1. Introduction

In [7], Professor Karatzas asks us the question “What can be said about the American version of the Asian option?”. Hence, in this paper we study the problem of pricing the perpetual American call on the time-average of the stock. Let us suppose that the price \( X^*(t) \) of the stock is given by

\[
X^*(t) = x \exp \left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \right\}, \quad t \geq 0,
\]

where \( x \geq 0 \) is an initial price, a mean rate \( \mu \) and a volatility \( \sigma \) are positive constants, and \( \{ W(t); 0 \leq t < \infty \} \) is a 1-dimensional Brownian motion on a complete filtrated probability space \( (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}^W_t) \). Further we assume that there exists the dividend payment per dollar invested in the stock, and its rate is a positive constant \( \delta \). Then the discounted payoff process for the perpetual American call on the time-average of the stock is given by

\[
\tilde{Y}(t) = \begin{cases} (x - k)^+, & t = 0, \\ e^{-r \tau} \left( \frac{1}{\tau} \int_0^\tau X^*(s) ds - k \right)^+, & 0 < t < \infty, \\ 0, & t = \infty, \end{cases}
\]

where \( r > 0 \) is the risk-less interest rate and \( k > 0 \) is the exercise price.

According to [9, §1.7], we adopt the risk-neutral martingale measure \( \mathbb{P}_0 \) which is equivalent to \( \mathbb{P} \) on \( (\mathcal{F}^T_t; 0 \leq t \leq T) \) for every finite \( T > 0 \), where \( \mathcal{F}^T_t \) is a sub-\( \sigma \)-algebra of the completion of \( \mathcal{F}^T_t \) with respect to \( \mathbb{P} \). Then, under \( \mathbb{P}_0 \), \( X^*(t) \) becomes

\[
X^*(t) = x \exp \left\{ \left( r - \delta - \frac{\sigma^2}{2} \right) t + \sigma W_0(t) \right\}, \quad t \geq 0,
\]

where \( \{ W_0(t); 0 \leq t < \infty \} \) is the \( \mathbb{P}_0 \)-Brownian motion \( W_0(t) = W(t) + t(\mu + \delta - r)/\sigma \).

The purpose of this paper is to prove that the value of the perpetual American call on the time-average of the stock is given by

\[
\hat{X}(t) = \operatorname{ess} \sup_{\tau \geq 0} E_0 \left[ e^{r \tau} \tilde{Y}(\tau) \bigg| \mathcal{F}^T_{\tau} \right] = \sup_{\tau \geq 0} E_0 \left[ e^{-r \tau} \left( y + \int_0^\tau X^*(u) du \right) \frac{x^*}{t + \tau} - k \right]^{+}, \quad \hat{x} = X^*(t)
\]

and to find the associated free-boundary problem.

For a general arbitrage-based theory for the pricing of American contingent claims, we can refer [3, 7, 9, 11]. The article [5] has characterized the value of the American put option in terms of the unique solution of the associated free-boundary problem. The perpetual American contingent claims include the “Russian option” and “Integral option”. The article [10] has studied the integral option, and found the analytic solution of the associated free-boundary problem. The Russian option has been analyzed by [4, 13–15]. In particular, the articles [13, 14] have found the analytic solution of the associated free-boundary problem.
The plan of this paper is as follows: Sect. 2 presents some properties of the expected payoff function \( g^* \) defined by (2.3). Then \( g^* \) is characterized by a unique solution of the associated free-boundary problem in Sect. 3. In Sect. 4 we prove that the value of the perpetual American call on the time-average of the stock is given by (1.1), and Sect. 5 concludes this paper.

2. Optimal Stopping Problem

In this section we investigate the some properties of the value function \( g^* \) defined as (2.3). First of all we have the following lemma which permits us to use the theory of optimal stopping for continuous-parameter processes.

Lemma 2.1.

\[
\begin{align*}
(i) \quad & \lim_{t \to \infty} \hat{Y}(t) = \hat{Y}(\infty) = 0, \quad P_0\text{-a.s.} \\
(ii) \quad & \mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} \hat{Y}(t) \right] \leq x \left( 1 + \frac{\sigma^2}{2\delta} \right), \quad x \geq 0.
\end{align*}
\]

Proof. Define \( \hat{A}(t) = e^{-\gamma t} \int_0^t X'(u)du, \ t > 0 \). Then we have

\[
\mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} \hat{A}(t) \right] \leq x \mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} 1 \int_0^t e^{\sigma W(t) - (\beta + \frac{\gamma^2}{2})t} du \right] \leq x \frac{x}{2\delta} \int_0^{\infty} e^{-\gamma t} dt \leq \frac{x}{\delta}\frac{x}{2\delta}.
\]

Thus Chebyshev’s inequality gives, for any \( \varepsilon > 0 \),

\[
\sum_{n=0}^{\infty} P_0 \left\{ \sup_{0 \leq t \leq \infty} \hat{A}(t) > \varepsilon \right\} \leq \frac{2x}{\delta}\frac{x}{2\delta} < \infty,
\]

and hence, it follows from Borel–Cantelli’s lemma that

\[
\limsup_{t \to \infty} \hat{A}(t) = 0 \quad P_0\text{-a.s.},
\]

which implies (i). Also, thanks to [8, Exercise 3.5.9, p.197], we obtain

\[
\mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} \hat{Y}(t) \right] \leq \mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} \hat{A}(t) \right] \leq x \mathbb{E}_0 \left[ \sup_{0 \leq t \leq \infty} \exp \left( \sigma W(t) - (\beta + \frac{\gamma^2}{2})t \right) \right] \leq x \left( 1 + \frac{\sigma^2}{2\delta} \right).
\]

Let us define the optimal expected payoff function

\[
g^*(t,x,y) = \sup_{r \in \mathcal{R}} \mathbb{E}_0 \left[ e^{-rt} \left( \frac{y + \int_0^t X'(u)du}{t + \int_0^t 1 du} - k \right) \right], \quad t > 0, \ x,y \geq 0
\]

and \( g^*(0,x,y) = \lim_{t \downarrow 0} g^*(t,x,y) \) for \( x,y \geq 0 \), where \( \mathcal{R} \) is the set of stopping times taking values in \([0,\infty)\) and \( \mathbb{E}_0 \) denotes the expectation under the probability measure \( P_0 \). Then from (2.2) we know for \( t > 0 \) and \( x, y \geq 0 \)

\[
g^*(t,x,y) \leq \frac{y}{t} + x \left( 1 + \frac{\sigma^2}{2\delta} \right),
\]

and we have the following:

