On the First Eigenvalue of the Combinatorial Laplacian for a Graph

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The eigenvalues of the combinatorial Laplacian of graphs with boundaries and infinite graphs without boundary are studied. For a graph with boundary $G=(V \cup \partial V, E \cup \partial E)$, a sharp lower bound of the first eigenvalue $\lambda_1(G)$ is given provided $G$ satisfies a general condition, the so-called non-separation property. For an infinite graph $G$ without boundary, the bottom of the spectrum, i.e., the infimum of the spectrum of the combinatorial Laplacian of $G$, denoted $\lambda_0(G)$, is estimated as

$$\lambda_0(G) \leq \frac{1}{4} \mu(G)^2 \exp(\mu(G)),$$

where $\mu(G)$ is the exponential growth of $G$. As a corollary, if $G$ is subexponential, $\lambda_0(G) = 0$. On the contrary, $\lambda_0(G) > 0$ is shown for a simply connected infinite graph $G$ with degree $\geq 4$ at each vertex.

KEYWORDS: graph, combinatorial Laplacian, eigenvalue, exponential growth

§1. Introduction

In analysis on Riemannian manifolds, there are many studies on the Laplacian (cf. [Bd], [Bl], [U], [N.U]). Some analogues are known in graph theory concerning eigenvalues of the adjacency matrix or the associated Laplacian (see [Bl], [D], [D.K], [Fn], [Fa], [M.W], [S] and see [C.D.S] for a general reference). In this respect, our aim is to obtain more analogues in graph theory regarding graphs as Riemannian manifolds.

A graph is a collection of vertices together with a collection of edges joining pairs of vertices. Dodziuk [D] and Friedman [Fn] considered the combinatorial Laplacian $\Delta$ on a graph $G = (V \cup \partial V, E \cup \partial E)$ with boundary and studied the Dirichlet eigenvalue problem. In this paper, we deal with the first eigenvalue $\lambda_1(G)$ of this Dirichlet problem.

The celebrated Faber-Krahn inequality for the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian for a bounded domain $\Omega$ in the Euclidean space $\mathbb{R}^n$ is that:

$$\lambda_1(\Omega) \text{ Vol}(\Omega)^{2/n} \geq \lambda_1(\Omega^*) \text{ Vol}(\Omega^*)^{2/n},$$

and the equality holds only for the case $\Omega$ is a ball $\Omega^*$ in $\mathbb{R}^n$. This is proved by the so-called symmetization technique (cf. [Bd]). In this paper, we first consider a symmetization for a graph and show a sharp lower bound of the first eigenvalue for the combinatorial Laplacian of a graph satisfying the non-separation property (cf. §2 and Theorem 6.1):

$$\lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6m^2},$$

where $m$ is the number of edges in $E \cup \partial E$. In this connection, we make the following conjecture

**Conjecture 1.** The first eigenvalue $\lambda_1(G)$ of the Dirichlet eigenvalue problem of a graph $G = (E \cup \partial E, V \cup \partial V)$ with the non-separation property satisfies the following inequality:

$$\lambda_1(G) \geq 2 - 2 \cos \left(\frac{\pi}{m}\right),$$

where $m$ is the number of edges of $E \cup \partial E$. Moreover, the equality holds if and only if $G$ is the graph of type $A_{m+1}$ (cf. Example 1 in §2).

Note that our estimate is asymptotically optimal in the sense that it has the same behavior of $2 - 2 \cos (\pi/m)$, the first eigenvalue of the graph of type $A_{m+1}$ when $m$ tends to $\infty$. So we propose that the above conjecture might be a natural analogue, in graph theory, of the Faber-Krahn inequality in Riemannian geometry.

Next we study the asymptotic behavior of an infinite graph at infinity. Brooks [B1] defined the exponential growth $\mu(M, g)$ of volume of a metric ball $B_p(r)$ of radius $r$ centered at some point $p$ in a complete Riemannian
manifold \((M, g)\) when \(r\) tends to infinity. He showed the bottom \(\lambda_0(M, g)\) of the spectrum of the Laplacian can be estimated as \(\lambda_0(M, g) \leq \left(1/4\right) \mu (M, g)^2\) if the volume of \((M, g)\) is infinite. As its corollary, if \((M, g)\) is subexponential, i.e., \(\mu (M, g) = 0\), then \(\lambda_0(M, g) = 0\). They imply that the analytic quantity can be estimated above by the volume growth. Moreover, McKean [M] showed that if \((M, g)\) is an \(n\)-dimensional simply connected Riemannian manifold whose curvature is bounded below by a negative constant \(k\), then \(\lambda_0(M, g) \approx \left((n - 1)^2/4\right)k\).

We aim to show the analogue in graph theory of the above in Riemannian geometry. We define in §7, two quantities of an infinite graph \(G\) without boundary. One is the bottom of the spectrum of the combinatorial Laplacian \(\Delta\), say \(\lambda_0(G)\), which is by definition the infimum of \(\lambda_1(L)\), where \(L \subseteq G\) vary in all finite subgraphs of \(G\) with boundaries. The other is the exponential growth, \(\mu(G)\), measures how the graph \(G\) spreads into branches at infinity. Then we shall show (cf. Theorem 7.10) that

\[
\lambda_0(G) \leq \frac{1}{4} \mu (G)^2 \exp \left(\mu (G)\right).
\]

As its corollary (cf. Corollary 7.11), if \(G\) is subexponential, i.e., \(\mu (G) = 0\), then \(\lambda_0(G) = 0\). This result means \(\lambda_0(G) = 0\) if \(G\) spreads slowly of polynomial order at infinity and contrasts entirely with Theorem 7.9, which is a discrete analogue of McKean’s theorem generalizing Sunada’s result [S] in the case of a tree \(T_{d+1}\) of degree \(d + 1 \geq 3\). Indeed, it holds that \(\lambda_0(T_{d+1}) > 0\) if \(d + 1 \geq 3\) (cf. Corollary 7.7). Our result in Corollary 7.11 has the following intuitive meaning in a population growth model: Any active society can gain at least one new member within a fixed period of time, if one regards \(\lambda_0\) as the base of activity of the society. This is because, if one regards \(V(r)\) as the number of members of a society at time \(r\), one may regard \(\mu\) as the growth ratio of the number of its members.

