On Congruent Hypersurfaces of Hermitian Symmetric Spaces

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1 Introduction

Calabi [1] proved that any Kähler submanifold of complex space forms $S$ has metric rigidity, i.e., its congruence class in $S$ is determined by its first fundamental form alone. This work has been developed by several authors in different directions (e.g., Green [3], Romero [6], Suyama [7], Umehara [9]). Romero [6] and Umehara [9] extended the rigidity theorem to indefinite complex space forms. Green [3] proved a remarkable theorem that the metric rigidity is still valid in a certain generic sense even if the ambient space is a general Kähler manifold; only one cannot find any differential-geometric conditions that assure the metric rigidity. At the same time, there are given examples without metric rigidity in [3]. Such examples were found also by Suyama [7]: he showed that complex quadrics $Q_m(C)$ naturally imbedded in $Q(C)$ do not have metric rigidity if $l \geq 2(m + 1)$. So the first fundamental form alone is not sufficient to determine the congruence class of a general Kähler submanifold. In [8], substantially generalizing Calabi’s theorem, we gave local differential-geometric conditions that determine the congruence class of a holomorphic mapping $f: M \rightarrow S$ in the case where $S$ is Hermitian symmetric and a higher order osculating space to $f$ is equal to the tangent space to $S$ at some point: we showed that in that case, the congruence class of $f$ is determined by the first fundamental form and certain functions on $M$ involving the curvature of $S$ and higher order covariant derivatives of $f$ along $f$. Thus, the result was still restricted and rather complicated.

In this article, we shall give a simpler, local differential-geometric condition for complex hypersurfaces of Hermitian symmetric spaces, generalizing Calabi’s theorem in our direction. Using our previous result and the classification of totally geodesic symmetric domains of bounded symmetric domains given in [4], we shall prove the following

**Theorem.** Let $S$ be a simply connected, complex $(n + 1)$-dimensional Hermitian symmetric space with metric $g$ and Riemannian curvature tensor $R$. Let $f$ and $f'$ be two holomorphic immersions of a connected, complex $n$-dimensional complex manifold $M$ into $S$. Then, $f$ and $f'$ are congruent in $S$, i.e., there exists a holomorphic and isometric transformation $F$ of $S$ such that $F \circ f = f'$ on $M$, if and only if

$$f^*g = f'^*g, \quad f^*R = f'^*R$$

on $M$. (1)

2 Proof of Theorem

If $f$ and $f'$ are congruent in $S$, it is obvious that they satisfy (1). We shall prove the converse.

We first assume that $f$ is not totally geodesic in $S$ and prove the theorem in this case. Denote by $T_pS$ the complexified tangent space to $S$ at $q \in S$ and $T^*_pS$ its complex linear subspace of type $(1, 0)$. Similarly, we define $T_pM$ for $p \in M$. Let $D$ and $D'$ be the covariant differential along $f$ and $f'$ respectively. We take a point $o$ of $M$ such that at least one of $(D_{\partial f}/DF_{\partial f}(\partial/\partial z_i))_o$, $1 \leq i, j \leq n$, is not contained in $f_*T_oM$, where $(z^1, \ldots, z^n)$ is a holomorphic coordinate system valid in an open neighborhood of $o$ of $M$. We may assume that $(D_{\partial f}/DF_{\partial f}(\partial/\partial z_i))_o \in f_*T_oM$ for every $p \in U$ by linearly changing the holomorphic coordinate system and shrinking $U$ if it is necessary. Fixing such a coordinate system, we denote by $f_i$ the holomorphic vector field $f_*(\partial/\partial z_i)$ along $f$ on $U$ and by $f_0$ the smooth vector field $D_{\partial f}/DF_{\partial f}(\partial/\partial z_i)$ along $f$ on $U$. Similarly, we define $f_0$, $f_1$, $\ldots,$ $f_n$ for $f': f'_i = f_*(\partial/\partial z_i), f'_0 = D_{\partial f}/DF_{\partial f}$. The set $(f_0)_p, (f_1)_p, \ldots, (f_n)_p$ forms a linear basis of $T^*_pS$ at every $p \in U$. Now we denote by $G(f)$, the family of functions $g(f, f_i)$ and $R(f, f_i, f_j, f_k)$ on $U$, $1 \leq i, j, k, l \leq n$, and similarly define $G(f')$, for $f'$. Note that $G(f) = G(f')$ is equivalent to the condition (1).