Proposition 2.2. For each \( t > 0 \) and \( x, y \geq 0 \),

(i) \( t \mapsto g^*(t,x,y) \) is a non-increasing, convex and continuous function with

\[
-\frac{1}{t + \Delta t} \left[ \frac{y}{t} + x \left( 1 + \frac{\sigma^2}{2\delta} \right) \right] \leq \frac{g^*(t + \Delta t,x,y) - g^*(t,x,y)}{\Delta t} \leq 0.
\]

(ii) \( x \mapsto g^*(t,x,y) \) is a non-decreasing, convex and uniformly continuous function with

\[
0 \leq \frac{g^*(t,x + \Delta x,y) - g^*(t,x,y)}{\Delta x} \leq 1 + \frac{\sigma^2}{2\delta}
\]

(iii) \( y \mapsto g^*(t,x,y) \) is a non-decreasing, convex and uniformly continuous function with
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\[ 0 \leq \frac{g^*(t, x, y + \Delta y) - g^*(t, x, y)}{\Delta y} \leq \frac{1}{t}. \]  

(2.6)

**Proof.** The monotonicity and convexity of each function are obvious. Fix \( t > 0 \) and \( x, y \geq 0 \). Because \( z_1^+ - z_2^+ \leq (z_1 - z_2)^+ \) for any \( z_1, z_2 \in \mathbb{R} \), we have

\[
g^*(t, x, y) - g^*(t + \Delta t, x, y) \leq \sup_{\tau \in \delta} E_0 \left[ e^{-rt} \left( \frac{y + \int_0^\tau X^s(u)du}{\tau + t} - k \right)^+ \right] \leq \sup_{\tau \in \delta} E_0 \left[ e^{-r \Delta t} \frac{y + \int_0^\tau X^s(u)du}{\tau + t} - k \right] \leq \frac{\Delta t}{t + \Delta t} \sup_{\tau \in \delta} \left[ \frac{e^{-r \tau} y}{\tau + t} + \sup_{\tau \in \delta} \left[ \frac{e^{-r \tau}}{\tau} \int_0^\tau X^s(u)du \right] \right].
\]

Moreover a few modification in the calculation above yields (2.5) and (2.6).

The following property will be used for the monotonicity of the stopping boundary.

**Proposition 2.3.** The function \( t \mapsto t(g^*(t, x, y) + k) \) is non-decreasing for each \( x, y \geq 0 \).

**Proof.** From (2.7) we have

\[
g^*(t, x, y) - g^*(t + \Delta t, x, y) \leq \frac{\Delta t}{t + \Delta t} \sup_{\tau \in \delta} E_0 \left[ e^{-r \tau} \left( \frac{y + \int_0^\tau X^s(u)du}{\tau + t} - k \right)^+ \right] \leq \frac{\Delta t}{t + \Delta t} \sup_{\tau \in \delta} \left[ \frac{e^{-r \tau} y}{\tau + t} + \frac{e^{-r \tau}}{\tau} \int_0^\tau X^s(u)du \right].
\]

and thus

\[
t \{ g^*(t, x, y) + k \} \leq (t + \Delta t) \{ g^*(t + \Delta t, x, y) + k \}.
\]

(2.8)

The next result ensures that

\[
\lim_{\tau \downarrow 0} g^* \left( t, X^s(t), \int_0^t X^s(s)ds \right) = g^*(0, x, 0).
\]

(2.9)

**Proposition 2.4.** \( \lim_{\tau \downarrow 0} g^*(t, x, t y) = g^*(0, x, 0) \) for every \( x \geq 0 \) and \( 0 \leq y \leq x \lor k \).

**Proof.** Fix \( x \geq 0 \). Because \( y \mapsto g^*(t, x, y) \) is non-decreasing, \( \lim_{\tau \downarrow 0} g^*(t, x, t y) \geq \lim_{\tau \downarrow 0} g^*(t, x, 0) = g^*(0, x, 0) \) for \( y \geq 0 \).

On the other hand, by the monotonicity of \( t \mapsto g^*(t, x, 0) \) and the monotone convergence theorem, we see easily that

\[
g^*(0, x, 0) = \sup_{\tau \in \delta} E_0 \left[ e^{-r \tau} \left( \frac{1}{\tau} \int_0^\tau X^s(u)du - k \right)^+ \right].
\]

Hence we have
Then, thanks to the theory of exponential functionals of Brownian motion, for
Proposition 2.5.
Therefore
\[ \lim_{t \downarrow 0} g^*(t, x, y) - g^*(0, x, 0) \leq \limsup_{t \downarrow 0} \mathbb{E}_0 \left[ e^{-rt} \left( \int_0^t X'(u) du - y \right) \right] \]
\[ \leq \mathbb{E}_0 \left[ \sup_{0 \leq t \leq \varepsilon} \left( y - \frac{1}{t} \int_0^t X'(u) du \right)^+ \right]. \]
Thus by letting \( \varepsilon \downarrow 0 \) we obtain
\[ \lim_{t \downarrow 0} g^*(t, x, y) - g^*(0, x, 0) \leq \mathbb{E}_0 \left[ \limsup_{t \downarrow 0} \left( y - \frac{1}{t} \int_0^t X'(u) du \right)^+ \right] = (y - x)^+, \]
and hence \( \lim_{t \downarrow 0} g^*(t, x, y) \leq g^*(0, x, 0) \) for \( 0 \leq y \leq x \).

**Proposition 2.5.** \( g^*(t, x, y) > 0 \) for \( t, y \geq 0 \) and \( x > 0 \).

**Proof.** Fix \( x > 0 \). For \( y > kt \) we get \( g^*(t, x, y) \geq (\frac{y}{k} - k)^+ > 0 \). Let \( Y \equiv kt \) and define
\[ \tau_1 = \inf \left\{ s > 0 \left| \frac{y + \int_0^s X'(u) du}{s + t} \geq k + 1 \right. \right\}. \]
Then, thanks to the theory\(^1\) of exponential functionals of Brownian motion, for \( T > 0 \),
\[ \mathbb{P}_0\{\tau_1 \leq T\} = \mathbb{P}_0 \left\{ \int_0^T X'(u) du \geq (k + 1)(s + t) - y, \quad 0 < 3s \leq T \right\} \]
\[ \geq \mathbb{P}_0 \left\{ \int_0^T X'(u) du \geq \frac{(k + 1)(T + t) - y}{x} \right\} = \mathbb{P}_0 \left\{ A^o \left( \frac{\pi T}{x} \right) \geq \frac{(k + 1)(T + t) - y}{4x} - \sigma^2 \right\} > 0, \]
where \( A^o(t) = \int_0^t e^{2(W(s) + \nu)} ds, \quad \nu = 2(r - \delta)/\sigma^2 - 1 \) and a real-valued Brownian motion \( \tilde{W}(t) = \sigma W(t)/\sigma^2 \). Thus we have

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\(^1\) Cf. [17, Proposition 2.6.2, p. 43]
\[ g^*(t, x, y) \geq E_0 \left[ e^{-r(t \wedge T)} \left( \frac{y + \int_0^{t \wedge T} X^u(u) du}{T} - k \right) \right] \]
\[ \geq E_0 \left[ e^{-r(t)} \left( \frac{y + \int_0^{t} X^u(u) du}{T} - k \right) \right] \]
\[ \geq E_0 \left[ e^{-r(t)} \left( \frac{y + \int_0^{t} X^u(u) du}{t} - k \right) \right] \]
\[ \geq e^{-rT} P_0 \{ t_1 \leq T \} > 0. \]

For the behavior of \( g^* \) as \( t, x, y \to \infty \), we have the following.