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§2. Preliminaries

Following Dodziuk [D], Brooks [Bs2] and Friedman [Fn], we prepare some notations concerning the combinatorial Laplacian on graphs with boundaries.

2.1 Graphs with boundaries

A graph with boundary means an undirected graph \(G = \{V \cup \partial V, E \cup \partial E\}\) such that (i) each edge in \(E\) has both endpoints in \(V\), (ii) each edge in \(\partial E\) has exactly one endpoint in \(V\) and one in \(\partial V\) and (iii) any vertex which has exactly one edge is in \(\partial V\). We call vertices in \(V\) (resp. \(\partial V\)) the interior (resp. boundary) vertices, and similarly for the edges. In the figures illustrating the following examples, black circles indicate the vertices in \(\partial V\), and white circles those in \(V\).

Example 1. (The graph of type \(A_{m+1}\)) This is a graph consisting of \(m + 1\) vertices connecting each other by \(m\) edges as a segment. See Figure 1.

Example 2. (A ball of radius \(k\) in a tree \(T_{d+1}\) of degree \(d + 1 (d \geq 2)\)) Let \(T_{d+1}\) be a tree of degree \(d + 1\). We fix any vertex \(v_0\) in \(T_{d+1}\). For a positive integer \(k\), the graph consisting of all vertices of distance \(k\) from \(v_0\) is called a ball of radius \(k\). See Figure 2.

2.2 The combinatorial Laplacian

Let \(C^0(G)\) be the set of all real valued functions \(f\) on \(V \cup \partial V\) satisfying \(f(x) = 0\) for \(x \in \partial V\). Let \(C^1(G)\) be the space of all functions \(\varphi\) defined on the set of all directed edges of \(G\) and satisfying

\[
\varphi([x, y]) = -\varphi([y, x]),
\]

where \([x, y]\), \(x, y \in V \cup \partial V\) denotes a directed edge in \(E \cup \partial E\) beginning at \(x\) and ending at \(y\). Here we fix once and for all a direction for each edge of \(G\).

We define the inner products as follows:

\[
\begin{aligned}
(f_1, f_2) &:= \frac{1}{2} \sum_{x \in V} m(x)f_1(x)f_2(x), \\
(\varphi_1, \varphi_2) &:= \sum_{\sigma \in E \cup \partial E} \varphi_1(\sigma)\varphi_2(\sigma),
\end{aligned}
\]

for \(f_1, f_2 \in C^0(G)\) and \(\varphi_1, \varphi_2 \in C^1(G)\). Here \(m(x), x \in V \cup \partial V\) is the number of edges in \(E \cup \partial E\) emanating from \(x\). The coboundary operator

\[
df([x, y]) := f(y) - f(x),
\]
maps $C_0^0(G)$ into $C^1(G)$.

The combinatorial Laplacian is defined by

$$\Delta f = d^* df, \quad f \in C_0^0(G),$$

where $d^*$ is the adjoint of $d$ with respect to the above inner products. We get by definition

$$(\Delta f_1, f_2) = (df_1, df_2), \quad f_1, f_2 \in C_0^0(G).$$

Moreover we obtain:

Lemma 2.3. For all $f \in C_0^0(G)$, we have

$$\Delta f(x) = - \frac{2}{m(x)} \sum_{y \sim x} f(y) + 2f(x), \quad x \in V,$$

where $y \sim x$ means that $x$ and $y$ are connected by an edge in $E \cup \partial E$.

The proof follows from a simple calculation using (2.1), (2.2) and the definition of $df$.

The eigenvalue $\lambda$ of $\Delta$ on $C_0^0(G)$ means that there exists a non-vanishing function $f \in C_0^0(G)$ such that $\Delta f(x) = \lambda f(x), \ x \in V$, where $f$ is called the eigenfunction with the eigenvalue $\lambda$. This means that $f$ and $\lambda$ satisfy the Dirichlet eigenvalue problem:

$$\begin{align*}
\Delta f(x) &= \lambda f(x), \quad x \in V, \\
\quad f(x) &= 0, \quad x \in \partial V.
\end{align*}$$

Note that each eigenvalue of the Dirichlet eigenvalue problem is positive. Because each $f \in C_0^0(G)$ satisfies

$$\langle \Delta f, f \rangle = (df, df) \geq 0,$$

and $0$ cannot be an eigenvalue of $\Delta$ on $C_0^0(G)$ since $df([x, y]) = 0$ for all $[x, y] \in E \cup \partial E$, $f$ is constant on $V \cup \partial V$.

Let us denote the eigenvalues of $\Delta$ on $C_0^0(G)$ as

$$0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_k(G),$$

where $k$ is the number of $V$.

2.3 The geometric realization of a graph

Following Friedman [Fn], we define the geometric realization $\mathcal{G}$ of a graph $G = (V \cup \partial V, E \cup \partial E)$ with boundary by gluing an interval of length 1 between two vertices $x$ and $y$ in $V \cup \partial V$ connected by an edge $e = [x, y] \in E \cup \partial E$. We get then a metric space which is a 1-dimensional manifold except for vertices and we can define the notion of differentiability of a function excepting vertices.
Let $C^1(\mathcal{G})$ be the set of all real valued continuous functions on $\mathcal{G}$ which are differentiable on $\mathcal{G} - (V \cup \partial V)$ and have compact supports in $\mathcal{G} - (V \cup \partial V)$. The space $C^0(\mathcal{G})$ can be embedded in $C^1(\mathcal{G})$ by extending each $f \in C^0(\mathcal{G})$ as an edgewise linear function on $\mathcal{G}$ denoted by the same letter $f$. In the sequel, we always identify $C^0(\mathcal{G})$ with its image in $C^1(\mathcal{G})$. Then, for $f \in C^0(\mathcal{G})$, it holds that

$$\langle df, df \rangle = \int_{\mathcal{G}} |\nabla f|^2 d\mu,$$

where $\nabla f(x)$ is the usual differential of $f$ at $x \in \mathcal{G} - (V \cup \partial V)$ and $d\mu$ is the usual Lebesgue measure on $\mathcal{G}$. Because,

$$\langle df, df \rangle = \sum_{[x,y] \in E \cup \partial E} (f(y) - f(x))^2 = \sum_{[x,y] \in E \cup \partial E} |\nabla f|^2 d\mu = \int_{\mathcal{G}} |\nabla f|^2 d\mu,$$

for all $f \in C^0(\mathcal{G})$.