**Proposition 1** Let $Z$ be an arbitrary holomorphic vector field on $U$. Then the condition $G(f) = G(f')$ implies the following identities on $U$:

$$R(f_0, f_0, f_0, f_0) = R(f'_0, f'_0, f'_0, f'_0),$$

(2)

$$R(f_0, f_0, f_0, f'Z) = R(f'_0, f'_0, f'_0, f'_Z),$$

(3)

$$R(f_0, f_0, f'_Z, f'_Z) = R(f'_0, f'_0, f'_Z, f'_Z).$$

(4)
\[ R(f_0, \overline{f_0} Z, f_0, \overline{f_0} Z) = R(f_0, \overline{f_0} Z, f_0, \overline{f_0} Z), \]
\[ R(f_0, f_0 Z, f_0 Z, f_0 Z) = R(f_0, f_0 Z, f_0 Z, f_0 Z). \]

In the following, we always assume \( G(f) = G(f') \) and adopt the range of indices \( 1 \leq i, j, k \leq n \) and \( 0 \leq \lambda, \mu \leq n \).

**Lemma 1** If \( f \) is not totally geodesic, then \( f' \) is also not totally geodesic.

**Proof.** By the identities
\[ g(f_0, \overline{f_0}) = \partial / \partial z^i g(f_i, \overline{f_i}), \quad (1 \leq i \leq n), \]
\[ g(f_0, f_0) = \partial / \partial z^i g(f_i, f_i) - g(f_i, R(f_i, \overline{f_i})), \]
and \( G(f) = G(f') \), we have \( g(f_0, \overline{f_0}) = g(f_0, \overline{f_0}) \) and \( g(f_0, f_0) = g(f_0, \overline{f_0}) \). So the Gramian determinant \( g(f_i, f_i) \) equals to \( g(f_i, f_i) \), which implies our assertion.

**Lemma 2** Let \( \gamma^i, \gamma^j, \rho^i, \rho^j \) be the smooth functions on \( U \) such that \( \partial_{\beta} f_0 = \Sigma_i \gamma^i \partial_{\beta} f_i, \partial_{\beta} f_0 = \Sigma_i \gamma^i f_0, \partial_{\beta} f_i = \Sigma_i \rho^i f_i \) on \( U \), then \( \gamma^i = \gamma^j \) and \( \rho^i = \rho^j \).

**Proof.** Every \( \gamma^i \) is determined by the left-hand sides of the identities
\[ g(D_{\beta} f_0, f_0) = \partial / \partial z^i g(f_0, f_0) \]
\[ g(D_{\beta} f_0, f_0) = \partial / \partial z^i g(f_0, f_0) - g(f_0, R(f, \overline{f})). \]
So we see first that if \( 1 \leq \mu \leq n \), \( \gamma^i \) is determined by \( G(f) \) in view of (7). Before examining \( \gamma^0 \), we consider \( \rho^i \).

If we put \( i = 1 \) in the identity
\[ R(f_0, \overline{f_0}, f_1, \overline{f_1}) = \partial / \partial z^i R(f_1, \overline{f_1}, f_1, \overline{f_1}) - R(f_1, \overline{f_1}, D_{\beta} f_1, \overline{f_1}), \]
we have
\[ 2R(f_0, \overline{f_0}, f_1, \overline{f_1}) = \partial / \partial z^i R(f_1, \overline{f_1}, f_1, \overline{f_1}). \]

This means that \( G(f) \) uniquely determines \( R(f_0, \overline{f_0}, f_1, \overline{f_1}) \) and also \( \rho^i \), since \( \rho^i \) are determined by \( g(f_0, R(f_i, \overline{f_i}) f_1) = -R(f_0, f_1, \overline{f_1}, f_1). \) Thus, we have \( \rho^i = \rho^j \) by the condition \( G(f) = G(f') \). Now since \( \gamma^0 (1 \leq \mu \leq n) \) and \( R(f_1, f_1, f_0, f_0) \) are determined by \( G(f) \) as we have just shown, so is each term of the right-hand side of (11) and hence also \( R(f_0, f_1, f_1, f_0) = -g(f_0, R(f, \overline{f})). \) Then by (7) and (8), we see that \( G(f) \) determines the left-hand sides of (9) and (10) also in the case \( \mu = 0 \). Therefore, we have \( \gamma^0 = \gamma^0 \) for all \( i, \lambda, \mu \).