**Proposition 2.6.**

(i) \( \lim_{t \to \infty} g^*(t, x, y) = 0, \quad x, y \geq 0. \)

(ii) \( \lim_{t \to \infty} \frac{g^*(t, x, y)}{x} = \sup_{t \in \delta} E_0 \left[ \frac{e^{-rT} T}{\int_0^{t} X^u(u) du} \right] \quad t > 0, \quad y \geq 0. \)

(iii) \( \lim_{t \to \infty} \frac{g^*(t, x, y)}{y} = \frac{1}{t}, \quad t, x \geq 0. \)

**Proof.** For \( t > 1 \), we have

\[ g^*(t, x, y) - \frac{y}{t} \leq \sup_{t \in \delta} E_0 \left[ e^{-rT} \frac{y + \int_0^{t} X^u(u) du}{\tau + t} \right] \leq \frac{y}{t} \]
\[ \leq \sup_{t \in \delta} E_0 \left[ \frac{e^{-rT}}{\tau + t} \int_0^{t} X^u(u) du \right] \leq \sup_{t \in \delta} E_0 \left[ \frac{1}{\tau + t} \int_0^{t} X^u(u) du + \frac{e^{-rT}}{\tau + t} \int_0^{t} X^u(u) du \right] \]
\[ \leq \frac{1}{\sqrt{t}} \left( 1 + \frac{\sigma^2}{2\delta} \right) + \sup_{t \geq \sqrt{t} - 1} E_0 \left[ \frac{e^{-rT}}{\tau + t} \int_0^{t} X^u(u) du \right]. \]

Hence, by the dominated convergence theorem and (2.1), we get

\[ \lim_{t \to \infty} g^*(t, x, y) \leq E_0 \left[ \lim_{T \to \infty} \sup_{t \in \delta} \frac{e^{-rT}}{\int_0^{t} X^u(u) du} \right] = 0. \]

To obtain (ii) and (iii) we may use the following inequalities:

\[ \sup_{t \in \delta} E_0 \left[ \frac{e^{-rT}}{\tau + t} \int_0^{t} X^u(u) du \right] \leq \frac{\int g^*(t, x, y) du}{x} \leq \frac{\int g^*(t, x, y) du}{x} \leq \frac{1}{t} + \sup_{t \in \delta} E_0 \left[ \frac{e^{-rT}}{\tau + t} \int_0^{t} X^u(u) du \right]. \]

**3. Free Boundary Problem**

In this section we characterize \( g^* \) as a unique solution to the corresponding free-boundary problem. Let us define the continuation region

\[ C = \left\{ (t, x, y) \in (0, \infty)^3 \left| g^*(t, x, y) > \left( \frac{y}{t} - k \right)^+ \right. \right\} \]

and consider its sections

\[ C(t, y) = \left\{ x \geq 0 \left| g^*(t, x, y) > \left( \frac{y}{t} - k \right)^+ \right. \right\}, \quad t, y > 0. \]
Because $g^*$ is continuous, $\mathcal{C}$ is open in $(0, \infty)^3$ and each $\mathcal{C}(t,y)$ is open in $(0, \infty)$. Further, since $x \mapsto g^*(t,x,y)$ is non-decreasing, we note that $\mathcal{C}(t,y) = (c(t,y), \infty)$, where

$$c(t,y) = \inf \{x \in \mathcal{C}(t,y) : x \geq 0\}. \quad (3.1)$$

We are now in a position to estimate the several features of the stopping boundary $c(\cdot)$.

**Proposition 3.1.** For each $t, y > 0$, $c(t,y) \leq \left(\frac{y}{t} + r \left(\frac{y}{t} - k \right)\right) \mathbf{I}_{\left[y > k\right]}$.

**Proof.** If $y \leq kt$, then by Proposition 2.5 we have $g^*(t,x,y) > 0 = (\frac{y}{t} - k)^+$ for all $x > 0$, that is, $c(t,y) = 0$ for $y \leq kt$. Let $(t, x, y) \in (0, \infty)^3$ with $y > kt$ and $x > \frac{y}{t} + rt(\frac{y}{t} - k)$, and define

$$A_1(s) = y + \int_0^s X^*(u) du - k(s + t), \quad s \geq 0,$$

$$A_2(s) = X^*(s) - \left( r + \frac{1}{s + t} \right) \left( \int_0^s X^*(u) du \right) + r k(s + t), \quad s \geq 0,$$

$$\tau_i = \inf \{ s > 0 | A_i(s) \leq 0 \}, \quad i = 1, 2.$$

Then $\tau_1, \tau_2 > 0$ $\mathbf{P}_1$-a.s. because $A_1(0) = y - kt > 0$ and $A_2(0) = x - \frac{y}{t} - rt(\frac{y}{t} - k) > 0$. Itô formula for convex functions\(^2\) gives

$$g^*(t,x,y) \geq \mathbf{E}_0 \left[ e^{r(\tau_1 \wedge \tau_2 \wedge T)} \left( \frac{y + \int_0^{\tau_1 \wedge \tau_2 \wedge T} X^*(u) du}{\tau_1 \wedge \tau_2 \wedge T + t} - k \right)^+ \right]$$

$$= \left( \frac{y}{t} - k \right)^+ + \mathbf{E}_0 \left[ \int_0^{\tau_1 \wedge \tau_2 \wedge T} \frac{e^{-rs}}{s + t} A_2(s) \mathbf{I}_{\{A_2(s) > 0\}} ds \right] \geq \left( \frac{y}{t} - k \right)^+.$$

Hence $c(t,y) \leq \frac{y}{t} + rt(\frac{y}{t} - k)$ for $y > kt$. \hspace{1cm} \Box

**Proposition 3.2.** For each $t, y > 0$, the functions $t \mapsto c(t,y)$ and $y \mapsto c(t,y)$ are non-increasing and non-decreasing, respectively.

