Multiplication Maps of Complete Linear Systems on Projective Toric Surfaces*

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Received February 20, 2006; final version accepted July 28, 2006

We obtain an algebro-geometric proof of the surjectivity of the multiplication map of complete linear systems of ample divisor and nef divisor on a nonsingular projective toric surface. As a consequence, we show that on a nonsingular toric Fano 3-fold, the multiplication map of complete linear systems of the anti-canonical divisor and a nef divisor is surjective.

KEYWORDS: toric variety

1. Introduction

Fakhruddin [1] showed that for an ample line bundle $L$ and a globally generated, that is, generated by its global sections, line bundle $E$ on a nonsingular toric surface, the multiplication map of their global sections $\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$ is surjective. We may ask how about on a singular toric surface, or more generally how about on a toric variety of higher dimension. For a help to answer these questions we give another proof of Theorem 1 in [1] in terms of algebraic geometry.

**Theorem 1.** (Fakhruddin[1]) Let $X$ be a nonsingular projective toric variety of dimension two. Let $L$ be an ample line bundle on $X$ and $E$ a globally generated line bundle on $X$. Then the multiplication map

$$\Gamma(L) \otimes \Gamma(E) \longrightarrow \Gamma(L \otimes E)$$

(1)

is surjective.

Let $T = (k^*)^n$ be an algebraic torus of dimension $n$ defined over an algebraically closed field $k$. Let denote $M = \text{Hom}_k(T, k^*)$ the group of characters of $T$. Then we have $M \cong \mathbb{Z}^n$ and $T = \text{Spec } k[M]$. A normal algebraic variety $X$ is called toric if it contains an algebraic torus $T$ as a dense open subset, together with an algebraic action $T \times X \to X$ that extends the natural action of $T$ on itself.

Let $\mathcal{O}_X(D)$ be a line bundle on $X$ defined by a Cartier divisor $D$. If $\mathcal{O}_X(D)$ is globally generated, then there is a convex polytope $P_D = \text{Conv}\{m_1, \ldots, m_r\}$ in $M \otimes \mathbb{R} \cong \mathbb{R}^n$ with some $m_i \in M$ such that

$$\Gamma(\mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} k \, e(m),$$

where $e(m)$ denotes the character of $T$ corresponding to $m \in M$ (see, for instance, Lemma 2.3 [8] or Section 3.5 [3]).

We also know that a line bundle on a complete toric variety is nef if and only if it is generated by its global sections (see, for instance, Theorem 3.1 [6]). If $D_1$ and $D_2$ are nef divisors on a toric variety $X$, then the convex polytope corresponding to $D_1 + D_2$ coincides with the Minkowski sum $P_{D_1} + P_{D_2} = \{x_1 + x_2 : x_i \in P_{D_i} \ (i = 1, 2)\}$. Moreover, the surjectivity of the multiplication map

$$\Gamma(\mathcal{O}_X(D_1)) \otimes \Gamma(\mathcal{O}_X(D_2)) \longrightarrow \Gamma(\mathcal{O}_X(D_1 + D_2))$$

is equivalent to the equality

$$P_{D_1} \cap M + P_{D_2} \cap M = (P_{D_1} + P_{D_2}) \cap M.$$

If $D$ is an ample divisor, then $\dim(P_D) = \dim X$. Conversely, any convex polytope $P$ in $M \otimes \mathbb{R}$ with $\dim P = \text{rank } M$ defines a polarized toric variety $(X, D)$ with $\dim X = \dim P$ so that the set of global sections of $\mathcal{O}_X(D)$ is a vector space with $\{e(m) : m \in P \cap M\}$ as a basis.

If $L$ is a globally generated line bundle on a toric variety $X$, then there exist a surjective morphism $\pi : X \to Y$ onto a toric variety and an ample line bundle $A$ on $Y$ such that $L = \pi^*A$.

When you try to prove the surjectivity of the multiplication map (1) in terms of convex polytopes, first you take a polytope corresponding to an ample divisor, which is characterized by $X$, and next you may face a difficulty in deciding

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* 2000 Mathematics Subject Classification. Primary 14M25; Secondary 14J40, 52B20.
which polytope corresponds to a nef divisor. In fact, it is very difficult to do it from the data of the convex polytope corresponding to an ample divisor. In general, there is neither inclusions nor similarities among convex polytopes corresponding nef divisors on $X$. Fakhruddin [1] succeeded it in the case that $X$ is a nonsingular toric surface. We cannot seem to deal with the case that $X$ is a singular toric variety.

Note that the surjectivity of the map (1) does not depend on a choice of a field $k$ as explained above. In this paper, we set $k = \mathbb{C}$.

If $X$ is a nonsingular toric variety, then $\operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$ and the group of algebraic 1-cycles $Z_1(X) \cong H_2(X, \mathbb{Z})$. They are dual to each other by the natural pairing $H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$. Let $\operatorname{NE}(X)$ be the convex cone generated by effective 1-cycles in $H_2(X, \mathbb{R})$ and the dual cone $\operatorname{PA}(X)$, called nef cone, in $H^2(X, \mathbb{R})$. By definition, a line bundle $L$ on $X$ is nef if and only if the class $[L]$ is contained in $\operatorname{PA}(X)$. Thus we have only to show the surjectivity of the map (1) for a line bundle $E$ with $[E] \in \operatorname{PA}(X) \cap \operatorname{Pic}(X)$. Now we happen to face another difficulty in choosing a good Cartier divisor $D$ with $E \cong \mathcal{O}_X(D)$. In this paper we really choose a suitable divisor $D$ such that we can easily show the surjectivity of the map (1).

As a consequence of our procedure, we obtain the following theorems.

**Theorem 2.** Let $X$ be a nonsingular toric Fano variety of dimension three. Then for a globally generated line bundle $E$ on $X$, the multiplication map

$$\Gamma(E) \otimes \Gamma(\mathcal{O}_X(-K_X)) \to \Gamma(E \otimes \mathcal{O}_X(-K_X))$$

is surjective.

**Theorem 3.** Let $X$ be a nonsingular projective toric variety of dimension three with the nef anti-canonical divisor $-K_X$. Then for an ample line bundle $A$ on $X$, the multiplication map

$$\Gamma(A) \otimes \Gamma(\mathcal{O}_X(-K_X)) \to \Gamma(A \otimes \mathcal{O}_X(-K_X))$$

is surjective.

We may employ the same procedure in the simplicial $\operatorname{NE}(X)$ case for a nonsingular toric variety $X$ of higher dimension. We will treat the case in the other paper.

### 2. Projective Toric Varieties

#### 2.1 Basic concepts

In this section we recall the fact about toric varieties needed in this paper following Oda’s book [8], or Fulton’s book [3]. In this paper all varieties will be defined over the complex number field $\mathbb{C}$.

Let $N$ be a free $\mathbb{Z}$-module of rank $n$, $M$ its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. We denote the same (,) as the pairing of $M_\mathbb{R}$ and $N_\mathbb{R}$ defined by scalar extension. Let $T_N := N \otimes_\mathbb{Z} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \operatorname{Hom}_\mathbb{Z}(T_N, \mathbb{C}^*)$ is the character group of $T_N$ and $T_N = \operatorname{Spec} \mathbb{C}[M]$. For $m \in M$ we denote $\theta(m)$ as the character of $T_N$. Let $\Delta$ be a finite complete fan in $N$ consisting of strongly convex rational polyhedral cones $\sigma$ in $N_\mathbb{R}$, that is, with a finite number of elements $v_1, \ldots, v_s$ in $N$ we can write as

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s$$

and it satisfies that $\sigma \cap (-\sigma) = \{0\}$. Then we have a complete toric variety $X = T_N \operatorname{emb} (\Delta) := \cup_{\sigma \in \Delta} U_{\sigma}$ of dimension $n$ (see Section 1.2 [8], or Section 1.4 [3]). Here $U_{\sigma} = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee := \{y \in M_\mathbb{R} : \langle y, x \rangle \geq 0 \text{ for all } x \in \sigma\}$ is the dual cone of $\sigma$. For the origin $\{0\} \in \Delta$, the affine open set $U_{\{0\}} = \operatorname{Spec} \mathbb{C}[M]$ is the unique dense $T_N$-orbit. We note that a toric variety is always normal.

If $|\Delta| := \cup_{\sigma \in \Delta} \sigma = N_\mathbb{R}$, then the variety $X$ is complete. Set $\Delta(s) := \{|s| \in \Delta : \dim \sigma = s\}$. Then $\tau \in \Delta(s)$ corresponds to the $T_N$-orbit $\operatorname{Spec} \mathbb{C}[\tau^\vee \cap M]$ and its closure $Y(\tau)$, which is also a $T_N$-invariant subvariety of dimension $n - s$. Hence $\Delta(1)$ corresponds to $T_N$-invariant irreducible divisors. If for any cone $\sigma \in \Delta$ there exist a part of $\mathbb{Z}$-basis $v_1, \ldots, v_s$ in $N$ such that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s,$$

then the toric variety $X$ is nonsingular.

