.** We do not state the definition of the (sl(\( N, \mathbb{C} \)), \( \rho \))-weight system in this paper, but state some properties needed later. Let \( \lambda \) and \( \mu \) be finite-dimensional irreducible representations of sl(\( N, \mathbb{C} \)). Then the following properties hold.

**Properties**

1. \( W_{\lambda \otimes \mu}(\ldots) = W_{\lambda \otimes \mu}(\ldots) + W_{\lambda \otimes \mu}(\ldots), \)
2. \( W_{\lambda}(\lambda) = W_{\rho_0}(\lambda) - \frac{1}{2} W_{\rho_0}(\lambda), \)
3. \( W_{\lambda}(\mu) = W_{\rho_0}(\mu) - W_{\rho_0}(\mu), \)
4. \( W_{\rho_0}(\mu) = N^2 - 1, \)
5. \( W_{\rho_0}(\mu) = N(N^2 - 1). \)

(1) is derived from a property of the representations. (4) and (5) are derived from the calculations along the original definition of the weight system. With respect to the definition of the weight system, (2) and (3), refer [1]. (See also [13] with respect to the definition of the weight system.)

Property (1) and the depiction of \( E_\rho \) transform the left side of the equation in Theorem 3.1 into as follows:
\[ W_{\rho_0} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right), \]

where \( \frac{N}{4} = \sum_{j=0}^{n} t_j \). Therefore it suffices to show that
\[ W_{\rho_0}(\downarrow) = W_{\rho_0}(\downarrow) - \frac{1}{N} W_{\rho_0}(\downarrow). \]

Indeed, \( W_{\rho_0}(\downarrow) \) is presented by the following equation:
\[ W_{\rho_0}(\downarrow) = x[W_{\rho_0}(\uparrow) - \frac{1}{N} W_{\rho_0}(\uparrow)] \quad (6) \]
for some \( x(\neq 0) \in \mathbb{C} \). The reason is as follows. \( W_{\rho_0}(\downarrow) \) is an intertwiner, which is an element of \( \text{Hom}_{\mathfrak{sl}(\mathbb{C})}(V_N \otimes \mathfrak{sl}(\mathbb{C}), V_N) \), where \( V_N \in \mathbb{C}^N \) is the representation space of \( \rho_0 \). In general, for any Jacobi diagram \( D \) the weight system \( W_{\rho}(D) \) is an intertwiner, because \( W_{\rho} \) satisfies the following condition:
\[ W_{\rho} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right) = W_{\rho} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right), \]
where the strands in the equation above present oriented strands or unoriented dashed lines. Note that the element of \( W_{\rho_0}(\downarrow) \) satisfies the relation
\[ W_{\rho_0}(\downarrow) \circ W_{\rho_0}(\uparrow) = W_{\rho_0}(\downarrow). \]

It is easy to check that the right side of (6) substituted \( x = 1 \) is an element of \( \text{Hom}_{\mathfrak{sl}(\mathbb{C})}(V_N \otimes \mathfrak{sl}(\mathbb{C}), V_N) \). The dimension of \( \text{Hom}_{\mathfrak{sl}(\mathbb{C})}(V_N \otimes \mathfrak{sl}(\mathbb{C}), V_N) \) is one. Hence Eq. (6) holds for some \( x \).

From Properties (2), (4) and Eq. (6), it follows that \( x^3 = 1 \). Moreover by Properties (3), (5) and Eq. (6), we get \( x^3 = 1 \). Thus we get \( x = 1 \). This completes the proof. \( \square \)

Note that for a Jacobi diagram \( D \) based on \( S^1 \) the \( (\mathfrak{sl}(\mathbb{C}), \rho) \)-weight system \( W_{\rho} \) satisfies the following formula by Eq. (7):
\[ W_{\rho}(D) = W_{\rho} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right) = W_{\rho_0} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right). \]

Namely, in this case it suffices to insert one modified Young symmetrizer \( E_{\rho} \) into the parallel copies of a circle as in the right side of the above relation.

