A Bifurcation Phenomenon for the Periodic Solutions of the Duffing Equation without Damping Terms

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We investigate a bifurcation phenomenon for the periodic solutions of the Duffing equation without damping terms:

$$\frac{d^2u}{dt^2}(t) + \kappa u(t) + \alpha u^3(t) = f_\lambda(t), \quad t \in \mathbb{R}$$

(1.1)

where $\kappa$ and $\alpha$ are positive constants and $f_\lambda(t)$ ($\lambda > 0$) is a given family of $T$-periodic external force parameterized by $\lambda$. We show an existence of not only $T$ and exact $2T$-periodic solutions but also exact $mT$-periodic solutions ($m \geq 3$) bifurcated from a specific $T$-periodic solution.

KEYWORDS: bifurcation, Duffing equation, damping term, periodic solution

1. Introduction

The Duffing equation without damping terms has the form

$$\frac{d^2u}{dt^2}(t) + \kappa u(t) + \alpha u^3(t) = f_\lambda(t), \quad t \in \mathbb{R}$$

(1.1)

where $\kappa$ and $\alpha$ are positive constants and $f_\lambda(t)$ ($\lambda > 0$) is a given family of $T$-periodic external force parameterized by $\lambda$.

In the case where $\kappa = 0.1$, $\alpha = 1$ and $f_\lambda(t) = \lambda \cos(2\pi t)$ or in the case where $\kappa = 1$, $\alpha = 1$ and $f_\lambda(t) = \lambda(\cos(2\pi t) + 0.5)$, by numerical computations, it seems to us that not only a period-doubling bifurcation phenomenon occurs but also a “period-quadruple” bifurcation phenomenon and a “period-octuple” bifurcation phenomenon do.

Therefore, in this paper, we aim to prove the existence of bifurcation phenomenon as above for (1.1) rigorously.

For the Duffing equation which includes a damping term:

$$\frac{d^2u}{dt^2}(t) + \mu \frac{du}{dt} + \kappa u(t) + \alpha u^3(t) = f_\lambda(t), \quad t \in \mathbb{R}$$

(1.2)

where $\mu$, $\kappa$ and $\alpha$ are positive constants and $f_\lambda(t)$ ($\lambda > 0$) is a given family of $T$-periodic external force parameterized by $\lambda$, it is well-known that for any $\lambda$ there exists at least one $T$-periodic solution of (1.2). So, Komatsu–Kotani–Matsumura [9] have tried to detect a bifurcation phenomenon around a “linear probe” $\{(\lambda, u_{\lambda})\}_{\lambda > 0}$ inserted into the product space $(\lambda, u)$, which is defined by

$$u_{\lambda}(t) := \lambda U(t), \quad U(t): a \text{ given } T\text{-periodic smooth function},$$

$$f_\lambda(t) := \frac{d^2u_{\lambda}}{dt^2}(t) + \mu \frac{du_{\lambda}}{dt} + \kappa u_{\lambda}(t) + \alpha u_{\lambda}^3(t).$$

(1.3)

Here we note that $u = u_1$ is a trivial solution of (1.2) corresponding to $f_\lambda$ for any $\lambda$. They have shown an existence of $T$ and exact $2T$-periodic solutions and nonexistence of $mT$-periodic solutions ($m \geq 3$) bifurcated from this specific $T$-periodic solution $u_1(t)$.

Let us return to our subject. An existence of periodic solutions of (1.1) has been shown under suitable situations (for example, see [1,4–8]). So, we also try to detect a bifurcation phenomenon around a “linear probe” $\{(\lambda, u_{\lambda})\}_{\lambda > 0}$ inserted into the product space $(\lambda, u)$, which is defined by

$$u_{\lambda}(t) := \lambda U(t), \quad U(t): a \text{ given } T\text{-periodic smooth function},$$

$$f_\lambda(t) := \frac{d^2u_{\lambda}}{dt^2}(t) + \kappa u_{\lambda}(t) + \alpha u_{\lambda}^3(t).$$

(1.4)

Here we note that $u = u_1$ is also a trivial solution of (1.1) corresponding to $f_\lambda$ for any $\lambda$. In this paper, we will show an existence of not only $T$ and exact $2T$-periodic solutions but also exact $mT$-periodic solutions ($m \geq 3$) bifurcated from this specific $T$-periodic solution $u_1(t)$.
Therefore, the bifurcation phenomenon for the Duffing equation without damping terms is different from that for the Duffing equation which includes a damping term.

The main theorem is stated precisely in Sect. 2. The proof of our theorem is similar to that in [9]. In Sect. 3, we first reduce the bifurcation problem above to the linearized eigenvalue problem of Sturm–Liouville equation by Crandall–Rabinowitz’s theorem on bifurcation theory (see [3]). Section 4 is devoted to Sturm–Liouville eigenvalue problems with the extended periodic boundary conditions (see [2]). We define the function \( \Delta(\Lambda) \) and prove that the eigenvalues occur at the roots of \( \Delta(\Lambda) = 2 \).

In Sect. 5, we investigate the asymptotic behavior of \( \Delta(\Lambda) \) as \( \Lambda \to \infty \). Using the results of Sect. 5, we prove our theorem in Sect. 6.

2. Main Theorem

We state our main theorem precisely.

We consider the Duffing equation without damping terms:

\[
\frac{d^2u}{dt^2}(t) + \kappa u(t) + \alpha u^3(t) = f_\lambda(t), \quad t \in \mathbb{R}
\]  
(2.1)

where \( \kappa \) and \( \alpha \) are positive constants and \( f_\lambda(t) \) is a \( \lambda \)-periodic smooth function. Here we note that \( u = u_\lambda \) is a trivial solution of (2.1) corresponding to \( f_\lambda \) for any \( \lambda \).