#### 2.2 Ample cone

Let $\operatorname{Pic}(X)$ be the group of all invertible sheaves modulo isomorphism. The map $D \mapsto \mathcal{O}_X(D)$ gives a homomorphism from the group of Cartier divisors onto $\operatorname{Pic}(X)$. Let $A_{n-1}(X)$ denote the group of all Weil divisors modulo a subgroup of divisors $[\operatorname{div}(f)]$ of rational functions. The map $D \mapsto [D]$ determines an injective homomorphism
Pic(X) ↦ A_{n-1}(X).

Since X is toric, any \( m \in M \) determines a principal Cartier divisor div(\( e(m) \)), which gives a homomorphism from \( M \) to the group Div(X) of \( T_X \)-invariant Cartier divisors. Let \( \{D_1, \ldots, D_d\} \) be the set of all \( T_X \)-invariant irreducible divisors.

Then there is an intersection pairing between the line bundles and the 1-cycles,

\[
\text{Pic} C \quad \text{interior of} \quad PA \quad L^2 \quad /C1 \quad \text{Pic} \quad \text{induced from the natural pairing}
\]

\( \text{nef cone} \) called the \( \text{N} \) \( X \) is nonsingular, then the two rows in (2) coincide.

In this paper we also assume that \( \text{toric variety} \ X \) is nonsingular. Let \( Z_1(X) \) be the group of algebraic 1-cycles on \( X \).

Then there is an intersection pairing between the line bundles and the 1-cycles,

\[
\langle \cdot, \cdot \rangle : \text{Pic}(X) \times Z_1(X) \rightarrow \mathbb{Z},
\]

induced from the natural pairing

\[
\langle \cdot, \cdot \rangle : H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \rightarrow \mathbb{Z}.
\]

\( L \in \text{Pic}(X) \) is said to be \textit{numerically equivalent to zero} (denoted \( L \equiv 0 \)) if \( (L \cdot z) = 0 \) for all \( z \in Z_1(X) \). Set \( N^1(X) := [\text{Pic}(X)/\equiv] \otimes \mathbb{Z}. \)

Similarly, \( z \in Z_1(X) \) is said to be \textit{numerically equivalent to zero} (denoted \( z \equiv 0 \)) if \( (L \cdot z) = 0 \) for all \( L \in \text{Pic}(X) \). Set \( N_1(X) := [Z_1(X)/\equiv] \otimes \mathbb{Z}. \)

Since \( X \) is rational in our case, we have \( \text{Pic}(X) \otimes \mathbb{R} \equiv N^1(X) \equiv H^2(X, \mathbb{R}). \)

Let \( \text{NE}(X) \) denote the convex cone generated by effective 1-cycles in \( N_1(X) \), and \( \overline{\text{NE}}(X) \) the closure of \( \text{NE}(X) \) in \( N_1(X) \).

The closed cone in \( N_1(X)^* = [\text{Pic}(X) \otimes \mathbb{R}] \) dual to \( \overline{\text{NE}}(X) \) coincides with the \textit{pseudo-ample cone} \( PA(X) \), which is also called the \textit{nef cone}. Namely, \( \delta \in N_1(X)^* \) belongs to \( PA(X) \) (or, \( \delta \) is said to be \textit{nef}) if and only if \( (\delta : [C]) \geq 0 \) holds for every closed 1-dimensional irreducible subvariety \( C \).

By the Kleiman’s criterion any integral point contained in the interior of \( PA(X) \) corresponds to an ample line bundle. In our case \( X \) is toric, hence, we know more about \( \text{NE}(X) \).

**Proposition 1.** (Reid [9]) For any nonsingular projective toric variety \( X = T_X \text{emb}(\Delta) \) of dimension \( n \), there exist \( \tau_1, \ldots, \tau_s \in \Delta(n-1) \) such that

\[
\text{NE}(X) = \overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}[V(\tau_1)] + \cdots + \mathbb{R}_{\geq 0}[V(\tau_s)].
\]

Since our \( \text{NE}(X) \) is a convex polyhedral cone, the dual cone \( PA(X) \) is also a convex polyhedral cone.

Now we assume \( \dim X = 2 \). Let \( Z_1^2(X) \) be the subgroup of \( Z_1(X) \) generated by \( T_X \)-invariant curves. Then \( Z_1^2(X) \) coincides with \( \text{Div}(X) \) and \( N_1(X) = N^1(X) \).

Set

\[
H := \text{Pic}(X) \cong \mathbb{Z}^{d-2}.
\]

Then \( H_{\mathbb{R}} := H \otimes \mathbb{R} = N_1(X) = N^1(X) \). We note that \( H \) possesses the intersection pairing \( \langle \cdot, \cdot \rangle \).

In this paper we also assume that \( X \) is a nonsingular projective toric variety of dimension two. From the classification Theorem 1.28 in [8], \( X \) is obtained by successive equivariant blowing-ups from minimal ones, i.e., the projective plane \( \mathbb{P}^2 \) or the Hirzebruch surface \( F_a := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)) \).

If \( X \) is isomorphic to \( \mathbb{P}^2 \), then the statement of the Theorem is trivial. Since \( F_1 \) is obtained by blowing-up a point from \( \mathbb{P}^2 \), we may assume that \( X \) is obtained from the Hirzebruch surface \( F_a \) with \( a \geq 0 \). \( F_a \) is defined by the fan \( \Delta \) in \( N \cong \mathbb{Z}^2 \):

\[
(-1,a) \\
(0,1) \\
(0,-1) \\
\text{O} \\
(1,0)
\]
3. Case I: Simplicial $NE(X)$

First we treat the case when $X$ is obtained by blow-ups centered at $T_N$-invariant infinitely near points of one of the $T_N$-invariant point on the nonpositive section. Set $N = \mathbb{Z}^2$. Consider the fan $\Delta$ in $N$:

Here we set $l = d - 3$, and we use the same letters $A_i$ to the cones of dimension one in $\Delta$ and the corresponding $T_N$-invariant curves. For example, $F = \mathbb{R}_{\geq 0}(1, 0) = V(\mathbb{R}_{\geq 0}(1, 0))$. By definition we have the self-intersection numbers as

$$(D_0^2) = a \geq 0, \quad (D_2^2) \leq 0 \quad \text{and} \quad (F^2) = 0.$$  

Set $(A_i^2) = -a_i$ ($a_i \geq 0$) for $i = 1, \ldots, l + 1$. If $l \geq 2$, then all $a_i > 0$.

Since $F$ is linearly equivalent to a linear combination of $A_1, \ldots, A_l$ with positive coefficients and since $D_0$ is also a linear combination of $A_1, \ldots, A_{l+1}$, we have

$$H = \text{Pic}(X) = \bigoplus_{i=1}^{l+1} \mathbb{Z}[A_i]$$  

and

$$NE(X) = \mathbb{R}_{\geq 0}[A_1] + \cdots + \mathbb{R}_{\geq 0}[A_{l+1}].$$  

Let $\{e_1, \ldots, e_{l+1}\}$ be the dual basis to $\{[A_1], \ldots, [A_{l+1}]\}$ in $H$ with respect to the pairing $(\cdot)$. Then we have

$$\text{PA}(X) \cap H = \mathbb{Z}_{\geq 0}e_1 + \cdots + \mathbb{Z}_{\geq 0}e_{l+1}.$$  

This implies that $\text{PA}(X) \cap H$ is generated by $\{e_1, \ldots, e_{l+1}\}$ as a semi-group and that any nef line bundle is a linear combination of $e_1, \ldots, e_{l+1}$ with non-negative coefficients in $\text{Pic}(X)$. We have to show that for every ample line bundle $L$ on $X$, the multiplication map

$$\Gamma(L) \otimes \Gamma(\mathcal{O}_X(e_i)) \longrightarrow \Gamma(L(e_i))$$

is surjective for $i = 1, \ldots, l + 1$.

**Lemma 1.** For every ample line bundle $L$ on $X$ the multiplication map

$$\Gamma(L) \otimes \Gamma(\mathcal{O}_X(D_0)) \longrightarrow \Gamma(L(D_0))$$

is surjective.

**Proof.** Since $(D_0^2) = a \geq 0, (D_0 \cdot F) = (D_0 \cdot A_1) = 1, (D_0 \cdot A_2) = \cdots = (D_0 \cdot A_{l+1}) = 0$, the divisor $D_0$ is nef, hence $\mathcal{O}_X(D_0)$ is generated by its global sections. Thus we have a section $s \in \Gamma(\mathcal{O}_X(D_0))$ with $(s)_0 = D_0$. This section defines an exact sequence
Since the class \(E\) is a linear combination of \(e_1,\ldots,e_{l+1}\) with non-negative coefficients in \(\text{Pic}(X)\) from the expression (6), it is enough to prove in the case when \([E] = e_i\).

In the case when \(a_1 = a_1 = 1, a_2 = \cdots = a_{i-1} = 2\), Corollary 1 gives a proof.