**Lemma 3** For an arbitrary holomorphic vector field \( Z \) on \( U \),
\[ R(f_0, f_0, f_1, f_1, f_0 Z) = R(f_0, f_1, f_1, f_1, f_0 Z), \]
\[ R(f_0, f_0, f_0 Z) = R(f_0, f_0, f_0 Z), \]
\[ R(f_0, f_0, f_0 Z) = R(f_1, f_1, f_1, f_1, f_0 Z), \]
\[ R(f_0, f_0, f_0 Z) = R(f_0, f_0, f_0 Z). \]

**Proof.** It suffices to prove the identities when \( Z = \partial / \partial z^k \). The identity (13) follows from
\[ 2R(f_0, f_1, f_1, f_1, f_0 Z) = \partial / \partial z^k R(f_1, f_1, f_1, f_1, f_0 Z). \]
in the same argument as in the proof of the above lemma. The identity (14) follows from
\[ R(f_0, f_1, f_0, f_0) = \partial / \partial z^i R(f_0, f_1, f_1, f_0) = R(D_0 f_0, f_1, f_1, f_0) \]
by (13) and Lemma 2. Similarly we obtain (15) in view of
\[ R(f_1, f_0, f_0, f_0) = \partial / \partial z^i R(f_1, f_1, f_1, f_0) - R(f_1, f_1, f_1, f_0, D_0 f_0). \]
Finally, the identity
\[ \partial / \partial z^i R(f_0, f_0, f_0, f_0) = 2R(f_0, f_0, f_0, f_0) + R(f_0, f_0, f_0, f_0) \]
together with (15) and Lemma 2 implies (16).

**Lemma 4** Let \( Z_1 \) and \( Z_2 \) be arbitrary holomorphic vector fields on \( U \). Then
\[ R(f_0 Z_1, f_0, f_1, f_0 Z_2) = R(f_0 Z_1, f_0, f_1, f_0 Z_2), \]
\[ R(f_0, f_0 Z_1, f_0, f_0 Z_2) = R(f_0, f_0 Z_1, f_0, f_0 Z_2). \]

**Proof.** It suffices to show the identities in case where \( Z_1 = \partial / \partial z^i \) and \( Z_2 = \partial / \partial z^k \). In view of (13) and Lemma 2, we obtain (18) by the identity
\[ R(f_0, f_0, f_0, f_0) = \partial / \partial z^i R(f_0, f_0, f_0, f_0) - R(f_0, f_0, f_0, f_0, D_0 f_0). \]
The identity (19) is a direct consequence of
\[ \partial / \partial z^1 R(f_i, f_0, f_0, f_0) = 2R(f_0, f_0, f_0, f_0). \]

Proof of Proposition 1. In order to prove (2), we first show \( R(f_0, f_0, f_0, f_0) = R(f_0, f_0, f_0, f_0) + 2R(f_0, f_0, f_0, f_0) \). In fact, this follows from the identity
\[ \partial / \partial z^1 R(f_0, f_0, f_0, f_0) = 2R(f_0, f_0, f_0, f_0) + 2R(f_0, f_0, f_0, f_0) \]
by Lemmas 2 and 3. Then, we obtain (2) from
\[ \partial / \partial z^1 R(f_0, f_0, f_0, f_0) = 2R(f_0, f_0, f_0, f_0) + 2R(f_0, f_0, f_0, f_0). \]
In a similar way, the identity (3) can be obtained from
\[ \partial / \partial z^1 R(f_0, f_0, f_0, f_0) = R(D_{\partial \bar{z}^1} f_0, f_0, f_0, f_0). \]
By Lemma 4, \( R(f_0, f_0, f_0, f_0) = R(f_0, f_0, f_0, f_0). \) The identity (4) follows from
\[ R(f_0, f_0, f_0, f_0) = R(D_{\partial \bar{z}^1} f_0, f_0, f_0, f_0) \]
implies (5). Finally (6) follows from
\[ R(f_0, f_0, f_0, f_0) = R(D_{\partial \bar{z}^1} f_0, f_0, f_0, f_0). \]
Thus, we have completed the proof of Proposition 1.