Moreover we assume that \( U^2(t) \) has \( N + 1 \) zero points \( \{t_i\}_{i=0}^N \) of \( n \)-th order on \( [t_0, t_0 + T] \) where \( N = 1 \) or \( 2 \) and \( t_0 < t_1 < \cdots < t_N = t_0 + T \) and set \( S_i := \int_{t_{i-1}}^{t_i} |U(s)|ds \).

In this paper, if a periodic solution has a period \( mT \), but not any of \( iT \ (1 \leq i \leq m - 1) \), this \( mT \)-periodic solution is called an exact \( mT \)-periodic solution.

Then our theorem is stated as follows:

**Theorem 2.1.** Under the above assumptions, for any \( m \ (m \in \mathbb{N}) \), there exist countably infinite exact \( mT \)-periodic solutions bifurcated from the probe \( \{u_\lambda\}_{\lambda > 0} \) except for the case where \( N = 2, m = 2 \) and \( S_1 = S_2 \).

**Remark 2.1.** (1) The bifurcation phenomenon for the Duffing equation without damping terms is different from that for the Duffing equation which includes a damping term.

(2) For the case where \( N = 2, m = 2 \) and \( S_1 = S_2 \), we do not know whether the bifurcation phenomenon occurs or not.

3. Reduction of the Problem

Arguments which we use in this section are similar to those in [9]. So, we state only the following lemma corresponding to [9, Lemma 3.2] which is used to prove our theorem.

As in [9], we look for the periodic solution of (2.1) in the form:

\[
u(t) = u_\lambda(t) + \lambda v(t) \]

(3.1)

where \( v(t) \) is a \( mT \)-periodic function. From (2.1), (2.2) and (3.1), \( v(t) \) must be a solution of the periodic problem:

\[
\begin{align*}
\frac{d^2v}{dt^2}(t) + \kappa v(t) + \lambda \left( U^2(t)v(t) + U(t)v^2(t) + \frac{1}{3} v^3(t) \right) &= 0, \\
v(t + mT) &= v(t), \quad t \in \mathbb{R}
\end{align*}
\]  
(3.2)

where \( \Lambda := 3\alpha \lambda^2 \). The bifurcation problem to (3.2) around the trivial solution \( v(t) \equiv 0 \) are reduced to the following linearized eigenvalue problem of (3.2) at \( v = 0 \) by the bifurcation theorem in Crandall–Rabinowitz [3, Theorem 1.7].

Namely, we have

**Lemma 3.1.** The existence of a \( mT \)-periodic solution bifurcated from \( u_\lambda(t) \) is reduced to the existence of \( \Lambda_0 \) which satisfies the following conditions:

(1) \( \Lambda = \Lambda_0 \) is a positive eigenvalue of the following linearized eigenvalue problem of (3.2) at \( v = 0 \):
The solution space of (3.3) at $\Lambda = \Lambda_0$ is one dimensional. Namely, the eigenvalue $\Lambda = \Lambda_0$ is simple.

4. Extended Oscillation Theorem

In this section, we consider the eigenvalue problem of the following Sturm–Liouville equations with the extended periodic boundary conditions:

$$\begin{align*}
\begin{cases}
(p(t)x')' + (\lambda r(t) - q(t))x = 0, \\
(x(0)) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x(T))
\end{cases}
\end{align*}
$$

where $\lambda$ is a real parameter and $\alpha, \beta, \gamma$ and $\delta$ are real constants such that $\alpha \delta - \beta \gamma = 1$ and $p', r$ and $q$ are real valued continuous functions on $[0, T]$ and $p(t) > 0, r(t) \geq 0$ on $[0, T]$. We allow the function $r$ to have zero points as isolated points on $[0, T]$. Moreover, it will be assumed that $p(0) = p(T)$. For brevity, we assume $p(0) = p(T) = 1$.

Let $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ be the solutions of the equation of (4.1) satisfying

$$\begin{pmatrix}
\varphi(0, \lambda) \\
\varphi'(0, \lambda)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix},$$

Let us define the function $\Delta(\lambda)$ as follows:

$$\Delta(\lambda) := \alpha \varphi(T, \lambda) + \beta \psi'(T, \lambda) + \gamma \psi(T, \lambda) + \delta \psi'(T, \lambda).$$

Then we have the following extended oscillation theorem which is essentially stated in [2, Chapt. 8, Problem 4].

**Theorem 4.1.**

1. $\lambda = \lambda_0$ is an eigenvalue of (4.1), if and only if

$$\Delta(\lambda_0) = 2.$$  

2. $\lambda = \lambda_0$ is a simple eigenvalue of (4.1), if and only if

$$\Delta(\lambda_0) = 2 \quad \text{and} \quad \frac{d\Delta}{d\lambda}(\lambda_0) \neq 0.$$  

3. If an eigenvalue $\lambda = \lambda_0$ of (4.1) is not simple, it follows that

$$\frac{d^2\Delta}{d\lambda^2}(\lambda_0) < 0.$$  

The following corollary is an immediate consequence of the above theorem.

**Corollary 4.1.** If $\lambda_1 < \lambda_2$ and $(\Delta(\lambda_1) - 2)(\Delta(\lambda_2) - 2) < 0$, then there exists a simple eigenvalue $\lambda_0$ of (4.1) in $(\lambda_1, \lambda_2)$.

We prove Theorem 4.1 at the end of this section.