Consider the case when \(a_1 \geq 1, a_j \geq 2 \) for \(j = 2,\ldots,i-1\) and \(a_i = 1\) with \(2 \leq i \leq l-1\). Then we have an expression as
\[ e_1 = [D_0], \quad e_2 = a_1 e_1 + [A_1], \]

\[ e_j = (a_1 - 1)e_1 + (a_2 - 2)e_2 + \cdots + (a_{j-2} - 2)e_{j-2} + (a_{j-1} - 1)e_{j-1} + [A_1 + \cdots + A_{j-1}] \]

for \( j \geq 3 \).

By the induction on \( j \), we can show that the map

\[ \Gamma(L) \otimes \Gamma(\mathcal{O}_X(e_j)) \rightarrow \Gamma(L(e_j)) \]

is surjective for \( j = 1, \ldots, i + 1 \) by using the expression (7) instead of Lemma 2 as in the proof of Corollary 1. Unfortunately, we cannot apply this argument for \( j \geq i + 2 \) because we have a negative coefficient \( a_i - 2 = -1 \) in the expression (7).

Now we note that \( A_i \) is an exceptional curve of the first kind. Thus we have a contraction morphism \( \pi_i : X \rightarrow X_i \) of \( A_i \) to a nonsingular toric surface \( X_i \). Let \( \Delta_i \) be the fan in \( N \) obtained by deleting \( A_i \) from \( \Delta \). Then \( X_i = T_{\text{Nemb}}(\Delta_i) \) and each \( A_j \) is the strict transform of \( A_j \) in \( X_i \). And we see that

\[ \text{NE}(X_i) = \mathbb{R}_{\geq 0}[\check{A}_1] + \cdots + \mathbb{R}_{\geq 0}[\check{A}_{i-1}] + \mathbb{R}_{\geq 0}[\check{A}_{i+1}] + \cdots + \mathbb{R}_{\geq 0}[\check{A}_{i+1}]. \]

Let \( \{\check{e}_1, \ldots, \check{e}_{i+1}\} \) be the dual basis of \( \text{Pic}(X_i) \). Then we have \( e_j = \pi^*_i(\check{e}_j) \) for \( j \neq i \) and

\[ \text{PA}(X_i) \cap \text{Pic}(X_i) = \mathbb{Z}_{\geq 0}\check{e}_1 + \cdots + \mathbb{Z}_{\geq 0}\check{e}_{i-1} + \mathbb{Z}_{\geq 0}\check{e}_{i+1} + \cdots + \mathbb{Z}_{\geq 0}\check{e}_{i+1}. \]

From this we see that every ample line bundle \( L \) on \( X_i \) is isomorphic to \( \pi^*_i(\check{L}) \otimes \mathcal{O}_X(ke_i) \) for some ample line bundle \( \check{L} \) on \( X_i \) and a positive integer \( k \). Thus we have a commutative diagram

\[
\begin{array}{ccc}
\Gamma(\mathcal{O}_X(e_j)) \otimes \Gamma(\pi^*_i(\check{L})) \otimes \Gamma(\mathcal{O}_X(ke_i)) & \rightarrow & \Gamma(\pi^*_i(\check{L}(\check{e}_j))) \otimes \Gamma(\mathcal{O}_X(ke_i)) \\
\downarrow & & \downarrow \\
\Gamma(\mathcal{O}_X(e_j)) \otimes \Gamma(L) & \rightarrow & \Gamma(L(e_j))
\end{array}
\]

Now we assume that the statement of Theorem holds for \( X_i \). Then we see that the map \( \Gamma(\pi^*_i(\check{L}) \otimes \mathcal{O}_X(e_j)) \rightarrow \Gamma(\pi^*_i(\check{L}(e_j))) \) is surjective for \( j \neq i \). From the expression (7) we can write \( \mathcal{O}_X(e_j) \) as \( \pi^*_j \check{E} \otimes \mathcal{O}_X(A_1 + \cdots + A_{i-1}) \) with a nef line bundle \( \check{E} \) on \( X \). Thus we see that the map \( \Gamma(\pi^*_j(\check{L}) \otimes \mathcal{O}_X(e_j)) \rightarrow \Gamma(\pi^*_i(\check{L}(e_j))) \) is also surjective from vanishing \( H^1(X, \pi^*_j(\check{L}(-A_1 - \cdots - A_{i-1}))) = 0 \), which follows from Lemma 3. Since \( \pi^*_i(\check{L}(\check{e}_j)) \otimes \mathcal{O}_X(e_j) \) is ample on \( X \), we see that the right-hand vertical arrow in (8) is surjective from the first part of this proof. Hence if we assume that the statement of Theorem holds for \( X_i \), then we see that the multiplication map \( \Gamma(\mathcal{O}_X(e_j)) \otimes \Gamma(L) \rightarrow \Gamma(L(e_j)) \) is surjective for \( j \geq i + 2 \) from the commutative diagram (8). By the induction on the number of the exceptional curves \( A_i \) of the first kind with \( 2 \leq i \leq l - 1 \), we obtain a proof from the diagram (8).

4. Case II: \((d-1)\)-gonal \text{NE}(X)

On the \( T_N \)-invariant negative section of the Hirzebruch surface \( \mathbb{F}_d \) there are two \( T_N \)-invariant points. Next we treat the case when \( X \) is obtained by blowing-ups at infinitely near points of these two points. The fan \( \Delta \) in \( N \) is of the form like this:

![Diagram](image-url)

Here we set \( l + m = d - 3 \). By definition \( (D_0^2) = a \geq 0 \). Set \( (B_j^2) = -b_j \) (\( b_j > 0 \)) for \( j = 1, \ldots, m \). We also have a
linear equivalence relation

$$F \sim \sum_{i=1}^{l} \alpha_i A_i - \sum_{j=1}^{m} \beta_j B_j$$

(9)

for some positive integers \(\alpha_i\) and \(\beta_j\). Since \(X\) is nonsingular, \(\alpha_1 = 1\). In the same way as in Section 3, we see

$$H = \text{Pic}(X) = \bigoplus_{i=1}^{l+1} \mathbb{Z}[A_i] \oplus \bigoplus_{j=1}^{m} \mathbb{Z}[B_j].$$

(10)

On the other hand, about the cone \(\text{NE}(X)\) we can say only

$$\text{NE}(X) = \mathbb{R}_{\geq 0}[A_1] + \cdots + \mathbb{R}_{\geq 0}[A_{l+1}] + \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[B_1] + \cdots + \mathbb{R}_{\geq 0}[B_m].$$

(11)

Let \(\{e_1, \ldots, e_{l+1}, f_1, \ldots, f_m\}\) be the dual basis to \([A_1], \ldots, [A_{l+1}], [B_1], \ldots, [B_m]\) in \(H\) with respect to the pairing (\(\cdot\)).

### 4.1 Elementary case

If we restrict ourselves to the case when \(\beta_j = 1\) for \(j = 1, \ldots, m\) in the Eq. (9), that is, the case when \(b_m = 1\) and \(b_j = 2\) for \(j = 1, \ldots, m - 1\), we have an expression of \(f_j\)’s.

**Lemma 4.** When \(b_m = 1\) and \(b_j = 2\) for \(j = 1, \ldots, m - 1\), we have

$$f_1 = [F] \quad \text{and} \quad f_{j+1} = f_j + [F + B_1 + \cdots + B_j] \quad \text{for} \quad j = 1, \ldots, m - 1.$$  

(12)

**Proof.** Easy.

Let \(E\) be a nef line bundle on \(X\). Then we may write in \(\text{PA}(X)\) as

$$[E] = \sum_{i=1}^{l+1} x_i e_i + \sum_{j=1}^{m} y_j f_j$$

(13)

for some non-negative integers \(x_i = (E \cdot A_i) \geq 0\) and \(y_j = (E \cdot B_j) \geq 0\). The inequality \((E \cdot F) \geq 0\) implies

$$\sum_{i=1}^{l+1} \alpha_i x_i - \sum_{j=1}^{m} \beta_j y_j \geq 0.$$  

(14)

From this observation we have the following.

**Proposition 2.** When \(a_1 = a_l = 1, a_2 = \cdots = a_{l-1} = 2, b_1 = \cdots = b_{m-1} = 2, b_m = 1\), we have

$$\text{PA}(X) \cap H = \sum_{i=1}^{l+1} \mathbb{Z}_{\geq 0} e_i + \sum_{i,j} \mathbb{Z}_{\geq 0} (e_i + f_j).$$

In particular, in this case the multiplication map

$$\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$$

is surjective for every ample line bundle \(L\) and nef line bundle \(E\).

**Proof.** We note that the proposition is the case when \(\alpha_i = \beta_j = 1\) for all \(i\) and \(j\). Thus \(\{e_i, e_i + f_j; i = 1, \ldots, l + 1, j = 1, \ldots, m\}\) generates the semi-group \(\text{PA}(X) \cap H\) by the inequality (14).