Therefore, if $f \in C^0(\mathcal{G})$ is the eigenfunction of $\Delta$ with the eigenvalue $\lambda$, then

$$\lambda = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{\langle f, f \rangle}.$$

Indeed, by (2.2) and (2.4), it holds that

$$\lambda = \frac{(Af, f)}{(f, f)} = \frac{(df, df)}{(f, f)} = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)}.$$

For $f \in C^0(\mathcal{G})$, the ratio

$$\Re(f, \mathcal{G}) = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)}.$$

is called the Rayleigh quotient of $f$.

2.4 The non-separation property

In this subsection, we introduce the following notion:

**Definition 2.6.** It is said a graph with boundary $G = (V \cup \partial V, E \cup \partial E)$ has the non-separation property if each connected component of the complement, say $\mathcal{G} - \{v\}$, of each vertex $v \in V$ contains at least one boundary vertex in $\partial V$.

Fig. 3
Example 2.7. A simply connected graph $G$, i.e., containing no cycle, has the non-separation property because of property (iii) in the definition of a graph with boundary (cf. §2.1). The examples above (Fig. 3) illustrate the non-separation property: the first example satisfies the non-separation property but the second does not.

By definition of the non-separation property, we get immediately:

**Lemma 2.8.** Let $G = (V \cup \partial V, E \cup \partial E)$ be a graph with the non-separation property. Let $f$ be a continuous function on the geometric realization $\mathcal{G}$ vanishing at the boundary $\partial \mathcal{G}$. Then the level set of $f$, $\{p \in \mathcal{G}; f(p) = t\}$, contains at least two points.

3. Some formulas

We show that:

**Proposition 3.1.** For an edgewise linear function $f$ on $\mathcal{G}$ satisfying $f = 0$ on $\partial V$ for a graph $G = (V \cup \partial V, E \cup \partial E)$, we have

$$3 \int_{\mathcal{G}} f^2 d\mu = 2(f, f) + \sum_{x \sim y} f(x)f(y) = 3(f, f) - \frac{1}{2} (f, \Delta f).$$

**Proof.** For an edgewise linear function $f$ on $\mathcal{G}$ satisfying $f = 0$ on $\partial V$, we have

$$3 \int_{\mathcal{G}} f^2 d\mu = \sum_{[x,y] \in E \cup \partial E} 3 \int f^2 d\mu_x = \sum_{[x,y] \in E \cup \partial E} (f(x)^2 + f(y)^2 + f(x)f(y)) = \sum_{x \in V \cup \partial V} m(x)f(x)^2 + \sum_{x \sim y} f(x)f(y),$$

which is the first equality. For the second equality, notice that

$$\left(\frac{1}{2} \Delta - I\right)f(x) = -\frac{1}{m(x)} \sum_{y \sim x} f(y), \quad x \in V,$$

where $I$ is the identity operator. Then we have

$$\sum_{x \sim y} f(x)f(y) = \frac{1}{2} \sum_x f(x) \sum_y f(y) = -\frac{1}{2} \sum_x f(x) m(x) \left(\frac{1}{2} \Delta - I\right)f(x) = \frac{1}{2} \sum_x m(x)f^2(x) - \frac{1}{2} \sum_x m(x)f(x) \frac{1}{2} \Delta f(x) = (f, f) - \frac{1}{2} (f, \Delta f).$$

This implies the second equality by means of the first equality. \hfill \Box

Immediately, we obtain:

**Corollary 3.2.** Let $\lambda_i(G)$, $1 \leq i \leq k$ be the eigenvalue of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$:

$$\begin{align*}
\Delta f &= \lambda f \quad \text{on} \quad V, \\
f &= 0 \quad \text{on} \quad \partial V.
\end{align*}$$

If we extend the eigenfunction $f$ with the eigenvalue $\lambda_i(G)$ on $V \cup \partial V$ to an edgewise linear function on $\mathcal{G}$ denoted by the same letter, then we have

$$3 \int_{\mathcal{G}} f^2 d\mu = \left(3 - \frac{1}{2} \lambda_i(G)\right) (f, f). \quad (3.3)$$

Thus the eigenvalues $\lambda_i(G)$, $1 \leq i \leq k$, satisfy $0 < \lambda_i(G) < 6$, really $0 < \lambda_i(G) < 4$.

4. Co-area formula

To estimate the first eigenvalue $\lambda_1(G)$ of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$, we need the following co-area formula:
Proposition 4.1. Let \( f \in C_b(\mathcal{G}) \) and \( h \) a measurable function on \( \mathcal{G} \).

(i) Then we have the co-area formula:

\[
\int_{\mathcal{G}} h|\nabla f| d\mu = \int_{-\infty}^{\infty} \sum_{p \in \mathcal{G}, f(p) = t} h(p) dt.
\]

(ii) If we put \( \mathcal{G}_t = \{ p \in \mathcal{G} : f(p) > t \} \), then \( \text{Vol}(\mathcal{G}_t) \), the length of \( \mathcal{G}_t \), is differentiable in \( t \) almost everywhere and at any regular point \( t = f(x) \), i.e., \( |\nabla f|(x) > 0 \),

\[
\frac{d}{dt} \text{Vol}(\mathcal{G}_t) = -\sum_{p \in \mathcal{G}_t, f(p) = t} |\nabla f|^{-1}(p).
\]

Proof. Since \( \mathcal{G} \) is a finite union of edges in \( E \cup \partial E \), one may prove it on each edge. But this is a special case of the usual co-area formula for a Riemannian manifold with boundary. Indeed, for a regular value \( t = f(x) \), it holds that

\[
|\nabla f| d\mu = \begin{cases} f'(x) dx = dt, & \text{if } f'(x) > 0, \\ -f'(x) dx = -dt, & \text{if } f'(x) < 0. \end{cases}
\]