In order to apply Corollary 4.1 to Lemma 3.1, we set $p(t)$, $q(t)$, $r(t)$, $\lambda$ and $(\alpha, \beta, \gamma, \delta)$ as follows:

$$\begin{align*}
p(t) &:= 1, \\
q(t) &:= -\kappa, \\
r(t) &:= U^2(t), \\
\lambda &:= \Lambda, \\
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &:= \begin{pmatrix} \cos(2\pi k/m) & -\sin(2\pi k/m) \\ \sin(2\pi k/m) & \cos(2\pi k/m) \end{pmatrix}
\end{align*}$$

where $k \in \{1, 2, \ldots, m\}$ and $\gcd(k, m) = 1$. We note that (4.7) satisfies $\alpha \delta - \beta \gamma = 1$ and the solution of (4.1) in the case of (4.7) satisfies the boundary conditions of (3.3). Let $\Phi(t) := \begin{pmatrix} \Phi(t, \lambda_1) \\ \Phi(t, \lambda_2) \end{pmatrix}$ be the fundamental matrix for the equation of (4.1) in the case of (4.7) with initial condition $\Phi(0) = E$. Then the function $\Delta(\Lambda)$ corresponding to (4.3) is defined to be
\[ \Delta(\Lambda) := \alpha \phi_1(T, \Lambda) + \beta \phi'_1(T, \Lambda) + \gamma \phi_2(T, \Lambda) + \delta \phi'_2(T, \Lambda) \quad (4.8) \]

where \( \alpha, \beta, \gamma \) and \( \delta \) are defined by (4.7).

Taking notice the condition \( \gcd(k, m) = 1 \), we have the following proposition corresponding to Corollary 4.1.

**Proposition 4.1.** If \( \Lambda_1 < \Lambda_2 \) and \((\Delta(\Lambda_1) - 2)(\Delta(\Lambda_2) - 2) < 0\), then there exists a simple eigenvalue \( \Lambda_0 \) of (4.1) in the case of (4.7) in \((\Lambda_1, \Lambda_2)\). The eigenfunction corresponding to \( \Lambda_0 \) is exact \( mT \)-periodic solution of (3.3).

Next two sections, we investigate the asymptotic behavior of \( \Delta(\Lambda) \) as \( \Lambda \to \infty \) and apply Proposition 4.1 to prove Lemma 3.1.

**Proof of Theorem 4.1.** Since the theorem is essentially stated and proved in [2], we only sketch the proof briefly.

For the boundary conditions of (4.1) to hold, it is necessary and sufficient that there exist constants \((C_1, C_2) \neq (0, 0)\) such that \( C_1 \psi + C_2 \psi \) satisfies the boundary conditions of (4.1), which yields

\[
\begin{pmatrix}
\alpha \psi(T, \Lambda) + \beta \psi'(T, \Lambda) - 1 & \alpha \psi(T, \Lambda) + \beta \psi'(T, \Lambda) \\
\gamma \psi(T, \Lambda) + \delta \psi'(T, \Lambda) & \gamma \psi(T, \Lambda) + \delta \psi'(T, \Lambda) - 1
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

(4.9)

A necessary and sufficient condition for (4.9) to have a nontrivial solution is that the determinant of the coefficients matrix of (4.9) should vanish, which is equivalent to (4.4).

Furthermore, a necessary and sufficient condition for two independent solutions to satisfy the boundary condition of (4.1) is

\[
\begin{pmatrix}
\alpha \psi(T, \Lambda) + \beta \psi'(T, \Lambda) & \alpha \psi(T, \Lambda) + \beta \psi'(T, \Lambda) \\
\gamma \psi(T, \Lambda) + \delta \psi'(T, \Lambda) & \gamma \psi(T, \Lambda) + \delta \psi'(T, \Lambda)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

(4.10)

Thus, \( \Lambda = \Lambda_0 \) is a simple eigenvalue, if and only if (4.4) holds and (4.10) does not hold for \( \Lambda_0 \).

On the other hand, we have

\[
d\Delta = \int_0^T \left[ (\gamma \psi(T) + \delta \psi'(T)) \psi^2(\tau) + (\alpha \psi(T) + \beta \psi'(T)) \right] d\tau.
\]

(4.11)

The bracket in (4.11) does not change the sign if

\[
(\alpha \psi(T) + \beta \psi'(T) - (\gamma \psi(T) + \delta \psi'(T)))^2 + 4(\gamma \psi(T) + \delta \psi'(T))(\alpha \psi(T) + \beta \psi'(T)) \leq 0.
\]

(4.12)

(4.12) is equivalent to

\[
\Delta(\lambda)^2 - 4 \leq 0.
\]

(4.13)

Thus if \( \Delta(\lambda) = 2 \), the bracket in (4.11) has a fixed sign. Therefore \( d\Delta(\lambda)/d\lambda \) cannot vanish unless the bracket is identically zero, which is equivalent to the condition (4.10) if \( \Delta(\lambda) = 2 \). Thus (4.4) holds and (4.10) does not hold for \( \Lambda_0 \), if and only if \( \Delta(\Lambda_0) = 2 \) and \( d\Delta(\Lambda_0)/d\Lambda \neq 0 \). Therefore, we have (4.5).

It remains only to prove (4.6). If an eigenvalue \( \lambda = \Lambda_0 \) of (4.1) is not simple, it follows that

\[
d^2\Delta/d\lambda^2(\Lambda_0) = 2 \left( \psi_1(T, \Lambda_0) \psi'_1(T, \Lambda_0) - \psi'_1(T, \Lambda_0) \psi_1(T, \Lambda_0) \right)
\]

\[= 2 \left( \left( \int_0^T \psi(\tau, \Lambda_0) \psi(\tau, \Lambda_0) r(\tau) d\tau \right)^2 - \int_0^T \psi^2(\tau, \Lambda_0) r(\tau) d\tau \int_0^T \psi^2(\tau, \Lambda_0) r(\tau) d\tau \right). \]

(4.14)

Since \( \psi \) and \( \varphi \) are independent, the Schwarz inequality implies that (4.14) is negative, which proves (4.6).

This completes the proof of Theorem 4.1.

\[ \square \]

5. **Asymptotic Behavior of \( \Delta(\Lambda) \)**

First, we set \( R_\lambda(t, s) := \Phi_\lambda(t) \Phi_\lambda^{-1}(s) \) and \( \rho(t) := U^2(t) \). Without loss of generality, we may assume that zero points of \( \rho(t) \) are \( t_0 = t_1 < t_2 < \cdots < t_\nu = T \).