**Proof.** For $y \leq k(t + \Delta t)$, we have $c(t + \Delta t, y) = 0 \leq c(t,y)$. Let $y > k(t + \Delta t)$. Then, from (2.8),

$$g^*(t + \Delta t, x,y) - \left( \frac{y}{t + \Delta t} - k \right)^+ \geq \left( \frac{t}{t + \Delta t} \right) g^*(t,x,y) - \left( \frac{y}{t} - k \right)^+.$$

Hence $x \in \mathcal{C}(t + \Delta t, y)$ if $x \in \mathcal{C}(t,y)$, that is, $c(t + \Delta t, y) \leq c(t,y)$.

If $y \leq kt$, then $c(t, y) = 0 \leq c(t, y + \Delta y)$. Let $y > kt$. Then, from (2.6),

$$g^*(t, x,y + \Delta y) - \left( \frac{y + \Delta y}{t} - k \right)^+ \leq g^*(t, x,y) + \frac{\Delta y}{t} - \frac{y + \Delta y}{t} + k = g^*(t, x,y) - \left( \frac{y}{t} - k \right)^+.$$

Hence $x \in \mathcal{C}(t,y)$ if $x \in \mathcal{C}(t,y + \Delta y)$, which implies the second assertion. \hspace{1cm} \Box

**Proposition 3.3.** For each $t, y > 0$, the functions $t \mapsto c(t,y)$ and $y \mapsto c(t,y)$ are upper semicontinuous, and are left-continuous and right-continuous, respectively.

**Proof.** Fix $t, y > 0$. Let $\{t_n\}_{n=1}^{\infty} \subset (0, \infty)$ with $\lim_{n \to \infty} t_n = t$ and $\lim_{n \to \infty} c(t_n, y) = c_0$. Because $\mathcal{C}$ is open and $(t_n, c(t_n, y), y) \notin \mathcal{C}$ for every $n$, we have $(t, c_0, y) \notin \mathcal{C}$ and thus $c_0 \leq c(t,y)$. In other words, $\lim \sup_{y \to c_0} c(t,y) \leq c(t,y)$. This proves the upper semicontinuity of $c(\cdot, y)$, and thus $c(\cdot, y) = c(t, y)$ since $c(\cdot, y)$ is non-increasing. The roles of $t$ and $y$ in this argument may be exchanged to prove the assertions for the function $y \mapsto c(t,y)$. \hspace{1cm} \Box

For the simplicity of the notation, we set for each $t, x, y \geq 0$,

$$H^{t,x,y}(s) = \left( t + s, X^*(s), y + \int_0^s X^*(u) du \right), \quad s \geq 0.$$

Here is the fundamental result of this section.

**Theorem 3.4.** The optimal expected payoff function $g^*$ of (2.3) is the unique solution on $\overline{\mathcal{C}}$ of the initial-boundary value problem

\(^2\) See [8, Theorem 3.6.22, p. 214 and Problem 3.6.7(i), p. 204]
\[ Lf = 0 \text{ in } \mathcal{C} = \{(t, x, y) \in (0, \infty)^3 \mid x > c(t, y)\}, \quad \text{(3.3)} \]
\[ f(t, c(t, y), y) = \frac{y}{t} - k, \quad t, y > 0, \ c(t, y) > 0, \quad \text{(3.4)} \]
\[ f(t, 0, y) = \left(\frac{y}{t} - k\right)^+, \quad t, y \geq 0, \quad \text{(3.5)} \]
\[ |f(t, x, y)| \leq c_0 \left\{ \frac{y}{t} + x + 1 \right\} \text{ for some constant } c_0 > 0, \quad \text{(3.6)} \]
where \( Lf = \frac{\partial^2}{\partial x^2} f_x + (r - \delta) f_x + x f_y + f - rf \). In particular, the partial derivatives \( g^*_x, g^*_y, g^*_t \) and \( g^*_r \) exist and are continuous on \( \mathcal{C} \).

**Proof.** It follows from (2.3), (2.4) and (3.1) that \( g^* \) satisfies the conditions (3.4)--(3.6). In order to verify the equation (3.3) for \( g^* \), let \( \mathcal{C}_\varepsilon = (\varepsilon, \infty)^3 \cap \mathcal{C} \) for \( \varepsilon > 0 \), and let us take a point \( (t, x, y) \in \mathcal{C}_\varepsilon \) and a region \( \mathcal{R} = (t_1, t_2) \times (x_1, x_2) \times (y_1, y_2) \) with \( (t, x, y) \in \mathcal{R} \subset \mathcal{C}_\varepsilon \). We consider the initial-boundary value problem
\[ Lf = 0, \quad \text{in } \mathcal{R}, \quad f = g^*, \quad \text{on } \partial_0 \mathcal{R}, \]
where \( \partial_0 \mathcal{R} = \partial \mathcal{R} \setminus \{(t_1) \times (x_1, x_2) \times (y_1, y_2)\} \). The classical theory for parabolic equations guarantees the existence of a unique solution \( f \) with \( f_x, f_y, f_t \) and \( f \) continuous. We have to prove that \( f \) and \( g^* \) agree on \( \mathcal{R} \).

Let us define the stopping times
\[ \tau^* = \inf \{ s > 0 \mid H^{t,\xi}(s) \notin \mathcal{C} \}, \]
\[ \tau = \inf \{ s > 0 \mid H^{t,\xi}(s) \notin \mathcal{R} \} \leq (t_2 - t) + \tau^*, \]
and the process \( N(s) = e^{-(r/\tau^*)} f(H^{t,\xi}(s \wedge \tau)), s \geq 0 \). By Itô formula we see that \( N(\cdot) \) is a bounded \( \mathbb{P}_0 \)-martingale, and thus
\[ f(t, x, y) = N(0) = \mathbb{E}_0[N(t_2 - t)] = \mathbb{E}_0\left[ e^{-rt^*} f(H^{t,\xi}(\tau^*)) \right]. \]

Since \( H^{t,\xi}(t) \in \partial_0 \mathcal{R} \), thanks to [9, Theorem D.9, p. 355 and Theorem D.12, p. 358] and the optional sampling theorem, we have
\[ f(t, x, y) = \mathbb{E}_0\left[ e^{-rt^*} g^*(H^{t,\xi}(\tau^*)) \right] = \mathbb{E}_0\left[ e^{-rt^*} g^*(H^{t,\xi}(\tau^*) \wedge \tau^*) \right] = g^*(t, x, y). \]
Thus \( f \) and \( g^* \) agree on \( \mathcal{R} \), and hence \( g^*_x, g^*_y, g^*_t \) and \( g^*_r \) are defined, continuous, and \( Lg^* = 0 \) at any point \( (t, x, y) \in \mathcal{C} \) because \( \mathcal{C} = \bigcup_{\varepsilon > 0} \mathcal{C}_\varepsilon = \lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon \) is open.