The surjectivity of the multiplication map follows from Lemma 4 by using the same argument in the proof of Corollary 1. \(\square\)

As the next step we consider the case when \(\beta_j = 1\) for all \(j\) but \(\alpha_i\) general. For non-negative integers \(y_j^{(i)}\) with \(\alpha_i = \sum_{j=1}^{m} y_j^{(i)}\), the element \(e_i + \sum_{j} y_j^{(i)} f_j\) is contained in \(\text{PA}(X)\) by (14).

**Lemma 5.** When \(a_1 \geq 1, a_2 \geq 2, \ldots, a_{k-1} \geq 2, a_k = 1\) for \(2 \leq k \leq l - 1\), it is satisfied that

$$\alpha_1 = 1 \leq \alpha_2 \leq \cdots \leq \alpha_k \quad \text{and} \quad \alpha_k > \alpha_{k+1},$$

and that \(\alpha_2 = a_1\),

$$\alpha_i = (a_{i-1} - 1)\alpha_{i-1} + (a_{i-2} - 2)\alpha_{i-2} + \cdots + (a_2 - 2)\alpha_2 + (a_1 - 1)\alpha_1$$

(15)

for \(3 \leq i \leq k\).

**Proof.** It is easily shown by using the equalities \(\alpha_1 = 1, \alpha_2 = a_1\) and

$$\alpha_{i+1} + \alpha_{i-1} = a_i \alpha_i \quad \text{for} \quad 2 \leq i \leq l.$$  

(16) \(\square\)
Proposition 3. When \( b_1 = \cdots = b_{m-1} = 2, b_m = 1 \), the semi-group \( \text{PA}(X) \cap H \) is generated by the set

\[
\left\{ e_i + \sum_j y_j^{(i)} f_j; 1 \leq i \leq l \text{ and all non-negative integers } y_j^{(i)} \text{ with } \sum_j y_j^{(i)} \leq \alpha_i \right\}.
\]

In particular, in this case the multiplication map

\[
\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)
\]

is surjective for every ample line bundle \( L \) and nef line bundle \( E \).

\textbf{Proof.} When \( b_1 = \cdots = b_{m-1} = 2, b_m = 1 \), it is satisfied that \( \beta_i = 1 \) for all \( j \). Thus we can easily find a generator of the semi-group \( \text{PA}(X) \cap H \) as in the statement from the inequality (14).

In the case when \( \alpha_1 = \alpha_2 = \cdots = \alpha_{l-1} = 2 \), Proposition 2 gives a proof of the surjectivity of the map (16).

Consider the case when \( \alpha_1 \geq 1, \alpha_2 \geq 2 \) for \( i = 2, \ldots, k - 1 \) and \( \alpha_k = 1 \) with \( 2 \leq k \leq m - 1 \). In this case we will show that the map (16) is surjective for \( [E] = e_1 + \sum_j y_j^{(i)} f_j (1 \leq i \leq k) \) with \( \sum_j y_j^{(i)} \leq \alpha_i \).

From the expression \( e_1 = [D_0] \) and Lemma 4 we see that the map (16) is surjective for \( [E] = e_1 \), or \([E] = e_1 + f_j (1 \leq j \leq m)\) by using the same argument of the proof of Corollary 1.

When \( i = 2 \) we have \( \alpha_2 = \alpha_1 \) and \( e_2 = a_1 e_1 + [A_1] \). For the set of non-negative integers \( y_j^{(2)} \) with \( \alpha_2 = a_1 \geq \sum_j y_j^{(2)} \), we have an expression

\[
e_2 + \sum_j y_j^{(2)} f_j = (a_2) e_1 + [A_1] + \sum_j y_j^{(2)} f_j = (a_2 - \sum_j y_j^{(2)}) e_1 + \sum_j y_j^{(2)} (e_1 + f_j) + [A_1].
\]

From this expression we see that the map (16) is surjective for \( [E] = e_2 + \sum_j y_j^{(2)} f_j \) with \( \sum_j y_j^{(2)} \leq \alpha_2 \).

When \( 3 \leq i \leq k \), we have the expressions (7) and (15) for \( e_i \) and \( \alpha_i \). We may write as

\[
e_i = a_i e_1 + a_i^2 e_2 + \cdots + a_i^{l-1} e_{l-1} + [A_1 + \cdots + A_{l-1}],
\]

\[
\alpha_i = a_i^l \alpha_1 + a_i^{l-2} \alpha_2 + \cdots + a_i^1 \alpha_{l-1}.
\]

Take the set of non-negative integers \( y_j^{(i)} \) with \( \sum_j y_j^{(i)} \leq \alpha_i \). Since \( \alpha_i = \sum_{s=1}^{l-1} a_i^s \alpha_s = \sum_j^l \sum_{s=1}^{l-1} a_i^s \alpha_s \), we can decompose \( y_j^{(i)} \) as sums of non-negative integers

\[
y_j^{(i)} = \sum_{s=1}^{l-1} \sum_{i=1}^{l} a_i^s y_j^{(i,s)}
\]

so that \( \sum_j y_j^{(i,s)} \leq \alpha_i \) for \( 1 \leq s \leq i - 1 \).

Then we have an expression of \( e_i + \sum_j y_j^{(i)} f_j \) as

\[
e_i + \sum_j y_j^{(i)} f_j = \sum_{s=1}^{l-1} \sum_{j=1}^{l} (e_i + \sum_j y_j^{(i,s)} f_j) + [A_1 + \cdots + A_{l-1}].
\]

By the induction on \( i \) we prove that the map (16) is surjective for \( [E] = e_i + \sum_j y_j^{(i)} f_j (1 \leq i \leq k) \) with \( \sum_j y_j^{(i)} \leq \alpha_i \).

Next we have to consider the case \( [E] = e_i, e_i + \sum_j y_j^{(i)} f_j \) with \( \sum_j y_j^{(i)} \leq \alpha_i \) for \( i > k \). Since \( (A_1')^{k} = -1 \), we have a contraction morphism \( \pi_1 : X \to X_k \). Here the fan \( \Delta_k \) in \( N \) obtained by deleting \( A_k \) from \( \Delta \) defines the toric surface \( X_k = T_{\text{NH}}(\Delta_k) \). Then \( A_i (i \neq k) \) and \( B_j \) are the strict transforms of the \( T_{\text{NH}} \)-invariant curves \( \tilde{A}_i \) and \( \tilde{B}_j \) on \( X_k \), and we have

\[
\text{Pic}(X_k) = \bigoplus_{i \neq k} \mathbb{Z}[\tilde{A}_i] \oplus \bigoplus_j \mathbb{Z}[\tilde{B}_j].
\]

Let \( \{ e_i, f_i; i \neq k \} \) be the dual basis to \( \{ \tilde{A}_i, \tilde{B}_j; i \neq k \} \) in \( \text{Pic}(X_k) \). Then \( e_i = \pi_1^* e_i \) and \( f_j = \pi_1^* f_j \). From the above argument and from \( H^1(X, \pi_1^* L(-A_1 - \cdots - A_{k-1})) = 0 \) we see that the map (16) is also surjective for \( L = \pi_1^* L \) and \( [E] = e_k + \sum_j y_j^{(k)} f_j \) with \( \sum_j y_j^{(k)} \leq \alpha_k \) such that \( L \) is ample on \( X_k \). For an ample line bundle \( L \), we can write \( L \cong \pi_1^* L \otimes B \) such that \( L \) is ample on \( X_k \) and \( B \) is nef with the form as a sum of several \( e_k + \sum_j y_j^{(k)} f_j \) with \( \sum_j y_j^{(k)} \leq \alpha_k \). If we assume that the statement of the proposition holds for \( X_k \), then the composite of the multiplication maps \( \Gamma'(\pi_1^* L) \otimes \Gamma(E) \otimes \Gamma'(B) \to \Gamma(\pi_1^* L \otimes E) \otimes \Gamma(E) \to \Gamma(L \otimes E) \) is surjective for \( [E] = e_k + \sum_j y_j^{(k)} f_j \) with \( \sum_j y_j^{(k)} \leq \alpha_k \) and \( i \geq k + 1 \) as in the proof of Theorem 4. Thus we see that the statement also holds for \( X \) by the induction on the number of the exceptional curves \( A_k \) of the first kind with \( 2 \leq k \leq l - 1 \). \qed

4.2 General case

As the second step, we consider the case when \( a_i = 1 \) for \( i = 1, \ldots, l \) and arbitrary \( \beta_j \) in the Eq. (9), that is, the case when \( a_1 = a_2 = 1 \) and \( a_i = 2 \) for \( i = 2, \ldots, l - 1 \). In this case the semi-group \( \text{PA}(X) \cap H \) is generated by \( e_i \)'s and \( \sum_j x_j^{(i)} e_i + f_j \) for all non-negative integers \( x_j^{(i)} \) with \( \sum_i x_j^{(i)} = \beta_j \).
Lemma 6. When $b_1 \geq 1, b_2 \geq 2, \ldots, b_{k-1} \geq 2, b_k = 1$ for $2 \leq k \leq m-1$, we have
\[
f_1 = [F], \quad f_2 = (b_1 - 1)f_1 + [F + B_1],
\]
\[
f_j = (b_1 - 2)f_1 + (b_2 - 2)f_2 + \cdots + (b_{j-2} - 2)f_{j-2} + (b_{j-1} - 1)f_{j-1} + [F + B_1 + \cdots + B_{j-1}]
\]
for $3 \leq j \leq k$. Moreover, we have
\[
\beta_2 \geq (b_1 - 1)\beta_1, \quad \beta_j \geq (b_1 - 2)\beta_1 + \cdots + (b_{j-2} - 2)\beta_{j-2} + (b_{j-1} - 1)\beta_{j-1}
\]
for $3 \leq j \leq k$.