In the case of \( N = 1 \), \( \rho(t) \) has two zero points of \( n \)-th order on \([0, T]\). So, there exist \( \xi, \xi \geq 1 \) such that

\[
\rho(t) = C_1 t^\xi (1 + C_2 t + O(t^{\xi+1})) \quad \text{as} \quad t \to +0,
\]

\[
\rho(t) = C_1 (T - t)^\xi (1 + C_2 (T - t)^\xi + O((T - t)^{\xi+1})) \quad \text{as} \quad t \to T - 0.
\]

(5.1)

We define \( \tilde{\rho}(t) := \rho(T - t) \), and the fundamental matrix for
by \( \Phi_A(t) = (\hat{\Phi}(t, \Lambda), \hat{\Phi}(t, \Lambda)) \) with initial condition \( \Phi_A(0) = E \). Then, making use of \( \Phi_A(t) \), we have

\[
\Phi_A(T) = R_A[T/2, T]^{-1}R_A[T/2, 0]
\]

\[
= \begin{pmatrix}
\hat{\phi}_2(T/2, \Lambda) & \hat{\phi}_2(T/2, \Lambda) \\
\hat{\phi}_1(T/2, \Lambda) & \hat{\phi}_1(T/2, \Lambda)
\end{pmatrix}
\begin{pmatrix}
\phi_1(T/2, \Lambda) & \phi_2(T/2, \Lambda) \\
\phi_1(T/2, \Lambda) & \phi_2(T/2, \Lambda)
\end{pmatrix},
\]

From (4.8) and (5.3), we have

\[
\Delta(\Lambda) = \alpha \left( \phi_2(T/2, \Lambda) \phi_1(T/2, \Lambda) + \phi_2(T/2, \Lambda) \phi_1(T/2, \Lambda) \right)
\]

\[
+ \beta \left( \phi_2(T/2, \Lambda) \phi_1(T/2, \Lambda) + \phi_1(T/2, \Lambda) \phi_1(T/2, \Lambda) \right)
\]

\[
+ \gamma \left( \phi_2(T/2, \Lambda) \phi_2(T/2, \Lambda) + \phi_2(T/2, \Lambda) \phi_2(T/2, \Lambda) \right)
\]

\[
+ \delta \left( \phi_1(T/2, \Lambda) \phi_2(T/2, \Lambda) + \phi_1(T/2, \Lambda) \phi_1(T/2, \Lambda) \right).
\]

We may consider \( \{\phi(T/2, \Lambda)\}_{i=1,2} \), since similar arguments hold for \( \{\hat{\phi}(T/2, \Lambda)\}_{i=1,2} \). On the interval \([0, T/2] \), we introduce the following change of variable and function, so-called Liouville transformation:

\[
\text{variable: } x = \int_0^t \sqrt{\rho(s)} ds,
\]

\[
\text{function: } g(x) = \rho(t)^{1/4} v(t).
\]

By this transformation, the equation of (3.3) is reduced to

\[
\frac{d^2 g}{dx^2}(x) + (\Lambda - Q(x)) g(x) = 0
\]

where \( Q(x) := -\kappa \rho^{-1}(t) - \rho^{-3/4}(t) (\rho^{-1/4})' \).

Let us set \( \Phi_i(x) := \rho^{1/4}(t) \phi_i(t) \) \((i = 1, 2) \). Then \( \{\Phi_i(x)\}_{i=1,2} \) satisfy (5.7). It holds from (5.1) that

\[
Q(i) = -\frac{n(n + 4)}{16} C_i^{-1} \Gamma^{-(n+2)} \left\{ 1 + \frac{16\kappa}{n(n + 4)^2} - \frac{n^2 + (-2\xi + 4)n + 4\xi(\xi - 1)}{n(n + 4)} C_2^\xi + O(t^{\xi+1}) \right\}
\]

as \( t \to 0 \). According to (5.5), we have the relation \( t \) and \( x \)

\[
t = C_i^{-1/(n+2)} \left( \frac{2}{n + 2} \right)^{2/(n+2)} x^{2/(n+2)}
\]

\[
\times \left\{ 1 \frac{1}{n + 2 + 2\xi} + C_i^{1-\xi/(n+2)} \left( \frac{2}{n + 2} \right)^{-2\xi/(n+2)} x^{2\xi/(n+2)} + O(x^{2\xi/(n+2)}) \right\}
\]

as \( x \to 0 \). Combining (5.8) with (5.9), we have the behavior of \( Q(x) \) near \( x = 0 \),

\[
Q(x) = Q_0(x) \left\{ 1 + \frac{16\kappa}{n(n + 4)} C_i^{2v(n+4)^2} x^{2\xi} \right. - \frac{8\xi(\xi - 1)}{n(n + 4)(n + 2\xi + 2)} C_i^{2\xi} x^{2\xi} + O(x^{2\xi+1}) \right\}
\]

as \( x \to 0 \), where \( Q_0(x) := -n(n + 4)v^2/(4\xi^2) \) and \( v := 1/(n + 2) \).

Instead of (5.7), we consider the following equation:

\[
\frac{d^2 g}{dx^2}(x) + (\Lambda - Q_0(x)) g(x) = 0.
\]

Then the solutions of (5.11) are explicitly given by \( A_n \lambda^{v/2} \sqrt{J_{-v}(\sqrt{\lambda}x)} \) and \( B_n \lambda^{-v/2} \sqrt{J_v(\sqrt{\lambda}x)} \), where \( J_v \) is the Bessel function of order \( v \) and \( A_n, B_n \) are determined so that \( \{\phi(T/2, \Lambda)\}_{i=1,2} \) satisfies the initial condition \( \phi_A(0) = E \), that is,

\[
A_n = \frac{1}{\sqrt{2}} \Gamma(1 - v)(n + 2)^{v/2} C_i^{v/2},
\]

\[
B_n = \frac{1}{\sqrt{2}} \Gamma(v)(n + 2)^{-v/2} C_i^{-v/2}.
\]