Also note that the modified Young symmetrizer \( E_{\rho} \) is normalized such that the following equation holds:
\[ W_{\rho}(\uparrow) = W_{\rho_0} \left( \begin{array}{c} \includegraphics[width=0.2\textwidth]{image} \end{array} \right) \text{ on } V_{\rho}. \]

4. Reconstruction

4.1 \( (\mathfrak{sl}(\mathbb{C}), \rho) \)-weight system

As in the previous section, the \( (\mathfrak{sl}(\mathbb{C}), \rho) \)-weight system can be depicted through the \( (\mathfrak{sl}(\mathbb{C}), \rho_0) \)-weight system, where \( \rho_0 \) is the fundamental representation of \( \mathfrak{sl}(\mathbb{C}) \). In this subsection, we reconstruct the formulas of the \( (\mathfrak{sl}(\mathbb{C}), \rho_0) \)- and the \( (\mathfrak{sl}(\mathbb{C}), \rho) \)-weight system by using that, where \( \text{ad} \) is the adjoint representation of \( \mathfrak{sl}(\mathbb{C}) \). (Refer [11].) For convenience, \( W_{\mathfrak{sl}(\mathbb{C}),\rho} \) is also denoted by \( W_{\rho} \) in this subsection.
Theorem 4.1 (Le and Murakami [8]).

\[ W_{\rho_0}(\downarrow) = -\frac{1}{N} W_{\rho_0}(\uparrow) + W_{\rho_0}(\prec). \]

Reconstruction. The highest weight of \( \rho_0 \) is \((0, \cdots, 0, 1)\). It is easy to check that the constant \( m_{\rho_0} \) is 1. Therefore by Theorem 3.1 and Property (2),

\[ W_{\rho_0}(\downarrow) = \left( W_{\rho_0}(\downarrow) \otimes W_{\rho_0}(\downarrow) \right) \circ W_{\rho_0}(\downarrow) = -\frac{1}{N} W_{\rho_0}(\uparrow) + W_{\rho_0}(\prec). \]

Theorem 4.2 (joint work with Takamuki [11]).

\[ W_{ad}(\downarrow) = W_{\rho_0}(\prec) + W_{\rho_0}(\prec) - W_{\rho_0}(\prec) - W_{\rho_0}(\prec). \]

Reconstruction. The highest weight of \( ad \) is \((1, 0, \cdots, 0, 1)\), and so the modified Young symmetrizer \( E_{ad} \) is as follows:

\[ E_{ad} = \frac{2(N-1)}{N} \]

The second equality holds by the following relation:

where \( T_i = \frac{1}{N} \). This relation is derived from the following fact:

\[ \mathcal{Y}_{n-1} \cup \bigcup_{i=1}^{n-1} (i+1) \mathcal{Y}_{n-1} = \mathcal{Y}_n. \]

Here the equations below are verified naturally:

\[ \begin{aligned} \cdots &\uparrow \uparrow \uparrow \uparrow \quad \cdots = \frac{1}{N} \prec \\
N &\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \quad N-1 \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \quad N \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{aligned} \quad (8) \]

\[ \begin{aligned} \cdots &\uparrow \uparrow \uparrow \uparrow \quad \cdots = \downarrow \downarrow \downarrow \downarrow \downarrow \\
N-1 &\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \quad \cdots \end{aligned} \quad (9) \]

because the relations below are hold as the representation space:

\[ \frac{1}{N} \sum_{\sigma \in \pi} (-1)^{\text{sign}(\sigma)} \sigma(V_N^{\otimes N}) \simeq \mathbb{C} \simeq \frac{1}{N} \sum_{i=1}^{N} e_i \otimes e^i(V_N^{\otimes N} \otimes V_N), \]