Proof. We can easily check by definition and by using the equalities $\beta_2 = b_1\beta_1 - 1$ and $\beta_{j+1} + \beta_{j-1} = b_j\beta_j$ for $2 \leq j \leq m - 1$.

Proposition 4. When $a_1 = a_i = 1, a_i = 2$ for $2 \leq i \leq l - 1$ and $b_1 \geq 1, b_2 \geq 2, \ldots, b_{k-1} \geq 2, b_k = 1$ for $2 \leq k \leq m - 1$, the multiplication map
\[
\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)
\]
is surjective for every ample line bundle $L$ and nef line bundle $E$.

Proof. We can prove by using the same argument of the proof of Proposition 3 from Lemmas 2 and 6.

As the final step in the Section 4, we consider the case when $a_i$'s and $b_j$'s in the Eq. (9) are arbitrary.

Lemma 7. Assume that $a_1 \geq 1, a_2 \geq 2, \ldots, a_{k-1} \geq 2, a_k = 1$ for $2 \leq k \leq l - 1$. Let $L$ be an ample line bundle and $E$ a nef line bundle on $X$. Then the multiplication map
\[
\Gamma(L) \otimes \Gamma(E_f) \to \Gamma((L \otimes E)_{f_1, \ldots, f_k})
\]
is surjective for $1 \leq s \leq k - 1$.

Moreover, if we let $\pi_k : X \to X_k$ be the blowing-down of $A_k$ at a point, then the map (18) is surjective for $L = \pi_k^*L$ such that $L$ is an ample line bundle on $X_k$.

Proof. If $[E] = \sum_i x_i e_i$, then the map (18) is surjective since the multiplication map $\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$ is surjective and since $H^1(X, L \otimes E(-A_1 \cdots - A_{j-1})) = 0$. When $L = \pi_k^*L$, since $E = \pi_k^*E$ with a nef bundle $E$ on $X_k$, the latter half of the statement also holds.

If $[E] = \sum_i x_i e_i + \sum_j y_j f_j$, choose a nef bundle $F$ with $[F] = \sum_i x_i e_i$. Then we have $E_{A_1, \ldots, A_s} \cong F_{A_1, \ldots, A_s}$. Thus we obtain a proof.

Theorem 5. Let $X$ be a nonsingular projective toric surface obtained by a succession of $T_\delta$-equivariant blowing-ups from the Hirzebruch surface $\mathbb{F}_d$ centered at infinitely near points of two $T_\delta$-invariant points on the negative section. For an ample line bundle $L$ and a nef line bundle $E$ on $X$, then, the multiplication map
\[
\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)
\]
is surjective.

Proof. Set $[E] = \sum_i x_i e_i + \sum_j y_j f_j$ in $\text{Pic}(X)$. Since $E$ is nef, we have $x_i \geq 0, y_j \geq 0$ and
\[
\sum_{i=1}^{l-1} a_i x_i - \sum_{j=1}^m \beta_j y_j \geq 0.
\]
For $X$ satisfying the condition that all $a_i = 1$ or all $\beta_j = 1$, we see that the statement holds from Propositions 3 and 4.

Assume that $a_1 \geq 1, a_2 \geq 2, \ldots, a_{k-1} \geq 2, a_k = 1$ for $2 \leq k \leq l - 1$. Let $\pi_k : X \to X_k$ be the blowing-down of $A_k$ at a point. If we assume that the statement holds for $X_k$, then the multiplication map $\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$ is surjective for $E \cong \pi_k^*E$ and $L \cong \pi_k^*L$ with ample $L$ and nef $E$ on $X_k$. The condition $E \cong \pi_k^*E$ is the same as $x_k = 0$. When $x_k = 1$, we have $E \cong \pi_k^*E \otimes \Theta_{x_k(A_1 + \cdots + A_{k-1})}$ from the expressions (7) and (15). Thus we have the surjectivity of the multiplication map in this case. When $x_k \geq 2$, we have $E \cong F \otimes \Theta_{x_k(A_1 + \cdots + A_{k-1})}$ with nef $F$ satisfying $(F \cdot A_k) = x_k - 1$. By the induction on $x_k$ we see that the map is surjective for arbitrary nef $E$ and $L \cong \pi_k^*L$ with ample $L$ on $X_k$. If $L$ is ample, then we have $[L] = \sum_i x_i e_i + \sum_j y_j f_j$ with $x_i, y_j > 0$ and $\sum_i a_i x_i - \sum_j \beta_j y_j > 0$. When $x_k = 1$, we have $L \cong \pi_k^*L \otimes \Theta_{x_k(A_1 + \cdots + A_{k-1})}$ from the expressions (7) and (15). Thus we have the surjectivity of the multiplication map in this case by using Lemmas 3 and 7. By the same way we see that the map is surjective for ample $L$ and nef $E$ if we assume the theorem holds for $X_k$. By the induction on the number of the exceptional curves $A_k$ of the first kind with $2 \leq k \leq l - 1$, we have a proof.
5. Case III: General NE($X$)

A nonsingular projective toric surface $X$ is in general given by a fan $\Delta$ in $N$ of the form like this up to isomorphisms of $N$: 

\[ F = \sum_{i=1}^{l} a_i A_i - \sum_{j=1}^{m} B_j + \sum_{k=1}^{n} g_k C_k, \]  

(19) 

\[ D_0 = \sum_{i=2}^{l+1} a'_i A_i + \sum_{j=1}^{m} b'_j B_j - \sum_{k=1}^{n} f_k C_k \]  

(20) 

for some positive integers $a_i, a'_i, b_j, b'_j, g_k, f_k$. Since $X$ is nonsingular, $a_1 = \gamma_1 = 1, a'_2 = \gamma_2 = 1$. In the same way as in Section 4, we see 

\[ H = \text{Pic}(X) = \bigoplus_{i=1}^{l+1} \mathbb{Z}[A_i] \oplus \bigoplus_{j=1}^{m} \mathbb{Z}[B_j] \oplus \bigoplus_{k=1}^{n} \mathbb{Z}[C_k]. \]  

(21) 

On the other hand, about the cone NE($X$) we can say only 

\[ \text{NE}(X) = \mathbb{R}_{\geq 0} [A_1] + \cdots + \mathbb{R}_{\geq 0} [A_{l+1}] + \mathbb{R}_{\geq 0} [F] + \mathbb{R}_{\geq 0} [D_0] \]  

+ \mathbb{R}_{\geq 0} [B_1] + \cdots + \mathbb{R}_{\geq 0} [B_m] + \mathbb{R}_{\geq 0} [C_1] + \cdots + \mathbb{R}_{\geq 0} [C_n]. \]

Let $\{e_1, \ldots, e_{l+1}, f_1, \ldots, f_m, g_1, \ldots, g_n\}$ be the dual basis to $\{[A_1], \ldots, [A_{l+1}], [B_1], \ldots, [B_m], [C_1], \ldots, [C_n]\}$ in $H$ with respect to the pairing $.$

Let $E$ be a nef line bundle on $X$ with an expression 

\[ [E] = \sum_{i=1}^{l+1} x_i e_i + \sum_{j=1}^{m} y_j f_j + \sum_{k=1}^{n} z_k g_k \]  

(22) 

in Pic($X$). Then $x_i, y_j, z_k \geq 0$ and 

\[ ([E] \cdot F) = \sum_{i=1}^{l} \alpha_i x_i - \sum_{j=1}^{m} \beta_j y_j + \sum_{k=1}^{n} \gamma_k z_k \geq 0, \]  

(23) 

\[ ([E] \cdot D_0) = \sum_{i=2}^{l+1} \alpha'_i x_i + \sum_{j=1}^{m} \beta'_j y_j - \sum_{k=1}^{n} \gamma'_k z_k \geq 0. \]  

(24) 

From these inequalities we see that $e_i$'s are nef for $i = 1, \ldots, l + 1$.

First we consider the case when $n = 1$, that is, $c_1 = 1, \gamma_1 = \gamma'_1 = 1$.

**Lemma 8.** When $n = 1$, we have an expression
Proof. The expression is given by a direct calculation.

When \(a_1 = \cdots = a_{i-1} = 2\) and \(a_i = 1\) the surjectivity of the multiplication map is obtained by the induction on \(i\).