Making use of these solutions, we note that \( \{\Phi_i(x)\}_{i=1,2} \) also satisfy the following integral equation:
\[
\Phi_1(x) = \Lambda^{-1/(2\nu)}A(\sqrt{\Lambda}x) + \frac{1}{\sqrt{\Lambda}} \int_0^y \left( A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x) - A(\sqrt{\Lambda)x)B(\sqrt{\Lambda}s) \right) \tilde{Q}(s) \Phi_1(s)ds,
\]
\[
\Phi_2(x) = \Lambda^{-1/(2\nu)}B(\sqrt{\Lambda}x) + \frac{1}{\sqrt{\Lambda}} \int_0^y \left( A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x) - A(\sqrt{\Lambda)x)B(\sqrt{\Lambda}s) \right) \tilde{Q}(s) \Phi_2(s)ds
\]
where \(\tilde{Q}(x) := Q(x) - Q_0(x), A(y) := A_n\sqrt{J_{-1}(y)} \) and \(B(y) := B_n\sqrt{J_0(y)} \). Taking notice that
\[
A_n B_n \frac{2}{\pi} = \frac{1}{\sin \nu \pi}
\]
and \(A(y), B(y)\) have the asymptotic properties
\[
A(y) = A_n \sqrt{\frac{2}{\pi}} \cos \left( y - \frac{1 - 2\nu}{4} \pi \right) + o(y^0) \quad \text{as } y \to \infty,
\]
\[
B(y) = B_n \sqrt{\frac{2}{\pi}} \cos \left( y - \frac{1 + 2\nu}{4} \pi \right) + o(y^0) \quad \text{as } y \to \infty,
\]
we have the following lemma.

**Lemma 5.1.** \(\Phi_1(x)\) satisfies that
\[
|\Phi_1(x) - \Lambda^{-1/(2\nu)}A(\sqrt{\Lambda}x)| = o(\Lambda^{-1/(2\nu)}) \quad \text{as } \Lambda \to \infty,
\]
for any fixed \(x\). \(\Phi_2(x)\) satisfies that
\[
|\Phi_2(x) - \Lambda^{-1/(2\nu)}B(\sqrt{\Lambda}x)| = o(\Lambda^{-1/(2\nu)}) \quad \text{as } \Lambda \to \infty,
\]
for any fixed \(x\).

**Proof.** We consider only \(\Phi_1(x)\), since similar arguments hold for \(\Phi_2(x)\).

Let us define the successive approximations \(\{\Phi_1^{(l)}(y)\}_{l=0}^\infty\) by
\[
\begin{align*}
\Phi_1^{(0)}(y) &:= \Lambda^{-1/(2\nu)}A(y), \\
\Phi_1^{(l)}(y) &:= \frac{1}{\Lambda} \int_0^y (A(z)B(y) - A(y)B(z)) \tilde{Q}\left( \frac{z}{\sqrt{\Lambda}} \right) \Phi_1^{(l-1)}(z)dz, \quad \text{for } l \geq 1,
\end{align*}
\]
where \(y := \sqrt{\Lambda}x\).

First, we deal with the case of \(\xi > 1\). From (5.18) with \(l = 1\), it follows that
\[
|\Phi_1^{(1)}(y)| = \frac{1}{\Lambda} \int_0^y (A(z)B(y) - A(y)B(z)) \tilde{Q}\left( \frac{z}{\sqrt{\Lambda}} \right) \Lambda^{-1/(2\nu)}A(z)dz
\]
\[
\leq \frac{1}{\Lambda} \int_0^y \left( |A(z)||B(y)| + |A(y)||B(z)| \right) \tilde{Q}\left( \frac{z}{\sqrt{\Lambda}} \right) \Lambda^{-1/(2\nu)}|A(z)|dz.
\]
Let \(y_0 > 0\) be any fixed number. From (5.10), there exist a positive constant \(\tilde{C}_Q\) such that the inequality
\[
\left| \tilde{Q}\left( \frac{z}{\sqrt{\Lambda}} \right) \right| \leq \tilde{C}_Q e^{-2\nu \Lambda^{1-2\nu}}
\]
holds for any \(0 \leq z \leq y_0\) and for \(\Lambda\) large enough. From the definition of \(A(z)\) and \(B(z)\), there exist some positive constants \(C_i\) and \(C_J\) such that the inequalities
\[
|A(z)||B(y)| + |A(y)||B(z)| \leq \frac{1}{2\nu} C_i (z^{1/2\nu} y^{1/2\nu} + y^{1/2\nu} z^{1/2\nu})
\]
and
\[
|A(z)| \leq C_J z^{1/2\nu}
\]
hold for any \(0 \leq z \leq y \leq 1\). From (5.19), (5.20), (5.21) and (5.22), it holds for any \(0 < y \leq 1\) that
\[
|\Phi_1^{(1)}(y)| \leq \Lambda^{-1/(2\nu)} \Lambda^{-2\nu} \tilde{C}_Q C_i \frac{1}{2\nu} C_J \frac{3}{2\nu} e^{-2\nu \Lambda^{1/2\nu} + y^{1/2\nu} z^{1/2\nu}}
\]
\[
= \Lambda^{-1/(2\nu)} \Lambda^{-2\nu} C_J \tilde{C}_Q C_i \frac{1}{2\nu} e^{-2\nu \Lambda^{1/2\nu}}
\]
\[
< \Lambda^{-1/(2\nu)} \Lambda^{-2\nu} C_J \tilde{C}_Q C_i \frac{1}{2\nu} e^{-2\nu \Lambda^{1/2\nu}}.
\]
In the same way, by induction we obtain that

$$|\Phi_1^{(0)}(y)| \leq \Lambda^{-(1-2v)/4} \Lambda^{-2v} C_J \left( \frac{\tilde{C}_Q C_v}{2 l^2} \right)^{1/2} y^{1/2+4v}$$