To establish uniqueness, suppose \( f \) is a solution to (3.3)--(3.6). For \( (t, y) \in (0, \infty)^2 \) and \( x > c(t, y) \), define
\[ \tau^* = \inf \{ s > 0 \mid X^y(s) \leq c \left( t + s, y + \int_0^s X^y(u) du \right) \}. \]

Itô formula shows that \( e^{-(r/\tau^*)} f(H^{t,\xi}(\cdot \wedge \tau^*)) \) is a \( \mathbb{P}_0 \)-martingale for some sequence \( \{\tau_n\}_{n=1}^{\infty} \) of stopping times with \( \tau_n \uparrow \infty \) a.s. as \( n \to \infty \). Hence
\[ f(t, x, y) = \mathbb{E}_0\left[ e^{-rt^*} f(H^{t,\xi}(\tau_n \wedge \tau^*)) \right]. \]

From (2.2) and (3.6),
\[ \mathbb{E}_0\left[ \sup_{T \geq 0} e^{-rt} \left| f(H^{t,\xi}(T)) \right| \right] \leq c_0 \mathbb{E}_0\left[ \sup_{T \geq 0} e^{-rt} \left( \frac{y + \int_0^T X^y(u) du}{T + t} + X^y(T) + 1 \right) \right] \leq c_0 \left\{ 1 + \frac{y}{t} + 2x \left( 1 + \frac{\sigma^2}{2\delta} \right) \right\} < \infty. \quad \text{(3.7)} \]
Thus, by the dominated convergence theorem,
\[ f(t, x, y) = \lim_{n \to \infty} \mathbb{E}_0\left[ e^{-r(\tau_n \wedge \tau^*)} f(H^{t,\xi}(\tau_n \wedge \tau^*)) \right] = \mathbb{E}_0\left[ e^{-rt^*} f(H^{t,\xi}(\tau^*)) \right] = \mathbb{E}_0\left[ e^{-rt^*} \left( \frac{y + \int_0^{\tau^*} X^y(u) du}{\tau^* + t} - k \right)^+ \right] = g^*(t, x, y), \]
since \( \tau^* \) is optimal from [9, Theorem D.12, p. 358].
Proposition 3.5. Fix \( y > 0 \). If \( c(\cdot, y) \) is continuous at point \( t \), then \( x \mapsto g^*(t, x, y) \) is of class \( C^1(0, \infty) \). In particular, 
\[ g^*_t(t, c(t, y) + y) = 0. \]

Proof. Let \( t, y > 0 \) with \( c(\cdot, y) \) continuous at point \( t \). If \( c(t, y) = 0 \), Theorem 3.4 implies this smoothness of \( g^*(t, \cdot, y) \). Assume \( c(t, y) > 0 \) [and thus \( y > k t \) from (3.2)]. Because \( g^*(t, x, y) = \gamma - k \) for \( 0 \leq x < c(t, y) \), we have \( g^*_t(t, c(t, y) + y) = 0 \). The convexity of \( x \mapsto g^*(t, x, y) \), which was proved in Proposition 2.2, implies that 
\[ g^*_t(t, c(t, y) + y) \geq 0. \]

Thus, it suffices to show \( g^*_t(t, c(t, y) + y) \leq 0 \). To this end, set \( x = c(t, y) \) and define
\[ \tau^*_x = \inf \{ s > 0 \mid X^{c+\xi}(s) \leq c(t + s, y + \int_0^s X^{c+\xi}(u) du) \}, \]
\[ \tau_x = \inf \{ s > 0 \mid X^{c+\xi}(s) \leq c(t + s, y) \} \]
for \( \xi > 0 \). Since \( y \mapsto c(t, y) \) is non-decreasing, \( \tau^*_x \leq \tau_x \) for every \( \xi > 0 \). Furthermore, because \( \xi \mapsto \tau_x \) is non-decreasing and \( s \mapsto c(s, y) \) is continuous in \((t - 2\xi, t + 2\xi)\) for some \( \xi > 0 \), we have
\[ \mathbf{P}_0 \left\{ \lim_{\xi \downarrow 0} \tau_{\xi} > 0 \right\} = \lim_{\xi \downarrow 0} \mathbf{P}_0 \left\{ \tau_{\xi} > 0 \right\} = \lim_{\eta \downarrow 0} \lim_{\xi \downarrow 0} \mathbf{P}_0 \left\{ \tau_x > \eta \right\}
= \lim_{\eta \downarrow 0} \lim_{\xi \downarrow 0} \mathbf{P}_0 \left\{ X^{c+\xi}(s) > c(t + s, y), \forall s \in [0, \eta \wedge \xi] \right\}
= \lim_{\eta \downarrow 0} \mathbf{P}_0 \left\{ \min_{0 \leq s \leq \eta \wedge \xi} X^t(s) - \frac{c(t + s, y)}{c(t, y) + \xi} > 0 \right\}
\leq \lim_{\xi \downarrow 0} \mathbf{P}_0 \left\{ \min_{0 \leq s \leq \xi} X^t(s) - \frac{c(t + \eta \wedge \xi, y)}{c(t, y) + \xi} > 0 \right\} = \mathbf{P}_0 \left\{ \lim_{\xi \downarrow 0} \inf_{x \in [0, \eta]} X^t(s) > 1 \right\} = 0, \]
where the last equality follows from the law of the iterated logarithm for Brownian motion\(^3\). Hence
\[ \tau_x \downarrow 0 \text{ P}_0\text{-a.s. as } \xi \downarrow 0. \]

We also have
\[ g^*(t, x + \xi, y) - g^*(t, x, y) \leq \mathbf{E}_0 \left[ e^{-r\tau^*_x} \left( \left( \frac{y + (x + \xi) \int_0^{\tau_x} X^t(u) du}{\tau^*_x + t} - k \right)^+ \right) \right] \]
\[ \leq \xi \mathbf{E}_0 \left[ e^{-r\tau^*_x} \int_0^{\tau_x} X^t(u) du \right] \leq \frac{\xi}{t} \mathbf{E}_0 \left[ \int_0^{\tau_x} X^t(u) du \right]. \]
Hence (3.8) and the monotone convergence theorem give
\[ \lim_{\xi \downarrow 0} g^*(t, x + \xi, y) - g^*(t, x, y) \leq \frac{1}{t} \mathbf{E}_0 \left[ \lim_{\xi \downarrow 0} \int_0^{\tau_x} X^t(u) du \right] = 0, \]
i.e. \( g^*(t, c(t, y) + y) \leq 0 \).

The next expression of the Doob–Meyer decomposition is useful for pricing and hedging the perpetual American call option.