When \(a_1 \geq 2, \ldots, a_{i-1} \geq 2, a_i = 1\) for \(2 \leq s \leq l\), we may consider the blow-down \(\pi_s: X \to X_s\) of the exceptional curve \(A_s\) of the first kind. From the expression of \(e_i\), we see that the multiplication map is surjective for \(i \leq s + 1\) and that the surjectivity also holds for \(L \cong \pi_s^* L\) with ample \(L\) on \(X_s\) and for \(i \leq s\) by using Lemma 3. If we assume that the statement holds on \(X_s\), then we see that the surjectivity holds for \(L \cong \pi_s^* L\) with ample \(L\) on \(X_s\) and for all \(i\). If \(L\) is ample, we have \([L] = \sum_i \xi_i e_i + \sum_j n_j f_j + \zeta g_1\) with positive coefficients. If \(\xi_i = 1\), then we have \(L \cong \pi_s^* L \oplus O_X(C_1 + A_1 + \cdots + A_{i-1})\) from our expression. Thus we have the surjectivity of the multiplication map in this case. By the induction on \(\xi_i\), we have the surjectivity of the multiplication map if we assume that the surjectivity holds on \(X_s\). Thus we have a proof of the proposition as in the proof of Theorem 3.

In the case when \(\beta_j = 1\) for all \(j\), that is, when \(b_1 = \cdots = b_{m-1} = 2, b_m = 1\), we also have generators of nef line bundles in \(\text{Pic}(X)\).

Lemma 9. When \(n = 1\) and \(b_1 = \cdots = b_{m-1} = 2, b_m = 1\), the semi-group \(\mathbb{PA}(X) \cap \text{Pic}(X)\) is generated by \(g_1 + f_j\) for \(1 \leq j \leq m\) and \(e_i + \sum_j y_j^{(i)} f_j\) for \(1 \leq i \leq l + 1\) and all non-negative integers \(y_j^{(i)}\) with \(\sum_j y_j^{(i)} \leq \alpha_i\).

In this case, the multiplication map

\[\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)\]

is surjective for an ample line bundle \(L\) and a nef line bundle \(E\) on \(X\).

Proof. It follows from the inequalities (23) and (24) because all \(\beta_j = 1\), \(\beta_j = j\) and \(\gamma_1 = \gamma'_1 = 1\).

Note that \(g_1 = [D_0]\). From Lemma 4, we have an expression

\[g_1 + f_1 = [D_0 + F]\quad \text{and}\quad g_1 + f_2 = g_1 + f_1 + [F + B_1], \ldots\]

The situation is very similar in the Subsection 4.1. From this expression we can show the surjectivity of the multiplication map as in the proof of Proposition 3.

Next we consider the case when \(\beta'_j \geq \beta_j\) and \(\gamma_k \geq \gamma'_k\) for all \(j\) and \(k\).

Proposition 5. When \(\beta'_j \geq \beta_j\) and \(\gamma_k \geq \gamma'_k\) for all \(j\) and \(k\), the multiplication map

\[\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)\]

is surjective for an ample line bundle \(L\) and a nef line bundle \(E\) on \(X\).

Proof. Note that \(\beta'_1 = \beta_1 = 1\) and \(\gamma_1 = \gamma'_1 = 1\) from the condition. We treated the case when \(n = 1\) and \(\beta_j = 1\) for all \(j\) in Lemma 9.

As the first step, we assume that \(n = 1\) and \(b_1 \geq 2, \ldots, b_{s-1} \geq 2, b_s = 1\) for \(2 \leq s \leq m - 1\). Then \(B_s\) is an exceptional curve of the first kind. Let \(\pi_s: X \to X_s\) be the blowing-down of \(B_s\). We assume that the statement holds for \(X_s\). Then the multiplication map \(\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)\) is surjective for \(L \cong \pi_s^* L\) and \(E \cong \pi_s^* \tilde{E}\) with ample \(\tilde{E}\) and nef \(\tilde{E}\) on \(X_s\). The condition \(E \cong \pi_s^* \tilde{E}\) is the same as \(y_s = 0\) in the expression (22).

When \(y_s = 1\), we may write \(E \cong E' \otimes O_X(F + B_1 + \cdots + B_{s-1})\) from the expression (17). Here \(E'\) has non-negative coefficients in the expression (22) and \(y_s = 0\). Moreover, we have

\[(E' \cdot F) = (E \cdot F) + \beta_1 - 1 \geq 0\]
\[(E' \cdot D_0) = (E \cdot D_0) - 1.\]

If \((E \cdot D_0) \geq 1\), then \(E'\) is nef and we have \(E \cong \pi_s^* \tilde{E}\) with nef \(\tilde{E}\) on \(X_s\). We have another expression \(E = E' \otimes O_X(B_1 + \cdots + B_{s-1})\) from (17) such that \(E'\) has non-negative coefficients in (22) and \(y_s = 0\), and

\[(E' \cdot F) = (E \cdot F) - 1\]
\[(E' \cdot D_0) = (E \cdot D_0) \geq 0.\]

If \((E' \cdot F) \geq 1\), then \(E' \cong \pi_s^* \tilde{E}\) with nef \(\tilde{E}\) on \(X_s\). If we set \((E' \cdot F) = (E \cdot D_0) = 0\), then all \(x_i = 0\) and \(y_j = 0\) for \(j \geq 2\) from the condition \(\beta'_j > \beta_j\) for \(j \geq 2\). This contradicts to \(y_s \neq 0\). Thus we have \((E' \cdot F) \geq 1\) or \((E \cdot D_0) \geq 1\). From this expression we see that the map \(\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)\) is surjective for this \(E\) with \(y_s = 1\) and \(L \cong \pi_s^* L\) with ample \(L\).
on $X_r$. By the induction on $y_r$ we see that the multiplication map is surjective for arbitrary nef $E$ and $L \cong \pi^* L$ with ample $L$ on $X_r$.

For an ample line bundle $L$ on $X$, we write

$$[L] = \sum_{i=1}^{l+1} \xi_i e_i + \sum_{j=1}^{m} \eta_j f_j + \zeta_1 g_1$$

with positive coefficients. When $\eta_j = 1$, we have $L \cong \pi^* L \times \mathcal{O}_X(F + B_1 + \ldots + B_{s-1})$ or $L \cong \pi^* L \times \mathcal{O}_X(B_1 + \ldots + B_{s-1})$ with ample $L$ on $X_s$ from the same reason as above. By the induction on $\eta_j$, we see that the statement holds in this case.

As the second step, we consider the case when $n \geq 2$ and $c_1 \geq 2, \ldots, c_{s-1} \geq 2, c_s = 1$ for $2 \leq t \leq n$. In general, we have an expression

$$g_1 = [D_0], \quad g_2 = (c_1 - 1)g_1 + [D_0 + C_1]$$

(25)

and for $k \geq 3$

$$g_k = (c_1 - 2)g_1 + \cdots + (c_{k-2} - 2)g_{k-2} + (c_{k-1} - 1)g_{k-1} + [D_0 + C_1 + \cdots + C_{k-1}]$$

Since $C_i$ is an exceptional curve of the first kind, i.e., $(C_i^2) = -1$, we have the contraction morphism $\rho_i : X \to X_i$ blowing down $C_i$. We assume that the statement holds for $X_i$. If a nef bundle $E$ is isomorphic to $\rho_i^* \mathcal{E}$ with nef $\mathcal{E}$ on $X_i$, then the multiplication map $\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$ is surjective for $L \cong \rho_i^* L$ with ample $L$ on $X_i$. Set $z_i > 0$ in the expression (22) of $E$. Then we have $(E \cdot F) \geq 1$ or $(E \cdot D_0) \geq 0$. In fact, if we set $(E \cdot F) = (E \cdot D_0) = 0$, then the equations (23) and (24) imply $z_i = 0$ from the condition $\beta_i > 0$ and $\gamma_i \geq 0$.

If $z_i = 1$, then we have $E \cong \rho_i^* \mathcal{E} \times \mathcal{O}_X(D_0 + C_1 + \cdots + C_{s-1})$ or $E \cong \rho_i^* \mathcal{E} \times \mathcal{O}_X(1 + \cdots + C_{s-1})$ with nef $\mathcal{E}$ on $X_i$. For $E$ with $z_i = 1$ and $L \cong \rho_i^* L$ with ample $L$ on $X_i$, hence, the multiplication map is surjective. By the induction on $z_i$, we see that the multiplication map of $\Gamma(L)$ and $\Gamma(E)$ is surjective for arbitrary nef $E$ and $L \cong \rho_i^* L$ with ample $L$ on $X_i$.

For an ample $L$ on $X$, we can also show the surjectivity of the multiplication map by using the same argument as in the first step.

We may restate this as the following.

**Corollary 2.** When $n = 1$, the multiplication map

$$\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$$

is surjective for an ample line bundle $L$ and a nef line bundle $E$ on $X$.

**Proof.** If we rename $F$ and $B_1$ in Proposition 5 as $D_0$ and $F$, then we have the corollary. 

Next we consider the case when $n \geq 2$.

**Lemma 10.** Set $n \geq 2$ and $t = \min\{k : (C_i^2) = -1\}$. Let $\rho_i : X \to X_i$ denote the birational morphism contracting $C_i$. For $L \cong \rho_i^* L$ with ample $L$ on $X_i$, then, the multiplication map

$$\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)$$

is surjective for a nef line bundle $E$ on $X$.