Therefore we have, for an interval \([a, b]\),

\[
\int_{a}^{b} h(x) |\nabla f(x)| dx = \int_{-\infty}^{\infty} \sum_{f(p) = t} h(p) dt,
\]

which yields (i). Putting \( h = 1/|\nabla f| \) at a regular value \( t = f(x) \),

\[
\text{Vol}(\mathcal{G}_{t+}) - \text{Vol}(\mathcal{G}_t) = -\int_{t}^{t+} \sum_{p \in \mathcal{G}_{t+}, f(p) = r} |\nabla f|^{-1}(p) dr,
\]

by (i). Therefore we obtain (ii). \( \square \)

5. Symmetrization

For a continuous function \( f \) on the geometric realization \( \mathcal{G} \) of a connected finite graph with boundary \( G = (V \cup \partial V, E \cup \partial E) \), we consider a symmetrization \( f^* \) on the interval \( \mathcal{G}^* \) of the real line \( \mathbb{R} \) with length \( \#(E \cup \partial E) \) (the number of edges in \( E \cup \partial E \)) as follows.

For every real \( t \in \mathbb{R} \), let us consider

\[
\mathcal{G}_t = \{ x \in \mathcal{G} : f(x) > t \}.
\]

Let us consider an interval

\[
\mathcal{G}_t^* = \{ x \in \mathbb{R} : |x| < r \},
\]

around the origin in the real line \( \mathbb{R} \) with which radius \( r \) is given by

\[
r = \frac{1}{2} \mu(\mathcal{G}_t).
\]

Here \( \mu \) is the Lebesgue measure of \( \mathcal{G} \). Also, let \( \mathcal{G}^* \) be \( \mathcal{G}_t^* \) for \( t < \min f \).

Let us define a continuous function \( f^* \) on \( \mathcal{G}^* \) by

\[
f^*(x) = t, \quad x \in \partial \mathcal{G}_t^*,
\]

where \( \partial \mathcal{G}_t^* = \{ x \in \mathbb{R} : |x| = r \} \). We call \( (f^*, \mathcal{G}^*) \), the symmetrization of \( (f, \mathcal{G}) \).

Then we have:

Proposition 5.1. Let \( (f^*, \mathcal{G}^*) \) be a symmetrization of \( (f, \mathcal{G}) \) for a continuous function \( f \) on the geometric realization for a graph \( G = (V \cup \partial V, E \cup \partial E) \).

(i) Then we have

\[
\int_{\mathcal{G}} f^2 d\mu = \int_{\mathcal{G}^*} f^2 d\mu,
\]

where \( d\mu \) are the Lebesgue measures on \( \mathcal{G} \) and \( \mathcal{G}^* \), respectively.

(ii) If \( f = 0 \) on \( \partial \mathcal{G} = \partial V \), then \( f^* = 0 \) on \( \partial \mathcal{G}^* \).

(iii) Assume that \( G \) satisfies the non-separation property. If a continuous function \( f \) on \( \mathcal{G} \) is differentiable on \( \mathcal{G} - (V \cup \partial V) \), it holds that
\[ \int_{\mathcal{G}} |\nabla f|^2 d\mu \geq \int_{\mathcal{G}^*} |\nabla f^*|^2 d\mu. \]

**Remark.** Proposition 5.1 does not mean that
\[ \mathcal{R}(f, \mathcal{G}) \geq \mathcal{R}(f^*, \mathcal{G}^*), \]
because \((f^*, f^*)\) and \(\mathcal{R}(f^*, \mathcal{G}^*)\) are not well defined.

**Proof.** We only give a proof of (iii) since (i) and (ii) are clear. For \(f\) on \(\mathcal{G}\), applying the co-area formula (Proposition 4.1) by putting \(h:= |\nabla f|\),
\[ \int_{\mathcal{G}} |\nabla f|^2 d\mu = \int_{-\infty}^{\infty} \left( \sum_{p \in \partial \mathcal{G}_t} |\nabla f| \right) dt, \] (5.2)
where we put \(\partial \mathcal{G}_t = \{ p \in \mathcal{G}; f(p) = t \}\). For a regular value \(t \in \text{Range}(f)\), by the Cauchy-Schwarz inequality,
\[ \sum_{p \in \partial \mathcal{G}_t} |\nabla f| \geq \frac{(\sum_{p \in \partial \mathcal{G}_t} 1)^2}{\sum_{p \in \partial \mathcal{G}_t} |\nabla f||^{-1}(p)}. \] (5.3)
Here, by Lemma 2.8, we have
\[ \sum_{p \in \partial \mathcal{G}_t} 1 \geq \sum_{p \in \partial \mathcal{G}_t} 1 = 2. \] (5.4)
Applying the co-area formula for the function \(h\) defined by
\[ h(x) = \begin{cases} 1, & f(x) \geq t, \\ 0, & f(x) < t, \end{cases} \]
we obtain
\[ \text{Vol}(\mathcal{G}_t) = \int_{\mathcal{G}_t} h(x) d\mu = \int_{t}^{\infty} \left( \sum_{p \in \partial \mathcal{G}_s} |\nabla f||^{-1}(p) \right) ds. \]
Differentiating both sides of the above in \(t\),
\[ \frac{d}{dt} \text{Vol}(\mathcal{G}_t) = -\sum_{p \in \partial \mathcal{G}_t} |\nabla f||^{-1}(p) \] (5.5)
at a regular value \(t\). Making use of (5.5) and the equality \(\text{Vol}(\mathcal{G}_t) = \text{Vol}(\mathcal{G}_t^*)\) which follows by definition,
\[ \sum_{p \in \partial \mathcal{G}_t} |\nabla f||^{-1}(p) = \sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*||^{-1}(p). \] (5.6)
Moreover, since \(|\nabla f^*|\) is constant on \(\partial \mathcal{G}_t^*\), the Cauchy-Schwarz inequality is in turn equality so we obtain
\[ \sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*||^{-1}(p) = \frac{(\sum_{p \in \partial \mathcal{G}_t} 1)^2}{\sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*||^{-1}(p)}. \] (5.7)
Substituting (5.4), (5.6) and (5.7) into (5.3), we have
\[ \sum_{p \in \partial \mathcal{G}_t} |\nabla f| \geq \sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*|(p), \]
Together with (5.2), we have
\[ \int_{\mathcal{G}} |\nabla f| d\mu \geq \int_{\mathcal{G}^*} |\nabla f^*| d\mu. \]
Thus we obtain (iii). \(\Box\)