(5.24)

for any $0 < y \leq 1$. On the other hand, for large $y$ ($y > 1$), from (5.23) with $y = 1$ and (5.20), we have

$$|\Phi_1^{(1)}(y)| < \Lambda^{-(1-2v)/4} \Lambda^{-2v} \left\{ C_J \frac{\tilde{C}_Q C_v}{2 l^2} + \int_1^y (|A(z)||B(y)| + |A(y)||B(z)|) \frac{\tilde{C}_Q \zeta_2 + |A(z)| l}{A(z)} dz \right\}. \tag{5.25}
$$

From the asymptotic property (5.15), there exist some positive constants $C_W$ and $C_A$ such that the inequalities

$$|A(z)||B(y)| + |A(y)||B(z)| \leq C_W$$

(5.26)

and

$$|A(z)| \leq C_A$$

(5.27)

hold for any $1 \leq z \leq y$. From (5.25), (5.26) and (5.27), it holds for any $y$ ($y > 1$) that

$$|\Phi_1^{(1)}(y)| < \left\{ \begin{array}{ll}
\Lambda^{-(1-2v)/4} \Lambda^{-2v} \left\{ C_J \frac{\tilde{C}_Q C_v}{2 l^2} + C_A C_W \frac{1}{1 - 4 \gamma_{-1 + 4v} - 1} \right\} & (v \neq \frac{1}{4}), \\
\Lambda^{-(1-2v)/4} \Lambda^{-2v} \left\{ C_J \frac{\tilde{C}_Q C_v}{2 l^2} + C_A C_W \tilde{C}_Q \log y \right\} & (v = \frac{1}{4}).
\end{array} \right. \tag{5.28}
$$

Let us define positive constants $L$, $L_0$, $M$ and $M_0$ as follows:

$$L := C_W \tilde{C}_Q \left| \frac{1}{1 - 4 \gamma_{-1 + 4v}} \right| \quad (v \neq \frac{1}{4}),$$

$$L_0 := \left\{ \sum_{j=1}^{\infty} C_J \left( \frac{\tilde{C}_Q C_v}{2 l^2} \right)^{1/2} \frac{1}{l^j} \frac{1}{1 - 4 \gamma_{-1 + 4v}} \right\} + C_A \quad (v \neq \frac{1}{4}),$$

$$M := C_W \tilde{C}_Q \quad (v = \frac{1}{4}),$$

$$M_0 := \left\{ \sum_{j=1}^{\infty} C_J \left( \frac{\tilde{C}_Q C_v}{2 l^2} \right)^{1/2} \frac{1}{l^j} \frac{1}{\delta M} \right\} + C_A \quad (v = \frac{1}{4}). \tag{5.29}
$$

where $\delta (0 < \delta < 1)$ is a positive constant such that the inequality $\log y < y^\delta$ holds for any $y$ ($y > 1$). Then the following inequality holds.

$$|\Phi_1^{(1)}(y)| < \left\{ \begin{array}{ll}
\Lambda^{-(1-2v)/4} \Lambda^{-2v} L_0 L & (v < \frac{1}{4}), \\
\Lambda^{-(1-2v)/4} \Lambda^{-2v} M_0 M y^\delta & (v = \frac{1}{4}), \\
\Lambda^{-(1-2v)/4} \Lambda^{-2v} L_0 L y^{-1+4v} & (v > \frac{1}{4}).
\end{array} \right. \tag{5.30}
$$

In the same way, by induction, we have

$$|\Phi_1^{(2)}(y)| < \left\{ \begin{array}{ll}
\Lambda^{-(1-2v)/4} \Lambda^{-2v} L_0 L & (v < \frac{1}{4}), \\
\Lambda^{-(1-2v)/4} \Lambda^{-2v} M_0 M \left( \frac{M}{\delta} \right) y^\delta & (v = \frac{1}{4}), \\
\Lambda^{-(1-2v)/4} \Lambda^{-2v} L_0 L y^{-(1+4v)} & (v > \frac{1}{4}).
\end{array} \right. \tag{5.31}
$$

Therefore, the series $\sum_{i=0}^{\infty} \Phi_1^{(i)}(y)$ is absolutely and uniformly convergent on any compact interval in $[0, \infty)$. Hence, $\Phi_1(x) := \sum_{i=0}^{\infty} \Phi_1^{(i)}(y)$ satisfies (5.13) and we can obtain the following estimate.

$$\left| \Phi_1(x) - \Lambda^{-(1-2v)/4} A(\sqrt{A} x) \right| \leq \sum_{i=1}^{\infty} \left| \Phi_1^{(i)}(\sqrt{A} x) \right|$$
Thus, we have
\[ \Phi_1(x) - \Lambda^{-1-2\nu/4} A(\sqrt{\Lambda} x) = o(\Lambda^{-1-2\nu/4}) \quad \text{as} \quad \Lambda \to \infty, \]
for any fixed \( x \).

Next we consider the case of \( \xi = 1 \). From (5.10), the coefficient of \( x^{-2+2\nu} \) of \( \tilde{Q}(x) \) is equal to 0. Therefore \( \tilde{Q}(x) = O(x^{-2+4\nu}) \) as \( x \to 0 \). This means that the proof of the case of \( \xi = 1 \) consists with that of the case of \( \xi = 2 \). Thus the proof is completed.