**Proof.** We will show by the induction on the number of $(−1)$-curves in $B_j$ and $C_k$.

When $n = 1$ and $b_1 \geq 2, \ldots, b_m \geq 2, b_m = 1$, the statement holds from Proposition 5. If $E$ has the coefficient $z_i = 0$ in the expression (22), then $E \cong \rho_i^* \mathcal{E}$ with nef $\mathcal{E}$ on $X_i$, hence the multiplication map is surjective. Set $s = \min\{j : (B_j^2) = -1\}$.

Consider the case when $n = 1$ and $b_1$ general. If $z_i = 1$ and $y_r > 0$, then we have $E \cong \rho_i^* \mathcal{E} \times \mathcal{O}_X(D_0 + F + B_1 + \cdots + B_{s-1})$ with nef $\mathcal{E}$ on $X_i$ from the expressions (17) and (25), hence the multiplication map is surjective. If $y_r = 0$, then $E \cong \pi^* \mathcal{E}^e$ such that $E^e$ is a nef bundle on $X_e$, where $\pi_e : X \to X_e$ is the contraction morphism of $B_e$. By the assumption of the induction, the statement holds for $X_e$. If $L$ has the coefficient $\eta_i = 1$ of $f_i$ in Pic$(X)$, then we have $L \cong \pi^* \mathcal{E} \times \mathcal{O}_X(F + B_1 + \cdots + B_{s-1})$ such that $L^e \cong \rho_i^* L$ with ample $L^e$ on $Y$, where $\rho_i : X_i \to Y$ is the contraction morphism of $C_i$. Since $\Gamma(L) \otimes \Gamma(E^e) \to \Gamma(L^e \otimes E^e)$ is surjective, by the induction on $\eta_i$, we have the surjectivity for $E$ with $z_i = 1$. By the induction on $z_i$, we have the surjectivity of the multiplication map.

Next we consider the case when $n \geq 2$. Set $u = \min\{k : (C_i^2) = -1\}$ on $X_u$. Let $\rho_u : X_r \to X_u$ denote the contraction morphism of $C_u$. We write $\rho = \rho_0 \circ \cdots \circ \rho_u : X \to X_u$. We will show that the multiplication map is surjective for $L \cong \rho^* L$ with ample $L$ on $X_u$, when $n = 2$.

If $z_t = 0$, then we have $E \cong \rho_t^* \mathcal{E}$ with nef $\mathcal{E}$ on $X_i$. If we consider $L$ and $E$ on $X_i$, then we have already shown it above.

Consider the case when $n = 0$. In this case we note that if $z_t > 0$, then $(E \cdot F) \geq 1$. Hence if $z_t = 1$, then we have $E \cong \rho_t^* \mathcal{E} \times \mathcal{O}_X(D_0 + C_1 + \cdots + C_{s-1})$ with nef $\mathcal{E}$ on $X_i$. Thus the map $\Gamma(\rho_t^* L) \otimes \Gamma(\mathcal{E}) \to \Gamma(\rho_t^* L \otimes \mathcal{E})$ is surjective from the above. Consequently, the multiplication map of $\Gamma(L)$ and $\Gamma(E)$ is surjective for nef $E$ with $z_t = 1$. By the
induction on \( z_t \), we have the surjectivity when \( m = 0 \).

Set \( m \geq 1 \). If \( z_t = 1 \) and \( y_t > 0 \), then we have \( E \cong \rho_t^* \bar{E} \otimes O_X(D_0 + F + B_1 + \cdots + B_{r-1}) \) with nef \( \bar{E} \) on \( X_t \), from the expressions (17) and (25), hence the multiplication map is surjective. If \( y_t = 0 \), then \( E \cong \pi_t^* E' \) such that \( E' \) is a nef bundle on \( X_t \). If \( L \) has the coefficient \( \eta_t = 1 \) of \( F_t \) in Pic(X), then we have \( L \cong \pi_t^* U \otimes O_X(F + B_1 + \cdots + B_{r-1}) \) such that \( U \) has positive coefficients except \( \xi_t = \bar{\xi_t} = 0 \). Since \( (L \cdot D_0) \geq 2 \) because \( m \geq 1 \) and \( \bar{L} \) is ample on \( X_{nr} \), we have \((L' \cdot F) = (L \cdot F) - 1 - (B_1^2) \geq 1 \) and \((L' \cdot D_0) = (L \cdot D_0) - 1 \geq 1 \), hence, \( L' \) is the pull back of an ample line bundle by the contraction morphism \( \bar{\rho} : X_t \to Y_t \) of \( \bar{C_t} \) and \( \bar{C_t} \). By the assumption of the induction, the surjectivity holds for \( X_t \), that is, the multiplication map of \( \Gamma(U') \) and \( \Gamma(E') \) is surjective. Thus we have the surjectivity of \( \Gamma(L) \otimes \Gamma(E) \to \Gamma(U \otimes E) \) for \( E \) with \( z_t = 1 \). By the induction on \( z_t \), we have a proof of the surjectivity of the multiplication map of \( \Gamma(\rho_t^* \bar{E}) \) and \( \Gamma(E) \) when \( n = 2 \).

From this we have the surjectivity of the multiplication map for \( L \cong \rho_t^* \bar{L} \) with ample \( \bar{L} \) on \( X_t \). As the same procedure, if \( z_t = 1 \) and \( y_t > 0 \), then we have \( E \cong \rho_t^* \bar{E} \otimes O_X(C_{r-1} + \cdots + C_1 + D_0 + F + B_1 + \cdots + B_{r-1}) \) with nef \( \bar{E} \) on \( X_t \), hence the multiplication map is surjective. If \( y_t = 0 \), then \( E \cong \pi_t^* E' \) such that \( E' \) is a nef bundle on \( X_t \). If \( L \) has the coefficient \( \eta_t = 1 \) and \( \xi_t > 0 \) in Pic(X), then we have \( L \cong \pi_t^* U \otimes O_X(C_{r-1} + \cdots + C_1 + D_0 + F + B_1 + \cdots + B_{r-1}) \) such that \( L' \cong \rho_t^* L' \) with ample \( L' \) on \( X_{nr} \), where \( \rho_t : X_t \to X_{nr} \) is the contraction morphism of \( C_t ' (C_t : \text{the strict transform of } C_t) \). Moreover, if \( \xi_t = 1 \), then \( L' \) is the pull back of an ample line bundle on \( Y_t \). From the above we have the surjectivity of \( \Gamma(L') \otimes \Gamma(E') \to \Gamma(U' \otimes E') \). By the induction on \( \eta_t \) we have the surjectivity for \( E \) with \( z_t = 1 \). By the induction on \( z_t \), we have a proof for \( L \cong \rho_t^* \bar{L} \) when \( n = 2 \).

For \( n \geq 3 \), we can use the induction on the number of \((-1\)-curves of \( B_t \) and \( C_t \) to prove the surjectivity for \( L \cong \rho_t^* \bar{L} \) with ample \( \bar{L} \) on \( Y_t \), i.e., \( z_t = \xi_t = 0 \) and next for \( L \cong \rho_t^* \bar{L} \) with ample \( \bar{L} \) on \( X_t \), i.e., \( z_t = 0 \).

From this lemma we have the main Theorem.

**Theorem 6.** Let \( X \) be a nonsingular projective toric variety of dimension two. For an ample line bundle \( L \) on \( X_t \) the multiplication map

\[
\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)
\]

is surjective.

**Proof.** Set \( N = \mathbb{P}^2 \). Let \( \Delta \) be a nonsingular complete fan in \( N \) such that \( X = T_{N \text{emb}}(\Delta) \). Since relatively minimal model of \( X \) is the projective plane \( \mathbb{P}^2 \) or the Hirzebruch surface \( \mathbb{F}_a = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(a)) \), we have an equivariant surjective morphism \( \pi : X \to \mathbb{F}_a \) of toric varieties unless \( X \cong \mathbb{P}^2 \). We may assume \( X \not\cong \mathbb{P}^2 \). Then we have the vertical line in the figure of \( \Delta \) in \( N \) corresponding the strict transform of a generic fiber of \( \mathbb{F}_a \). Set the lower part of the line \( D_0 \) and one of adjacent edges \( F \). Since \( X \) is nonsingular, we can take the edge \( F \) as horizontal. Thus we may assume that \( \Delta \) is given as the figure in this section.

Proposition 5, or equivalently Corollary 2, proves the case when \( n = 1 \) in the relations (19) and (20). When \( n \geq 2 \), Lemma 10 says that the multiplication map

\[
\Gamma(L) \otimes \Gamma(E) \to \Gamma(L \otimes E)
\]

is surjective for any nef line bundle \( E \) on \( X_t \) and \( L \cong \rho_t^* \bar{L} \) with ample \( \bar{L} \) on \( X_t \). Let \( L \) be an ample line bundle on \( X_t \) with the coefficient \( z_t = 1 \) of \( F_t \) in Pic(X). If \( L \) has the coefficient \( \eta_t > 0 \), then we have \( L \cong \rho_t^* L \otimes O_X(C_{r-1} + \cdots + C_1 + D_0 + F + B_1 + \cdots + B_{r-1}) \) with ample \( L' \) on \( X_{nr} \). From Lemma 10, we have the surjectivity.