6. Inequality of the first eigenvalue

Our main theorem is

**Theorem 6.1.** Let \(\lambda_1(G)\) be the first eigenvalue of the Dirichlet eigenvalue problem for a graph \(G = (V \cup \partial V, E \cup \partial E)\) with the non-separation property. Then we have
\[ \lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6(|E \cup \partial E|^2)}. \]

**Proof.** Let \(f\) be the eigenfunction with the eigenvalue \(\lambda_1(G)\) of the Dirichlet eigenvalue problem for a graph \(G = (V \cup \partial V, E \cup \partial E)\). Extend \(f\) to an edgewise linear function on \(\mathcal{G}\), denoted by the same letter \(f\), and take its
symmetrization \((f^*, g^*)\). By Propositions 3.1 and 5.1, we obtain
\[
\lambda_1(G) = \frac{\int_{g^*} |\nabla f|^2 d\mu}{(f,f)}
\geq \frac{3 - \frac{1}{2} \lambda_1(G) \int_{g^*} |\nabla f^*|^2 d\mu}{3 \int_{g^*} f^2 d\mu}
\geq \frac{3 - \frac{1}{2} \lambda_1(G) \int_{g^*} |\nabla f^*|^2 d\mu}{3 \mu_1(g^*)},
\]
where \(\mu_1(g^*)\) is the first eigenvalue of the Dirichlet eigenvalue problem of the usual Laplacian \(-d^2/dx^2\) for the interval \(g^*\) in \(\mathbb{R}\):
\[
\begin{cases}
- \frac{d^2}{dx^2} f = \mu f & \text{on } g^*, \\
f = 0 & \text{on } \partial g^*.
\end{cases}
\]
Therefore, we have
\[
\lambda_1(G) \geq \frac{6 \mu_1(g^*)}{6 + \mu_1(g^*)}.
\]
Since it is well known that
\[
\mu_1(g^*) = \frac{\pi^2}{\#(E \cup \partial E)^2},
\]
we obtain
\[
\lambda_1(G) \geq \frac{6 \pi^2}{\pi^2 + 6 \#(E \cup \partial E)^2},
\]
which is our theorem. \(\Box\)

**Remark 6.2.** For a graph of type \(A_{m+1}\) (cf. Example 1), the eigenvalues of \(\Delta\) are \(2(1 - \cos (j \pi/m)), j = 1, 2, \ldots, m - 1\). Then, the first eigenvalue \(\lambda_1(G)\) is
\[
\lambda_1(G) = 2 \left(1 - \cos \frac{\pi}{m}\right) \sim \frac{\pi^2}{m^2}, \quad \text{as } m \to \infty.
\]
In this case, our lower bound \(6 \pi^2/(\pi^2 + 6m^2)\) is asymptotically \(\pi^2/m^2\) as \(m \to \infty\). Therefore our estimate is optimal asymptotically as \(\#(E \cup \partial E) \to \infty\). This is a reason why we propose Conjecture 1 in the Introduction.

**Remark 6.3.** If \(G\) is a ball of radius \(k\) in a tree \(T_{d+1}\) of degree \(d + 1\) (cf. Example 2), the number of \(E \cup \partial E\) is
\[
\#(E \cup \partial E) = \frac{d + 1}{d - 1} (d^k - 1).
\]
Therefore, by Theorem 6.1, we get
\[
\lambda_1(G) \geq \frac{6 \pi^2}{\pi^2 + 6 \left[\frac{d + 1}{d - 1} (d^k - 1)\right]^2}.
\]

### 7. Asymptotic behavior of infinite graphs

Subsequently, we deal with infinite graphs and study asymptotic behaviors of several quantities.

**Definition 7.1.** For an infinite graph \(G = (V, E)\) without boundary, let us define the **bottom of the spectrum** as follows:
Combinatorial Laplacian for Graph

\[ \lambda_0(G) = \inf_{L \subset G} \lambda_1(L) , \]  

where \( L \subset G \) runs over all finite subgraphs of \( G \) with boundaries and \( \lambda_1(L) \) is the first eigenvalue of the Dirichlet eigenvalue of \( L \). Let us define the bottom of the essential spectrum as follows:

\[ \lambda^{es}_0(G) = \lim_k \lambda_0(G - K) , \]  

where \( K \) runs over finite subgraphs of \( G \) without boundaries, tending to \( G \), \( G - K \) is a subgraph of \( G \) with boundary which is defined by the complement of \( K \) in \( G \) and \( \lambda_0(G - K) \) is the above bottom of the spectrum for \( G - K \).

Note that, by definition,

\[ \lambda_0(G) \leq \lambda^{es}_0(G). \]

**Definition 7.4.** Let the exponential growth of \( G \) be defined as

\[ \mu(G) = \lim_{r \to \infty} \sup_r \frac{1}{r} \log V(r) , \]  

where for \( G = (V, E) \), \( V(r), r > 0 \), is the volume of a ball \( B_p(r) \) with center \( p \in V \) and radius \( r \). That is, \( B_p(r) = \{ x \in V; d(p, x) < r \} \), \( V(r) \) is the number of edges connecting two vertices in \( B_p(r) \) and \( d(x, y) \) is the distance of \( x \) and \( y \) in \( V \) which is defined by the length of the shortest path connecting \( x \) and \( y \).

Note that \( \mu(G) \) does not depend on choice of a point \( p \in V \).