We are now ready to investigate the asymptotic behavior of \( \Delta(\Lambda) \) as \( \Lambda \to \infty \). From (5.15) and Lemma 5.1, we have
\[
\begin{align*}
\phi_1(t, \Lambda) &= \Lambda^{-(1-2\nu/4)} \rho^{-1/4}(t)A_n \sqrt{2/\pi} \cos \left( \sqrt{\Lambda} \int_0^t \sqrt{\rho(s)} ds - \frac{1-2\nu}{4} \pi \right) + o(A^{-1-2\nu/4}), \\
\phi_2(t, \Lambda) &= \Lambda^{-(1+2\nu/4)} \rho^{-1/4}(t)B_n \sqrt{2/\pi} \cos \left( \sqrt{\Lambda} \int_0^t \sqrt{\rho(s)} ds - \frac{1+2\nu}{4} \pi \right) + o(A^{-(1+2\nu/4)})
\end{align*}
\]  
(5.34)
as \( \Lambda \to \infty \). In the same way, we have
\[
\begin{align*}
\tilde{\phi}_1(t, \Lambda) &= \Lambda^{-(1-2\nu/4)} \rho^{-1/4}(t)A_n \sqrt{2/\pi} \cos \left( \sqrt{\Lambda} \int_0^t \sqrt{\tilde{\rho}(s)} ds - \frac{1-2\nu}{4} \pi \right) + o(A^{-1-2\nu/4}), \\
\tilde{\phi}_2(t, \Lambda) &= \Lambda^{-(1+2\nu/4)} \rho^{-1/4}(t)B_n \sqrt{2/\pi} \cos \left( \sqrt{\Lambda} \int_0^t \sqrt{\tilde{\rho}(s)} ds - \frac{1+2\nu}{4} \pi \right) + o(A^{-(1+2\nu/4)})
\end{align*}
\]  
(5.35)
as \( \Lambda \to \infty \). It follows from (5.3), (5.34) and (5.35) that
\[ \Phi_\Lambda(T) = \begin{pmatrix}
A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0) & \Lambda^{-\nu} B_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 - \nu \tau) + o(A^{-\nu}) \\
\Lambda^{-\nu} A_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 + \nu \tau) + o(A^\nu) & A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0)
\end{pmatrix} \]
(5.36)
as \( \Lambda \to \infty \). From (4.8) and (5.36), we have
\[
\begin{align*}
\Delta(\Lambda) &= \alpha \left( A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0) \right) \\
&\quad + \beta \left( \Lambda^{-\nu} A_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 + \nu \tau) + o(A^\nu) \right) \\
&\quad + \gamma \left( \Lambda^{-\nu} B_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 - \nu \tau) + o(A^{-\nu}) \right) \\
&\quad + \delta \left( A_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0) \right)
\end{align*}
\]  
(5.37)
as \( \Lambda \to \infty \).

For \( N = 2 \), as in the case of \( N = 1 \), we have
\[
\Phi_\Lambda(T) = R_\Lambda[(t_1 + T)/2, T]^{-1} R_\Lambda[(t_1 + T)/2, t_1] R_\Lambda[t_1/2, t_1]^{-1} R_\Lambda[t_1/2, 0] 
\]  
\[
= \begin{pmatrix}
A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_2) + o(A^0) & \Lambda^{-\nu} B_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_2 - \nu \tau) + o(A^{-\nu}) \\
\Lambda^{-\nu} A_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_2 + \nu \tau) + o(A^\nu) & A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_2) + o(A^0)
\end{pmatrix} \\
\times \begin{pmatrix}
A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0) & \Lambda^{-\nu} B_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 - \nu \tau) + o(A^{-\nu}) \\
\Lambda^{-\nu} A_n^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 + \nu \tau) + o(A^\nu) & A_n B_n \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1) + o(A^0)
\end{pmatrix}
\]  
(5.38)
as \( \Lambda \to \infty \). From (4.8) and (5.38), we have
\[ \Delta(\Lambda) = a \left\{ A_n^2 B_n \left( \frac{2}{\pi} \right)^2 \left[ \cos(\sqrt{\Lambda} S_2) \cos(\sqrt{\Lambda} S_1) + \cos(\sqrt{\Lambda} S_2 - \nu \tau) \cos(\sqrt{\Lambda} S_1 + \nu \tau) + o(\Lambda^0) \right] \right\} \]
\[ + \beta \left\{ A_n^2 \Lambda B_n \left( \frac{2}{\pi} \right)^2 \left[ \cos(\sqrt{\Lambda} S_2 + \nu \tau) \cos(\sqrt{\Lambda} S_1) + \cos(\sqrt{\Lambda} S_2 \cos(\sqrt{\Lambda} S_1 + \nu \tau)) + o(\Lambda^0) \right] \right\} \]
\[ + \gamma \left\{ A_n^2 \Lambda B_n \left( \frac{2}{\pi} \right)^2 \left[ \cos(\sqrt{\Lambda} S_2) \cos(\sqrt{\Lambda} S_1 - \nu \tau) + \cos(\sqrt{\Lambda} S_2 - \nu \tau) \cos(\sqrt{\Lambda} S_1) + o(\Lambda^0) \right] \right\} \]
\[ + \delta \left\{ A_n^2 B_n^2 \left( \frac{2}{\pi} \right)^2 \left[ \cos(\sqrt{\Lambda} S_2 + \nu \tau) \cos(\sqrt{\Lambda} S_1 - \nu \tau) + \cos(\sqrt{\Lambda} S_2 \cos(\sqrt{\Lambda} S_1)) + o(\Lambda^0) \right] \right\} \] (5.39)
as \( \Lambda \to \infty \).

6. Proof of Theorem 2.1

We are now ready to prove Theorem 2.1. It is sufficient to see that \( \Delta(\Lambda) \) satisfies the assumption of Proposition 4.1 for each case.

In the case of \( N = 1 \)

For any \( m (m \geq 3) \) and any \( k (k \in \{1, 2, \ldots, m\}, \gcd(k, m) = 1 \) \), we have \( \beta = -\sin(2\pi k/m) \neq 0 \). From (5.12), (5.14) and (5.37), the asymptotic behavior of \( \Delta(\Lambda) \) is almost determined by

\[ \beta \Lambda^2 \frac{2}{\pi} \cos(\sqrt{\Lambda} S_1 + \nu \tau). \] (6.1)

For \( \Lambda \) large enough, the amplitude \( |\beta \Lambda^2 2/\pi| \) is much greater than 2, hence by Proposition 4.1 there exist countably infinite simple eigenvalues and the eigenfunctions corresponding to these ones are exact \( mT \)-periodic solution of (3.3). Therefore, in view of Lemma 3.1, Theorem 2.1 holds for the case of \( m \geq 3 \).