If \( \eta_t = 1 \), then \( L' \) is the pull back of an ample line bundle on \( X_{nr} \). If \( E \) has the coefficient \( y_t = 0 \) in the expression (22), then we have \( E \cong \pi_t^* E' \) with nef \( F \) on \( X_t \). Then we have the surjectivity by applying Lemma 10 for \( X_t \). If \( E \) has the coefficient \( z_t = 0 \), then we have \( E \cong \rho_t^* \bar{E} \) with nef \( \bar{E} \) on \( X_t \). We have again the surjectivity from Lemma 10 by exchanging the role of \( B_t \) and \( C_t \). If \( y_t = 1 \) and \( z_t \geq 1 \), then we have \( E \cong \pi_t^* E' \otimes O_X(C_{r-1} + \cdots + C_1 + D_0 + F + B_1 + \cdots + B_{r-1}) \) with nef \( E \) on \( X_t \). By applying Lemma 10 for \( X_t \), we have the surjectivity of the multiplication map of \( \rho_t^* \bar{L} \) and \( \pi_t^* E' \), hence, \( \bar{E} \) with \( y_t = 1 \). By the induction on \( y_t \), we have the surjectivity for \( L \) with \( \eta_t = 1 \). By the induction on \( \eta_t \), we have a proof.

6. An Application

A nonsingular projective variety \( Y \) is called Fano if the anti-canonical divisor \(-K_Y \) is ample. We can easily show that the anti-canonical bundle on a nonsingular toric Fano 3-fold \( X \) is normally generated, that is, the multiplication map

\[
\Gamma(\Theta_X(-K_X)) \otimes \Gamma(\Theta_X(-IK_X)) \to \Gamma(\Theta_X(-(l + 1)K_X))
\]

is surjective for all \( l \geq 1 \). Set \( L = \Theta_X(-K_X) \). Take general members \( S_1, S_2 \in |-K_X| \). Set \( C := S_1 \cap S_2 \). Then \( K_C = L_C \).

Since an ample line bundle on a nonsingular toric variety is very ample, \( L \) hence \( K_C \) is very ample. In other words, \( C \) is not hyperelliptic. By Noether’s theorem \( K_C = L_C \) is normally generated. By Fujita’s ladder theorem [2] we see that \( L = \Theta_X(-K_X) \) is normally generated.

Here we know more.
Proposition 6. Let $X$ be a nonsingular toric Fano 3-fold and $B$ a nef line bundle on $X$. Then the multiplication map
\[
\Gamma(B) \otimes \Gamma(\Theta(-K_X)) \rightarrow \Gamma(B \otimes \Theta_X(-K_X))
\] (26)
is surjective.

Proof. Let $D = \sum_i D_i$ be the divisor consisting of all $T_N$-invariant irreducible divisor on $X$. Then $\Theta_X(D) \cong \Theta_X(-K_X)$. Since $D$ is ample, the bundle $B(D)$ is also ample. Hence it corresponds to an integral convex polytope $P$ of dimension three in $M_\mathbb{R} \cong \mathbb{R}^3$ such that
\[
\Gamma(B(D)) \cong \bigoplus_{m \in P \cap M} \mathbb{C} e(m),
\] (27)
where $e(m)$ denote the character of the algebraic torus $T_N$ corresponding to $m$ in $M = \text{Hom}_{\mathbb{R}}(T_N, \mathbb{C}^*)$. The restriction $B(D)_{D_i}$ corresponds to a face $F_i \subset P$ of dimension two.

Since the vector space $\Gamma(B(D + K_X)) \cong \Gamma(B)$ has a basis consisting of $[e(m); m \in \text{Int}(P) \cap M]$, the space of the global sections $\Gamma(D, B(D))$ is parametrized by the lattice points in the boundary of $P$. A boundary lattice point $m$ in $\partial P \cap M$ is contained in a face $F_i$ of $P$. In other words, $e(m) \in \Gamma(D, B(D)_{D_i})$ is contained in $\Gamma(D_i, B(D)_{D_i})$ for some $D_i$. Since $B$ is globally generated, the restriction $B_{D_i}$ is also globally generated. From Theorem 1, on a nonsingular toric surface $D$, the multiplication map of $\Gamma(B_{D_i})$ and $\Gamma(\Theta_{D_i}(D))$ is surjective. Hence we can find $e(m_1) \in \Gamma(D_i, B_{D_i})$ and $e(m_2) \in \Gamma(D_i, \Theta_{D_i}(D))$ such that $m_1 + m_2 = m$. If we know the surjectivity of $\Gamma(X, B) \rightarrow \Gamma(D_i, B_{D_i})$, we can see that
\[
\Gamma(X, B) \otimes \Gamma(D, \Theta_{D}(D)) \rightarrow \Gamma(D, B(D))
\] (28)
is surjective.

We need a lemma.

Lemma 11. Let $B$ be a globally generated line bundle on a projective toric variety $X$. Let $D_0$ be an irreducible $T_N$-invariant divisor on $X$. Then we have
\[
H^j(X, B(-D_0)) = 0 \quad \text{for} \quad j \geq 1.
\]

Proof. From [5] we have an equivariant surjective morphism $\pi : X \rightarrow Y$ to a toric variety $Y$ with connected fibers and an ample line bundle $A$ on $Y$ such that $B \cong \pi^* A$. By definition $\pi_* \Theta_X \cong \Theta_Y$ and $R^j \pi_* \Theta_X = 0$ for $j \geq 1$. Set $\pi(D_0) = E$. Then $E$ is an irreducible $T$-invariant subvariety of $Y$. Thus we have $\pi_* \Theta_{D_0} \cong \Theta_E$. By taking the direct image of the exact sequence
\[
0 \rightarrow \Theta_X(-D_0) \rightarrow \Theta_X \rightarrow \Theta_{D_0} \rightarrow 0
\]
we see that $\pi_* \Theta_X(-D_0) \cong I_E$ and $R^j \pi_* \Theta_X(-D_0) = 0$ for $j \geq 1$, where $I_E$ denotes the ideal sheaf of $E$ on $Y$. Thus we have
\[
H^j(X, B(-D_0)) = H^j(X, \pi^* A \otimes \Theta_X(-D_0)) \cong H^j(Y, A \otimes I_E).
\]
We claim that $H^j(Y, A \otimes I_E) = 0$ for $j \geq 1$: Since $H^j(Y, A) = H^j(E, A_E) = 0$ for $j \geq 1$, it is enough to show the surjectivity of the restriction map $\Gamma(Y, A) \rightarrow \Gamma(E, A_E)$. If $E$ is a divisor of $Y$, then it is obvious because if the polarized toric variety $(Y, A)$ corresponds to an integral convex polytope $Q$, whose lattice points form a basis of $\Gamma(Y, A)$, then the polarized toric variety $(E, A_E)$ corresponds to a face of $Q$, whose lattice points also form a basis of $\Gamma(E, A_E)$. If not, take an irreducible divisor $E_0$ on $Y$ containing $E$. Then we can factor the restriction map as
\[
\Gamma(Y, A) \rightarrow \Gamma(E_0, A_{E_0}) \rightarrow \Gamma(E, A_E).
\]
We know that the restriction map in the left hand side is surjective. By induction on the codimension of $E$ in $Y$ we can verify the claim. \qed

We continue proving Proposition 6. From Lemma 11 we see the surjectivity of the restriction map $\Gamma(X, B) \rightarrow \Gamma(D_i, B_{D_i})$, hence, the map (28) is surjective. By tensoring $\Gamma(B)$ with the global sections of the exact sequence
\[
0 \rightarrow \Theta_X \rightarrow \Theta_X(D) \rightarrow \Theta_{D}(D) \rightarrow 0,
\]
the surjectivity of (26) follows. \qed

We note that if $D = \sum_i D_i$ is nef, then the restriction map $\Gamma(X, \Theta_X(D)) \rightarrow \Gamma(D_i, \Theta_{D_i}(D))$ is surjective. By the same argument, hence, we see that the multiplication map
\[
\Gamma(X, A) \otimes \Gamma(D, \Theta_{D}(D)) \rightarrow \Gamma(D, A(D)_{D})
\]
is surjective for an ample line bundle $A$ and a nef divisor $D = \sum_i D_i$. Thus we have the following proposition.
Proposition 7. Let $X$ be a nonsingular projective toric 3-fold. Assume that a reduced $T$-invariant divisor $D = \sum D_i$ is nef, in particular, when the anti-canonical divisor $D = -K_X$ is nef. Then for any ample line bundle $A$ on $X$, the multiplication map
\[
\Gamma(A) \otimes \Gamma(O_X(D)) \to \Gamma(A \otimes O_X(D))
\]
is surjective.

REFERENCES