Theorem 3.6. For \( t, y \geq 0 \) and \( x > 0 \), define
\[ M^{x,y}(s) = g^*(t, x, y) + \sigma \int_0^s e^{-ru} X^u(s) g^*_u(H^{x,y}(u)) dW(u), \quad s \geq 0, \]
and
\[ L^{x,y}(s) = \int_0^s e^{-ru} K^{x,y}(u) \{ X(s) < c + \int_0^s X(u) du \} \, du, \quad s \geq 0, \]
where
\[ K^{x,y}(s) = \left( \frac{r - 1}{t + s} \right) \left( y + \int_0^s X^t(u) du \right) - X^t(s) - rk(t + s), \quad s \geq 0. \]

\( ^3 \) See [8, Theorem 2.9.23, p. 112]
Then $M^{t,x,y}(\cdot)$ is a $\mathbf{P}_0$-martingale, $\Lambda^{t,x,y}(\cdot)$ is non-decreasing, and
\[
e^{-r_t}g^s(H^{t,x,y}(s)) = M^{t,x,y}(s) - \Lambda^{t,x,y}(s), \quad s \geq 0,
\]
for $t,x > 0$ and $y \geq 0$. In particular, $M^{0,x,y}(\cdot)$ is a $\mathbf{P}_0$-martingale, $\Lambda^{0,x,y}(\cdot)$ is non-decreasing, and
\[
e^{-r_t}g^s(H^{0,x,y}(s)) = M^{0,x,y}(s) - \Lambda^{0,x,y}(s), \quad s \geq 0,
\]
for $x > 0$.

Proof. It follows immediately from (3.2) that $\Lambda^{0,x,y}(\cdot)$ is non-decreasing for each $x > 0$, and $\Lambda^{t,x,y}(\cdot)$ is non-decreasing for each $t,x > 0$ and $y \geq 0$. Because $0 \leq g^*_t \leq 1 + \frac{y}{t}$ and $\mathbf{E}[\int_0^t (e^{-r_s}X'(u))^2 du] < \infty$ for every $T \geq 0$, we know that $M^{0,x,y}(\cdot)$ is a $\mathbf{P}_0$-martingale for each $x > 0$, and $\Lambda^{t,x,y}(\cdot)$ is a $\mathbf{P}_0$-martingale for each $t,x > 0$ and $y \geq 0$.

In order to prove (3.12), let $\xi \in C^\infty(\mathbb{R}^3 \to [0, \infty))$ with $\int_{\mathbb{R}^3} \xi(u,v,w) du dv dw = 1$ and supp $\xi \subset [0,1]^3$. For $\varepsilon > 0$, define
\[
g^\varepsilon(t,x,y) = \int_0^\infty \int_0^\infty \int_0^\infty g^s(t + \varepsilon u, x + \varepsilon v, y + \varepsilon w) \xi(u,v,w) du dv dw.
\]
Then $g^\varepsilon(\cdot)$ is of class $C^\infty(0, \infty)^3$. Integrating by parts, we have
\[
g^\varepsilon(t,x,y) = \int_0^\infty \int_0^\infty \int_0^\infty g^s_x(\alpha, \beta, \gamma) \xi_\alpha \left( \frac{\alpha-t}{\varepsilon}, \frac{\beta-x}{\varepsilon}, \frac{\gamma-y}{\varepsilon} \right) du dv dy
\]
and
\[
g^\varepsilon(t,x,y) = -\frac{1}{\varepsilon^3} \int_0^\infty \int_0^\infty \int_0^\infty \left\{ g^s_x(\alpha, \beta, \gamma) \xi_\beta \left( \frac{\alpha-t}{\varepsilon}, \frac{\beta-x}{\varepsilon}, \frac{\gamma-y}{\varepsilon} \right) \right\}_{\beta=0}^{\beta=\infty} du dv dy.
\]
where $N(\gamma) = \{ \alpha > 0 \mid c(\alpha+\gamma) < c(\alpha, \gamma) \}$ for each $\gamma > 0$, and the last equality follows from Proposition 3.5. Since $N(\gamma)$ is a countable set for each $\gamma > 0$ by Proposition 3.3, the second term in (14.14) vanishes. These formulas show that $g^\varepsilon_x$ and $\mathcal{L}g^\varepsilon$ are bounded on compact subsets of $(0, \infty)^3$ and
\[
g^\varepsilon(t,x,y) = \lim_{\varepsilon \downarrow 0} g^\varepsilon(t,x,y),
\]
\[
\mathcal{L}g^\varepsilon(t,x,y) = \lim_{\varepsilon \downarrow 0} \mathcal{L}g^\varepsilon(t,x,y), \quad \forall (t,x,y) \in (0, \infty)^3, \quad x \neq c(t,y).
\]
According to Itô formula, for all $s \geq 0$,
\[
e^{-r_s}g^s(H^{t,x,y}(s)) - g^s(t,x,y) = \int_0^s e^{-r_u} \mathcal{L}g^\varepsilon(H^{t,x,y}(u)) du + \sigma^2 \int_0^s e^{-r_u} X^\varepsilon(u) g^s_x(H^{t,x,y}(u)) dW_0(u)
\]
\[
\rightarrow \int_0^s e^{-r_u} \mathcal{L}g^\varepsilon(H^{t,x,y}(u)) du + \sigma^2 \int_0^s e^{-r_u} X^\varepsilon(u) g^s_x(H^{t,x,y}(u)) dW_0(u)
\]
\[
e^{-r_t}g^s(H^{t,x,y}(s)) - g^s(t,x,y) = -\Lambda^{t,x,y}(s) + M^{t,x,y}(s) - g^s(t,x,y).
\]
For the convergence in (3.15) we have used \( P_t[X_i(u) = c(t + u, y + \int_0^u X_i(\xi)d\xi)] = 0 \) for each \( u \in (0, \infty) \). Thus we obtain (3.12). Furthermore the same arguments as above give, for \( 0 < t \leq s \),
\[
e^{-\alpha t}g^t(H^{0,1,0}(s)) - e^{-\alpha t}g^t(H^{0,1,0}(t)) = \{M^{0,1,0}(s) - M^{0,1,0}(t)\} - \{\Lambda^{0,1,0}(s) - \Lambda^{0,1,0}(t)\}
\]
From (2.9) we get (3.13) by letting \( t \downarrow 0 \).

**Corollary 3.7** For \( t > 0 \) and \( x, y \geq 0 \),
\[
g^t(x, y) = E_0\left[ \int_0^\infty \frac{e^{-\alpha t}K^{t,\chi}(y)}{t + u} \cdot I_{\{X(u) < c(t+u,y) + \int_u^\infty X(\xi)d\xi\}} \right]
\]
where \( K^{t,\chi}(\cdot) \) is defined by (3.11).