The exponential growth \( \mu(G) \) measures how the graph \( G \) spreads into branches at infinity. If \( G = T_{d+1} \) is a tree of degree \( d + 1 \), then

\[ V(r) = \frac{d + 1}{d - 1} (d' - 1) , \]

and

\[ \mu(G) = \log d > 0, \quad d \geq 2. \]

T. Sunada determined (cf. [SI]) the spectrum \( \text{Spect}(A) \) of the adjacency operator \( A \), which is defined by

\[ A f(x) = \sum_{y \neq x} f(y), \quad x \in V , \]

for a function \( f \) on \( V \), as follows:

**Theorem 7.6.** (T. Sunada) Let \( G = T_{d+1} \) be a tree of degree \( d + 1 \) (\( d \geq 2 \)). Then the spectrum of \( A \) is

\[ \text{Spect}(A) = [-2\sqrt{d}, 2\sqrt{d}] . \]

Since our Laplacian \( \Delta \) is

\[ \Delta = 2 \left( I - \frac{1}{d + 1} A \right) , \]

in this case, we have:

**Corollary 7.7.** The bottom of the spectrum of a tree \( G = T_{d+1} \) of degree \( d + 1 \), \( d \geq 2 \), \( \lambda_0(G) \), is

\[ \lambda_0(G) = 2 \left( 1 - \frac{2\sqrt{d}}{d + 1} \right) \geq 2 \left( 1 - \frac{2\sqrt{2}}{3} \right) = 0.1144 \cdots > 0 . \]

More generally, we wish to obtain a discrete analogue of McKean's theorem of the lower estimate of the bottom of the spectrum of the Laplacian of a simply connected Riemannian manifold of negative curvature (cf. [Mi]):

**Theorem 7.8.** (H. P. McKean) Let \( (M, g) \) be an \( n \)-dimensional simply connected Riemannian manifold with sectional curvature bounded above by a negative constant \( -k \) \( (k > 0) \). Then the bottom \( \lambda_0(M, g) \) of the spectrum of the Laplacian \( \Delta \) of \( (M, g) \) satisfies

\[ \lambda_0(M, g) \geq \frac{(n - 1)^2}{4k} . \]

For a graph \( G \), one may regard

\[ K(x) = 2 - m(x) , \]

as "curvature" of \( G \) at vertex \( x \). For an infinite graph \( G \), we define two numbers \( k(G) \) and \( l(G) \) as follows:
\[ k(G) = \min \{m(x); x \in V\}, \]
\[ l(G) = \sup \{m(x); x \in V\}, \]

where \( l(G) \) may be infinity. So the "curvature" \( K \) of any \( G \) satisfies
\[ 2 - l(G) \leq K(x) \leq 2 - k(G), \]

for each vertex \( x \in V \). Now we obtain:

**Theorem 7.9.** Let \( G \) be a simply connected (i.e., admitting no closed path) infinite graph. Then
\[ \text{Spect}(\mathcal{A}) \subset 2 \left( 1 - 2 \left( \frac{1}{k(G)} \right)^{1/2} \left( 1 - \frac{1}{l(G)} \right)^{1/2} \right) \left( 1 + 2 \left( \frac{1}{k(G)} \right)^{1/2} \left( 1 - \frac{1}{l(G)} \right)^{1/2} \right). \]

In particular,
\[ \lambda_0(G) \geq 2 \left( 1 - 2 \left( \frac{1}{k(G)} \right)^{1/2} \left( 1 - \frac{1}{l(G)} \right)^{1/2} \right). \]

Also, if either (i) \( k(G) \geq 5 \), (ii) \( k(G) = 4 \) and \( l(G) < \infty \), or (iii) \( k(G) = l(G) = 3 \), then \( \lambda_0(G) > 0 \).

We may propose the following conjecture:

**Conjecture 2.** Let \( G \) be a simply connected infinite graph satisfying \( m(x) \geq k \geq 3 \) for each vertex \( x \in V \). Then
\[ \lambda_0(G) > 0. \]

Moreover, \( \lambda_0(G) \) can be estimated below by some positive constant depending only on \( k \).

However we shall also show

**Theorem 7.10.** For any infinite graph \( G \),
\[ \lambda_{\infty}^G(G) \leq \frac{1}{4} \mu(G)^3 \exp \mu(G). \]

**Corollary 7.11.** If \( G \) is subexponential, i.e., \( \mu(G) = 0 \), then \( \lambda_{\infty}^G(G) = 0 \).

**Remark 7.12.** That \( G \) is subexponential when \( \mu(G) = 0 \) means that \( G \) spreads slowly of polynomial order into branches at infinity.

The next example was provided by A. Katsuda:

**Example 3.** (Ladder with infinite steps) Let an infinite graph \( G \) be a ladder with infinite steps, as in Figure 4. Then \( m(x) = 3 \) for each vertex \( x \in V \), but \( \mu(G) = 0 \), so by Corollary 7.11, \( \lambda_0(G) = 0 \). This example shows that one needs simply connectedness of \( G \) in Theorem 7.9 and Conjecture 2.

8. Proof of Theorem 7.9

Sunada's idea proving Theorem 7.6 works well in general to prove Theorem 7.9. For any function \( f \) on \( V \) vanishing outside a finite subset of \( V \), we have also
\[ \Delta f = 2(1-P)f, \]
where \( Pf(x) = (1/m(x)) \sum_{y \in x} f(y), x \in V \) (cf. Lemma 2.3). Thus it suffices to show:

**Lemma 8.1.**
\[ |Pf,f| \leq 2 \left( \frac{1}{k(G)} \right)^{1/2} \left( 1 - \frac{1}{l(G)} \right) (f,f), \]
for all \( f \) on \( V \) vanishing outside a finite subset of \( V \).

**Proof.** For an oriented edge \( e = [x, y] \), let the origin of \( e \) be \( o(e) = x \) and the terminal of \( e \), \( t(e) = y \). Since \( G \) admits no cycle (i.e. no closed path), we may take an orientation on each edge of \( E \) in such a way that for each \( x \)

---

![Fig. 4](image-url)
There exists a unique \( e \in E \), such that \( x = o(e) \): choose an infinite branch of \( G \) and arrange \( G \) as in Figure 5. Then

(i) the correspondence \( e \in E \leftrightarrow o(e) = x \in V \) is one-to-one, and
(ii) for each \( x \in V \),

\[
\# \{ e \in E ; t(e) = x \} = m(x) - 1.
\]

Therefore, we obtain

\[
|(Pf, f)| = \frac{1}{2} | \sum_{x \in V} m(x) Pf(x) f(x) | \\
= \frac{1}{2} | \sum_{x \in V} \sum_{y \sim x} f(y) f(x) | \\
= \frac{1}{2} | \sum_{e \in E} \{ f(t(e)) f(o(e)) + f(o(e)) f(t(e)) \} | \\
= | \sum_{e \in E} f(t(e)) f(o(e)) | \\
\leq ( \sum_{e \in E} f(t(e))^2 )^{1/2} ( \sum_{e \in E} f(o(e))^2 )^{1/2}.
\]