Proof of Theorem. We first prove the theorem in the case where \( f \) is not totally geodesic in \( S \). If we recall [8, Theorem 3.2], the congruence class of \( f \) in \( S \) is determined by \( f^* g \) and \( R(D_f f, D_{\bar{f}} f, D_{\bar{f}} f, D_f f), |f|, |\bar{f}|, |f|, |\bar{f}| \leq 2 \), since \( f \) is infinitesimally full of order 2 in this case (see [8] for the notation and terminologies). Now, for arbitrary smooth functions \( \beta^i \) on \( U \), \( 0 \leq \alpha \leq n \), if we set \( W = \sum_{\alpha=0}^n \beta^i f_i \) and \( W' = \sum_{\alpha=0}^n \beta^i f_i' \), we have by Proposition 1
whence
\[ R(f_0, f_0, f_0, f_0) = R(f_0, f_0, f_0, f_0) \text{ on } U, \]
by Lemma 2, this implies
\[ R(D_{\partial \bar{z}^1} f_0, D_{\bar{z}^1} f_0, D_{\bar{z}^1} f_0, D_{\bar{z}^1} f_0) = R(D_{\partial \bar{z}^1} f_0, D_{\bar{z}^1} f_0, D_{\bar{z}^1} f_0, D_{\bar{z}^1} f_0) \text{ on } U \]
for every multi-indices \( I_1, I_2, I_3, I_4 \) with \( |I_1|, |I_2|, |I_3|, |I_4| \leq 2 \). Thus, we have completed the proof in this case.

Next, we have to consider the case where \( f \) is totally geodesic in \( S \). We write \( S = S_0 \times S_0 \times S_0, \) where \( S_0, S_0, \) and \( S_0 \) are Hermitian symmetric spaces of Euclidean type, compact type, and non-compact type respectively. Note that every Hermitian symmetric space of non-compact type (a bounded symmetric domain) does not admit any complex Euclidean space nor Hermitian symmetric space of compact type as its complex submanifold. Considering the duality, we see also that every Hermitian symmetric space of compact type does not admit any complex Euclidean space nor Hermitian symmetric space of non-compact type as its totally geodesic complex submanifold, because such a submanifold is always obtained by a Lie triple system. Now we may assume that \( M \) is a simply connected Hermitian symmetric space, taking the universal covering space of \( M \) if it is necessary. We can write \( M = \rho_0 \times \rho_0 \times \rho_0, \) where \( \rho_0, \rho_0, \) and \( \rho_0 \) are Hermitian symmetric spaces of Euclidean type, compact type, and non-compact type respectively. Then we must have \( f(M_0) \subset S_0, f(M_0) \subset S_0, f(M_0) \subset S_0 \) in an obvious meaning. Thus, it suffices to prove our assertion in the case where \( S \) is either of Euclidean type, compact type, or non-compact type. Our assertion is obvious when \( S \) is of Euclidean type. What we need for the other cases are substantiated in Ichmura [4]. In fact, one can find there all the connected, complete totally geodesic complex hypersurfaces of a Hermitian symmetric space of non-compact type: If we retain the notation in [4], such a complex hypersurface is either \( f : S' \times (I_1)_{p-1} \rightarrow S' \times (I_1)_{p} \) \( (p \geq 2) \) or \( f : S' \times (IV)_{p-1} \rightarrow S' \times (IV)_{p} \) \( (p \geq 2) \), where \( S' \) is a Hermitian symmetric space of non-compact type and \( f \) is expressed as \( f = f_1 \times f_2 \) by a holomorphic isometry \( f_1 \) of \( S' \) onto itself and a totally geodesic hypersurface \( f_2 : (I_1)_{p-1} \rightarrow (I_1)_{p} \) or \( f_2 : (IV)_{p-1} \rightarrow (IV)_{p} \). In each case, \( f_2 \) is unique up to congruence and hence so is \( f_2 \). Then, by the duality, the same conclusion is also valid for the case where \( S \) is of compact type. Thus, we have completed the proof of Theorem.
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