**Proof.** Fix \( t, x > 0 \) and \( y \geq 0 \). Theorem 3.6 yields
\[
g^t(x, y) = E_0[\frac{e^{-\alpha t}g^t(H^{t,\chi}(T))}{T} + E_0[\Lambda^{t,\chi}(T)]
\]
for \( t, x > 0 \) and \( y, T \geq 0 \). From (2.1) and (2.4), we have
\[
\lim_{T \to \infty} e^{-\alpha T} E_0[g^t(H^{t,\chi}(T))] \leq \lim_{T \to \infty} \left\{ \frac{e^{-\alpha T}g^t(H^{t,\chi}(T))}{T} + E_0[\Lambda^{t,\chi}(T)] \right\} + x \left( 1 + \frac{\sigma^2}{2\delta} \right) e^{-\alpha T} = 0.
\]
Thus, thanks to the monotone convergence theorem, we obtain (3.16) by letting \( T \to \infty \) in (3.17) for any \( t, x > 0 \) and \( y \geq 0 \).

Also we note that \( E_0[\sup_{0 \leq s \leq 1} g^t(H^{t,\chi}(T))] \leq E_0[g^t(H^{t,\chi}(T))] < \infty \) and
\[
E_0[\sup_{0 \leq s \leq 1} \Lambda^{t,\chi}(s)] \leq E_0\left[ \int_0^T \frac{e^{-\alpha t}g^t(H^{t,\chi}(s))}{t + u} \cdot (r + \frac{1}{t + u}) \cdot \left( y + \int_0^u X(\xi)d\xi \right) \right] \leq \frac{T}{t}\left( r + \frac{1}{t} \right) \cdot \left( y + \int_0^T E_0[X(s)]ds \right) < \infty.
\]
Since \( y \mapsto c(t, y) \) is right-continuous, by letting \( x \downarrow 0 \) in (3.17) and by the dominated convergence theorem,
\[
g^t(x, y) = e^{-\alpha t}E_0[g^t(T, 0, y)] + E_0[\lim_{t \to 0} \Lambda^{t,\chi}(T)]
\]
\[
e^{-\alpha t}\left( \frac{y}{t + T} + k \right) + \int_0^T \frac{e^{-\alpha t}g^t(H^{t,\chi}(s))}{t + u} \cdot \left( r + \frac{1}{t + u} \right) \cdot \left( y - rk(t + u) \right) \cdot I_{\{0 < c(t+u, y)\}}ds.
\]
Therefore we have (3.16) for \( x = 0 \) by letting \( T \to \infty \).

**Proposition 3.8.** Assume the function \( d : (0, \infty)^2 \to \mathbb{R} \) satisfy the following:

(i) \( d \) is a Lebesgue measurable function on \( (0, \infty)^2 \),

(ii) \( 0 \leq d(t, y) \leq \left( \frac{y}{t} + rt \cdot \frac{y}{t} - k \right) \cdot I_{\{y < k\}} \), for \( t, y > 0 \),

(iii) \( \psi_d(t, x, y) \geq \left( \frac{y}{t} - k \right)^+ \), for \( t, y > 0 \) and \( x \geq 0 \),

(iv) \( \psi_d(t, x, y) \leq \left( \frac{y}{t} - k \right)^+ \), for \( t, y > 0 \) and \( 0 \leq x \leq d(t, y) \),

where
\[
\psi_d(t, x, y) = E_0\left[ \int_0^\infty \frac{e^{-\alpha t}K^{t,\chi}(s)}{s + s} \cdot I_{\{X(s) < d(t+s, y) + \int_y^\infty X(\xi)d\xi\}} \right]
\]
and \( K^{t,\chi}(\cdot) \) is defined by (3.11). Then \( d = c \) on \( (0, \infty)^2 \).

**Proof.** From (3.1), (3.2) and (3.16) we see that \( c \) satisfies (i)–(iv). First we prove that \( \psi_d(t, x, y) \leq g^t(t, x, y) \) for \( t, y > 0 \) and \( x \geq 0 \). For \( x \leq d(t, y) \), we have \( \psi_d(t, x, y) = \left( \frac{y}{t} - k \right)^+ \leq g^t(t, x, y) \). Fix \( t, y > 0 \) and \( x > d(t, y) \), and define, \( s \geq 0 \),
\[
M_d(s) = e^{-\alpha s}\psi_d(H^{t,\chi}(s) + \int_0^s \frac{e^{-\alpha t}K^{t,\chi}(s)}{t + u} \cdot I_{\{X(s) < d(t+s, y) + \int_y^\infty X(\xi)d\xi\}} ds.
\]
From (3.18) we have
\[ e^{-rt} \psi_d(H^{x,y}(s)) = E_0 \left[ \int_s^\infty \frac{e^{-ru}}{t+u} K^{x,y}(u) I \left\{ X(u) - d(t+u) + \int_0^u X'(d) \right\} du \right] F^{(T)}_s, \]

for some \( T \in [s, \infty) \), and thus \( M_d(t) \) is a (restricted) \( P_0 \)-martingale. Let us define

\[ \tau_0 = \inf \left\{ s \geq 0 \mid X^x(s) \leq d \left( t + s, y + \int_0^s X^x(u) du \right) \right\}. \]

Then

\[
\psi_d(t, x, y) = M_d(0) = E_0[M_d(\tau_0 \wedge s)] = E_0[M_d(0) I_{\{\tau_0 \leq s\}}] + E_0[M_d(s) I_{\{\tau_0 > s\}}]
\]
\[
= E_0 \left[ e^{-r \tau_0} \left( y + \int_0^{\tau_0} X^x(u) du \right) \right] I_{\{\tau_0 \leq s\}} + E_0 \left[ e^{-r \tau_0} (y - k) \right] I_{\{\tau_0 > s\}}
\]
\[
+ \int_s^\infty \frac{e^{-ru}}{t+u} E_0 \left[ K^{x,y}(u) I \left\{ X^x(t) - d(t+u) + \int_0^u X'(d) \right\} I_{\{\tau_0 > s\}} \right] du
\]
\[
\leq g^*(t, x, y) + \int_s^\infty \frac{e^{-ru}}{t+u} \left( r + \frac{1}{t+u} \right) \left( y + E_0 \left[ \int_0^u X'(v) dv \right] \right) du
\]

By letting \( s \to \infty \) we obtain \( \psi_d(t, x, y) \leq g^*(t, x, y) \) for \( t, y > 0 \) and \( x \geq 0 \).