Here by (i) and (ii), we obtain

\[
\sum_{e \in E} f(o(e))^2 = \sum_{x \in V} f(x)^2 = \frac{1}{2} \sum_{x \in V} m(x) \frac{2}{m(x)} f(x)^2 \\
\leq \frac{2}{k(G)} (f, f),
\]

and

\[
\sum_{e \in E} f(t(e))^2 = \sum_{x \in V} (m(x) - 1) f(x)^2 = \frac{1}{2} \sum_{x \in V} m(x) \frac{2(m(x) - 1)}{m(x)} f(x)^2 \\
\leq 2 \left( 1 - \frac{1}{l(G)} \right) (f, f),
\]

since

\[
\frac{2(m(x) - 1)}{m(x)} = 2 \left( 1 - \frac{1}{m(x)} \right) \leq 2 \left( 1 - \frac{1}{l(G)} \right).
\]

Therefore, we obtain

\[
|(Pf, f)| \leq 2 \frac{1}{k(G)} \left[ 1 - \frac{1}{l(G)} \right] (f, f).
\]

\[\Box\]
9. Proof of Theorem 7.10

To prove Theorem 7.10, we follow Brooks' method [Bs1], which requires the geometric realization $G$ of an infinite graph $G$ realized in a Hilbert space, and we take the usual 1-dimensional Lebesgue measure $d\mu$ on $G$. Then Theorem 7.10 is reduced to prove:

**Theorem 9.1.** Assume that there exists a positive constant $\alpha$ such that

$$\alpha^2 \exp (2\alpha) < \lambda_0 (G - K) \quad \text{and} \quad \int_{\gamma - \gamma} \exp (-2\alpha \rho(x)) \, d\mu < \infty,$$

where $\rho(x) = d(p, x), x \in \gamma$. Then, we obtain

$$\int_{\gamma - \gamma} \exp (2\alpha \rho(x)) \, d\mu < \infty.$$

**Proof of Theorem 7.10 from Theorem 9.1.** Assume Theorem 7.10 would not hold. Then there exists a finite subgraph $K$ of $G$ such that

$$\frac{1}{4} \mu(G)^2 \exp (\mu(G)) < \lambda_0 (G - K).$$

Since a real valued function $f(x) = x \exp (x)$ on the positive real half-line is monotone increasing, there exists a positive number $\alpha$ such that $\mu(G)/2 < \alpha$ and $\alpha \exp (\alpha) < \sqrt{\lambda_0 (G - K)}$. Then, we have

$$\int_{\gamma} \exp (-2\alpha \rho(x)) \, d\mu \leq \sum_{j=1}^{\infty} \{V(r) - V(r-1)\} \exp (-2\alpha(r-1)) \leq \sum_{j=1}^{\infty} V(r) \exp (-2\alpha r) (\exp (2\alpha) - 1) < \infty,$$

by definition of $\mu(G)$. By Theorem 9.1,

$$\int_{\gamma - \gamma} \exp (2\alpha \rho(x)) \, d\mu < \infty,$$

which contradicts that $G - K$ is an infinite graph. $\square$

10. Proof of Theorem 9.1

**Lemma 10.1.** For $\alpha > 0$ and $j = 1, 2, \cdots$, define the functions $h_j$ on $\gamma$ by

$$h_j(x) = \min \{\alpha \rho(x), -\alpha \rho(x) + j\}, \quad x \in \gamma.$$

Then

(i) $|\nabla h_j| \leq \alpha, j = 1, 2, \cdots$,

(ii) $\int_{\gamma - \gamma} \exp (2h_j) \, d\mu < \infty, j = 1, 2, \cdots$,

(iii) $h_j$ increases pointwise to $\alpha \rho$ as $j \to \infty$, and

(iv) $\int_{\gamma} \exp (2h_j) \, d\mu \leq \frac{1}{2} \exp (2\alpha \exp (2h_j(u)) + \exp (2h_j(v)))$.

for each edge $e = [u, v]$.

**Proof.** For (i), it suffices to note that the function $\rho$ is of the following form on each edge $e = [u, v]$: In the case $\rho(u) = \rho(v)$,

$$\rho(x) = \begin{cases} \rho(u) + d(u, x), & x \in e \quad \text{and} \quad d(u, x) \leq \frac{1}{2}, \\ \rho(v) + d(v, x), & x \in e \quad \text{and} \quad d(v, x) \leq \frac{1}{2}. \end{cases}$$

In the case $\rho(u) < \rho(v)$,

$$\rho(x) = \rho(u) + d(u, x), \quad x \in e=[u, v].$$

For (ii), note that $h_j(x) = -\alpha \rho(x) + j$ at $x \in \gamma$ with large $\rho(x)$. So we obtain
\[
\int_{\gamma-x} \exp (2h_j) d\mu \leq C + \int_{\gamma-x} \exp (-2\alpha \rho(x) + 2j) d\mu < \infty,
\]
because of the assumption \(|\gamma-x| \exp (-2\alpha \rho(x)) d\mu < \infty\). The assertion (iii) follows from the definition of \(h_j\).
For (iv), note that \(\exp (2h_j)\) is convex in \(\rho(x)\) except only at \(\rho(x) = j/(2\alpha)\). If \(e = [u, v]\) does not contain such exceptional points, then
\[
\exp (2h_j(x)) \leq d(x, u) \rho(u) + (1 - d(x, u)) \rho(v), \quad x \in e = [u, v].
\]
In this case, we get
\[
\int_e \exp (2h_j) d\mu \leq \frac{1}{2} \left( \exp (2h_j(u)) + \exp (2h_j(v)) \right).
\]
If \(e = [u, v]\) contains the exceptional points \(x\) with \(\rho(x) = j/(2\alpha)\), by the definition of \(h_j\), we obtain
\[
\int_e \exp (2h_j) d\mu \leq \frac{1}{2} \left( \exp (-2\alpha \rho(u) + 2j) + \exp (2\alpha \rho(v)) \right)
\]
\[
\leq \frac{1}{2} \left\{ \exp (-2\alpha (\rho(u) - 1) + 2j) + \exp (2\alpha (\rho(u) + 1)) \right\}
\]
\[
= \frac{1}{2} \exp (2\alpha) \{ \exp (2h_j(u)) + \exp (2h_j(v)) \},
\]
where we used the fact \(|\rho(u) - \rho(v)| \leq 1\).