For the case of \( m = 1, 2 \), we have no choice but to determine \( k = 1 \). From (5.14) and (5.37), we have

\[ \Delta(\Lambda) = \begin{cases} 
\frac{2}{\sin \nu \tau} \cos(\sqrt{\Lambda} S_1) + o(\Lambda^0) & (m = 1), \\
-\frac{2}{\sin \nu \tau} \cos(\sqrt{\Lambda} S_1) + o(\Lambda^0) & (m = 2)
\end{cases} \] (6.2)
as \( \Lambda \to \infty \). Since \( |\pm 2/\sin \nu \tau| > 2 \), by Proposition 4.1, there exist countably infinite simple eigenvalues and the eigenfunctions corresponding to these ones are exact \( mT \)-periodic solution of (3.3). Therefore, in view of Lemma 3.1, Theorem 2.1 holds for the case of \( m = 1, 2 \).

This completes the proof of our theorem in the case of \( N = 1 \).

In the case of \( N = 2 \)

For any \( m (m \geq 3) \) and any \( k (k \in \{1, 2, \ldots, m\}, \gcd(k, m) = 1 \) \), we have \( \beta = -\sin(2\pi k/m) \neq 0 \). From (5.12), (5.14) and (5.39), the asymptotic behavior of \( \Delta(\Lambda) \) is almost determined by

\[ \beta \Lambda^2 \frac{2}{\pi} \left( \sin(\sqrt{\Lambda} S_2) \cos(\sqrt{\Lambda} S_1) + \cos(\sqrt{\Lambda} S_2 \cos(\sqrt{\Lambda} S_1 + \nu \tau)) \right) \]
\[ = \beta \Lambda^2 \frac{2}{\pi} \left( \sin(\sqrt{\Lambda}(S_1 + S_2) + \nu \tau) + \cos(\sqrt{\Lambda}(S_1 - S_2)) \cos \nu \tau \right). \] (6.3)

Let us define \( \Lambda_i \) and \( \hat{\Lambda}_i \), as follows:

\[ \Lambda_i := \left( \frac{2i - \nu \pi}{S_1 + S_2} \right)^2, \quad \hat{\Lambda}_i := \left( \frac{2i + 1 - \nu \pi}{S_1 + S_2} \right)^2 \quad (i \in \mathbb{N}). \] (6.4)

Then, it follows from (6.3) and (6.4) that

\[ \Lambda_i < \hat{\Lambda}_i, \]
\[ \Delta(\Lambda_i) = +\infty \quad \text{as} \quad i \to \infty, \] (6.5)
\[ \Delta(\hat{\Lambda}_i) = -\infty \quad \text{as} \quad i \to \infty. \]

From (6.5) and Proposition 4.1, there exist countably infinite simple eigenvalues and the eigenfunctions corresponding to these ones are exact \( mT \)-periodic solution of (3.3). Therefore, in view of Lemma 3.1, Theorem 2.1 holds for the case of \( m \geq 3 \).
For the case of \( m = 1, 2 \), we have \( k = 1 \). From (5.14) and (5.39), we have
\[
\Delta(\Lambda) = \begin{cases} 
\frac{2}{\sin^2 v\pi} \left( \cos(\sqrt{\Lambda}(S_1 + S_2)) + \cos(\sqrt{\Lambda}(S_1 - S_2)) \cos^2 v\pi \right) + o(\Lambda^0) \quad (m = 1), \\
-\frac{2}{\sin^2 v\pi} \left( \cos(\sqrt{\Lambda}(S_1 + S_2)) + \cos(\sqrt{\Lambda}(S_1 - S_2)) \cos^2 v\pi \right) + o(\Lambda^0) \quad (m = 2)
\end{cases}
\]  
(6.6)
as \( \Lambda \to \infty \). Let us define \( \Lambda_j \) and \( \tilde{\Lambda}_j \) as follows:
\[
\Lambda_j := \left( \frac{2\pi}{S_1 + S_2} \right)^2, \quad \tilde{\Lambda}_j := \left( \frac{(2j + 1)\pi}{S_1 + S_2} \right)^2 \quad (j \in \mathbb{N}).
\]  
(6.7)
Then, we have
\[
\limsup_{j \to \infty} \frac{2}{\sin^2 v\pi} \left( \cos(\sqrt{\Lambda}(S_1 + S_2)) + \cos(\sqrt{\Lambda}(S_1 - S_2)) \cos^2 v\pi \right) = \frac{2(1 + \cos^2 v\pi)}{\sin^2 v\pi} > 2 \quad (6.8)
\]
and
\[
\liminf_{j \to \infty} \frac{2}{\sin^2 v\pi} \left( \cos(\sqrt{\tilde{\Lambda}}(S_1 + S_2)) + \cos(\sqrt{\tilde{\Lambda}}(S_1 - S_2)) \cos^2 v\pi \right) = \begin{cases} 
-2 & (S_1 = S_2), \\
< -2 & (S_1 \neq S_2).
\end{cases} \quad (6.9)
\]
For the case of \( m = 1 \) and the case where \( m = 2 \) and \( S_1 \neq S_2 \), it follows from (6.6), (6.8), (6.9) and Proposition 4.1 that there exist countably infinite simple eigenvalues and the eigenfunctions corresponding to these ones are exact \( mT \)-periodic solution of (3.3). Therefore, in view of Lemma 3.1, Theorem 2.1 holds for the case of \( m = 1 \) and the case where \( m = 2 \) and \( S_1 \neq S_2 \).

This completes the proof of our theorem in the case of \( N = 2 \). 

\[ \square \]

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