\[ \frac{1}{N-1} \sum_{\sigma \in \pi_{n-1}} (-1)^{\text{sign}(\sigma)} \sigma(V_N^{\otimes N-1}) \simeq V_N^{\ast} \simeq \text{id}_{V_N^{\otimes N}}(V_N^{\ast}), \]

where \( e^i \in V_N^{\ast} \) is the dual base of the canonical \( i \)-th base \( e_i \) of \( V_N = \mathbb{C}^N \). Note that the right sides of (8) and (9) are the depictions of \( \sum_{i=1}^{N} e_i \otimes e^i \) and \( \text{id}_{V_N^{\otimes N}} \) respectively.
Hence,
\[ E_{ad} = \begin{vmatrix} 0 & 1 \\ -1/N & 0 \end{vmatrix}. \]

By using \( E_{ad} \) above, we can derive the equation in Theorem 4.2 as the reconstruction of Theorem 4.1. Note that
\[ W_{ad} \begin{pmatrix} \ \ \\ \ \ \end{pmatrix} = W_{p_0} \begin{pmatrix} \ E_{ad} \\ \ E_{ad} \end{pmatrix} + W_{p_0} \begin{pmatrix} \ E_{ad} \\ \ E_{ad} \end{pmatrix} - 2/N W_{p_0} \begin{pmatrix} \ E_{ad} \\ \ E_{ad} \end{pmatrix}. \]

\[ \square \]

4.2 The Kauffman polynomial

Let \( ad \) be the adjoint representation of \( \mathfrak{sl}(N, \mathbb{C}) \). According to [2] and [9], we can find immediately that the quantum \((\mathfrak{sl}(2, \mathbb{C}), ad)\)-invariant is the Kauffman polynomial. In [6], it was observed concretely. In this paper, we do not discuss it in detail, but mention it.

There exists the following propositions.

Proposition 4.1 (joint work with Hamai [6], [11]). Let \( \hat{Z} \) be the modified Kontsevich integral. Then \( \hat{W}_{ad} \circ \hat{Z} \) is the Kauffman polynomial (essentially in one variable). Namely,
\begin{align*}
\hat{W}_{ad}(\hat{Z}(\ \)) &= \hat{W}_{ad}(\hat{Z}(\ \ )) = (e^h - e^{-h})(\hat{W}_{ad}(\hat{Z}(\ ) - \hat{W}_{ad}(\hat{Z}(\ ))), \\
\hat{W}_{ad}(\hat{Z}(\ \ )) &= e^{2h} \hat{W}_{ad}(\hat{Z}(\)), \\
\hat{W}_{ad}(\hat{Z}(\ )) &= e^h + e^{-h} + 1,
\end{align*}

where \( \hat{W}_{ad}(D) = W_{ad}(D) e^{\text{deg}(D)} \) for an arbitrary Jacobi diagram \( D \).

The \((\mathfrak{sl}(2, \mathbb{C}), ad)\)-weight system is formulated by the following proposition.

Proposition 4.2 (Chmutov and Varchenko [2]).
\begin{align*}
W_{ad}(\{-\}) &= 2(W_{ad}(\ \ ) - W_{ad}(\ \ )) , \\
W_{ad}(\ ) &= N^2 - 1, \\
W_{ad}(D \sqcup D') &= W_{ad}(D) \cdot W_{ad}(D').
\end{align*}

Proposition 4.2 is also derived from Theorem 3.1. (Refer [11].) From Proposition 4.2 and the definition of the modified Kontsevich integral \( \hat{Z} \), we can get Proposition 4.1. (Refer [6] and [11] in detail. With respect to the modified Kontsevich invariant, see [13].)

Acknowledgements

I would like to thank Professor Jun Murakami and Professor Yoshiyuki Yokota for their helpful comments. I also would like to thank Professor Mitsuyoshi Kato for his encouragement.

REFERENCES

A Diagrammatic Construction of the \((\mathfrak{sl}(N, \mathbb{C}), \rho)\)-Weight System 51