Let \(\{\mathcal{K}_i\}_{i=1}^\infty\) be an increasing sequence of finite subgraphs of \(G - K\) such that \(\bigcup_i \mathcal{X}_i = \mathcal{G} - \mathcal{X}\). We set a sequence of functions on \(\mathcal{G} - \mathcal{X}\), \(\{\chi_i\}_{i=1}^\infty\), satisfying the following:

(i) \(\text{supp} (\chi_i) \subset \mathcal{X}_i \subset \mathcal{G} - \mathcal{X}\),

(ii) \(\chi_i(x) = 1\), if \(d(x, \mathcal{G} - \mathcal{X}_i) \geq 1\), \(x \in \mathcal{G} - \mathcal{X}\),

(iii) \(\chi_i(x) = d(x, \mathcal{G} - \mathcal{X}_i)\), if \(0 \leq d(x, \mathcal{G} - \mathcal{X}_i) \leq 1\).

Then we get

(iv) \(|\nabla \chi_i| \leq 1\),

(v) \(\text{supp} (\nabla \chi_i) \subset B_1(\partial \mathcal{X}_i) = \{ x \in \mathcal{X}_i; d(x, \partial \mathcal{X}_i) \leq 1 \}\).

Moreover, we obtain

(vi) \(\int_{\mathcal{X}_i - B_1(\partial \mathcal{X}_i)} \exp (2h_j) d\mu \leq \exp (2\alpha) (f, f)_{\mathcal{G} - \mathcal{X}}\),

where \((f, f)_{\mathcal{G} - \mathcal{X}}\) is the inner product relative to the graph \(G - K\) of the function \(f\) which is defined as

\[
f(x) = \exp (h_j(x)) \chi_i(x), \quad x \in \mathcal{G} - \mathcal{X},
\]

for each \(j\) and \(i\), since from (iv) of Lemma 10.1,
\[
\int_{\mathcal{X}_i - B_1(\partial \mathcal{X}_i)} \exp (2h_j) d\mu \leq \int_{\mathcal{X}_i - B_1(\partial \mathcal{X}_i)} \exp (2h_j) \chi_i^2 d\mu
\]
\[
\leq \frac{1}{2} \exp (2\alpha) \sum_{u \in \mathcal{V}} m(u) \exp (2h_j(u)) \chi_i(u)^2
\]
\[
= \exp (2\alpha) (f, f)_{\mathcal{G} - \mathcal{X}},
\]

where \(\mathcal{V}\) is the set of vertices of \(G\).

Now, we need to estimate
\[
\int_{\gamma-x} f^2 d\mu = \int_{\mathcal{G}-x} f^2 d\mu = \int_{B_1(\partial \mathcal{X}_i)} f^2 d\mu + \int_{\mathcal{X}_i - B_1(\partial \mathcal{X}_i)} f^2 d\mu.
\]

The second term of the right hand side of (10.2) can be estimated as follows:
\[
\int_{\mathcal{X}_i - B_1(\partial \mathcal{X}_i)} f^2 d\mu \leq C (f, f)_{\mathcal{G} - \mathcal{X}} \leq C \lambda_0 (G - K)^{-1} \int_{\gamma-x} |\nabla f|^2 d\mu,
\]
where \(C = \exp (2\alpha)\). Since \(\nabla f = \nabla (\exp (h_j) \chi_i) = \exp (h_j \nabla h_j + \nabla \chi_i)\).
\[ \int_{g^{-X}} |\nabla f|^2 \, d\mu = \int_{g^{-X}} \exp \left( 2h_j \right) \left( \chi_i^2 |\nabla h_j|^2 + 2 \chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2 \right) \, d\mu \leq \int_{g^{-X}} f^2 |\nabla h_j|^2 \, d\mu + \int_{g^{-X}} \exp \left( 2h_j \right) \left( 2 \chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2 \right) \, d\mu. \]

Since \( |2 \chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2| \leq 1 + 2\alpha \) by (i) of Lemma 10.1 and (iv) above, we have, together with (v) above,

\[ \int_{g^{-X}} |\nabla f|^2 \, d\mu \leq \int_{g^{-X}} f^2 |\nabla h_j|^2 \, d\mu + (1 + 2\alpha) \int_{B_i(\partial X)} \exp \left( 2h_j \right) \, d\mu. \]

Therefore, we obtain

\[ \int_{g^{-X}} f^2 \left( 1 - C\lambda_0(G - K)^{-1}\alpha^2 \right) \, d\mu \leq \int_{g^{-X}} f^2 \left( 1 - C\lambda_0(G - K)^{-1}\alpha^2 \right) |\nabla h_j|^2 \, d\mu \leq (1 + (1 + 2\alpha)C\lambda_0(G - K)^{-1}) \int_{B_i(\partial X)} \exp \left( 2h_j \right) \, d\mu. \]

Here, as \( i, j \to \infty \),

\[ \int_{B_i(\partial X)} \exp \left( 2h_j \right) \, d\mu \to \int_{B_i(\partial X)} \exp \left( 2\alpha \rho \right) \, d\mu, \]

which is finite because of the compactness of \( B_i(\partial X) \). Thus, recalling \( C = \exp \left( 2\alpha \right) \), if \( 1 - C\lambda_0(G - K)^{-1}\alpha^2 \) > 0, that is, \( \lambda_0(G - K) > \alpha^2 \exp \left( 2\alpha \right) \), then

\[ \int_{g^{-X}} \exp \left( 2h_j \right) \chi_i^2 \, d\mu = \int_{g^{-X}} f^2 \, d\mu \]

is bounded by a constant not depending on \( i \) and \( j \). By (iii) of Lemma 10.1, as \( i, j \to \infty \), we can conclude that \( \int_{g^{-X}} \exp \left( 2\alpha \rho \right) \, d\mu \) is finite, which is what we set out to prove. \( \square \)

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