Next we show that \( d \geq c \) on \((0, \infty)^2 \) and \( \psi_d = g^* \) on \([0, \infty) \times [0, \infty) \times (0, \infty)\). Assume that \( d(t, y) < c(t, y) \) for certain \( t, y > 0 \). Let us define \( x = (d(t, y) + c(t, y))/2 \),

\[ \tau_1 = \inf \left\{ s \geq 0 \mid X^x(s) \geq c \left( t + s, y + \int_0^s X^x(u) du \right) \right\}, \]

and \( \tau_2 = \tau_1 \wedge \tau_0 \). Since \( d(t, y) < x < c(t, y) \),

\[ \left( \frac{y}{t} - \frac{k}{t} \right) \geq g^*(t, x, y) \geq \left( \frac{y}{t} - \frac{k}{t} \right). \quad (3.20) \]

On the other hand, by virtue of (3.12) and (3.19), we have

\[ g^*(t, x, y) - \psi_d(t, x, y) = E_0 \left[ e^{-r \tau_2} \left( g^*(H^{x,y}(t_2 \wedge s)) - \psi_d(H^{x,y}(t_2 \wedge s)) \right) \right] + E_0 \left[ \Lambda^{x,y}(t_2 \wedge s) \right]. \quad (3.21) \]

But the first term in (3.21) is non-negative and the second term in (3.21) is strictly positive by (3.10), which means that (3.21) is inconsistent with (3.20). Hence \( d \geq c \) on \((0, \infty)^2 \), and thus we have

\[ 0 \leq g^*(t, x, y) - \psi_d(t, x, y) \]
\[ = E_0 \left[ \int_0^\infty \frac{e^{-rs}}{t+s} K^{x,y}(s) \right] \left\{ I \left\{ X^{x,c} \left( t+s+y+\int_0^u X'(d) \right) \right\} - I \left\{ X^{x,c} \left( t+s+y+\int_0^u X'(d) \right) \right\} \right\} ds \leq 0, \]

from (3.16) and (3.18). Therefore \( \psi_d(t, x, y) = g^*(t, x, y) \) for every \( t, y > 0 \) and \( x \geq 0 \).

Finally, in order to prove \( d = c \) on \((0, \infty)^2 \), we assume that \( c(t, y) < d(t, y) \) for certain \( t, y > 0 \). Let us set \( x = (c(t, y) + d(t, y))/2 \), and define

\[ \widehat{\tau}_0 = \inf \left\{ s \geq 0 \mid X^x(s) \geq d \left( t + s, y + \int_0^s X^x(u) du \right) \right\}, \]
\[ \widehat{\tau}_1 = \inf \left\{ s \geq 0 \mid X^x(s) \leq c \left( t + s, y + \int_0^s X^x(u) du \right) \right\} \]

and \( \widehat{\tau}_2 = \widehat{\tau}_0 \wedge \widehat{\tau}_1 \). Then

\[ g^*(t, x, y) - \psi_d(t, x, y) = E_0 \left[ e^{-r \widehat{\tau}_2} g^*(H^{x,y}(\widehat{\tau}_2 \wedge s)) \right] - E_0[M_d(\widehat{\tau}_2 \wedge s)] \]
\[ = -E_0 \left[ \int_0^{\widehat{\tau}_2} \frac{e^{-ru}}{t+u} K^{x,y}(u) du \right] < 0, \]
which is in contradiction with $\psi_d = g^*$. Thus we obtain $d = c$ on $(0, \infty)^2$.

As we have shown, we can get the stopping boundary $c(\cdot)$ by solving the integral equation as in the above proposition, and then we can obtain the value function $g^*(\cdot)$ by calculating the expectation (3.16) or solving the initial-boundary value problem (3.3)–(3.6). Moreover, the following result enable us to estimate the value function and the stopping boundary simultaneously.

**Theorem 3.9.** Assume that a pair of functions $f : (0, \infty) \times [0, \infty)^2 \to (0, \infty)$ and $d : (0, \infty)^2 \to [0, \infty)$ satisfies the following:

(i) $f$ is continuous on $(0, \infty) \times [0, \infty)^2$,
(ii) $t \mapsto d(t, y)$ is non-increasing and left-continuous on $(0, \infty)$ for each $y > 0$,
(iii) $y \mapsto d(t, y)$ is non-decreasing and right-continuous on $(0, \infty)$ for each $t > 0$,
(iv) $f_t, f_s, f_y$ and $f_{st}$ are defined and continuous on the open set $D = \{(t, x, y) \in (0, \infty)^3 \mid x > d(t, y)\}$,
(v) $0 \leq d(t, y) \leq \left(\frac{y}{t} + rt\right)^{-} 1_{\{y \leq k\}}, \quad$ for $t, y > 0$,
(vi) $L f = 0$ in $D$,
(vii) $f(t, x, y) \geq \left(\frac{y}{t} - k\right)^{+}$ for $x, y \geq 0$ and $t > 0$,
(viii) $f(t, x, y) = \frac{y}{t} - k \quad$ for $t, y > 0$ and $0 < x \leq d(t, y)$,
(ix) $f(t, 0, y) = \left(\frac{y}{t} - k\right)^{+} \quad$ for $t > 0$ and $y \geq 0$,
(x) $|f(t, x, y)| \leq c_0 \left(\frac{y}{t} + x + 1\right), \quad$ for $x, y \geq 0, t > 0$ and some constant $c_0 > 0$,
(xi) $f_y(t, d(t, y) + y) = 0, \quad$ for all $t > 0$,
(xii) $f_y(t, d(t, y) + y) = 0, \quad$ for all $t \in (0, \infty) \setminus N(y)$, where $N(y) = \{t \in (0, \infty) \mid d(t, y) < d(t, y)\}$.

Then $f = g^*$ on $(0, \infty) \times [0, \infty)^2$ and $d = c$ on $(0, \infty)^2$.

**Proof.** By means of Proposition 2.2, Proposition 3.1, Proposition 3.3, Theorem 3.4 and Proposition 3.5, we know that $(g^*, c)$ satisfies (i)–(xii). Fix $t, x > 0$ and $y \geq 0$, and use the mollification argument of Theorem 3.6 to get the formula

$$e^{-rt} f(H^{t,x,y}(s)) = M^{t,x,y}_f(s) - \Lambda^{t,x,y}_d(s), \quad s \geq 0,$$

where $M^{t,x,y}_f(\cdot)$ is the $P_0$-local martingale

$$M^{t,x,y}_f(s) = f(t, x, y) + \sigma \int_0^s e^{-ru} X^u(f(H^{t,x,y}(u))) dW_0(u),$$

$\Lambda^{t,x,y}_d(\cdot)$ is the non-decreasing process

$$\Lambda^{t,x,y}_d(s) = \int_0^s e^{-ru} K^{t,x,y}(u) \left\{ X^u(s - d(t, u, y) + \int_0^s X^u(\tilde{d}(t, u, y)) \right\} du,$$

and $K^{t,x,y}(\cdot)$ is defined by (3.11). Let $\{\tau_n\}_{n=1}^\infty$ be a sequence of stopping times with $\tau_n \uparrow \infty$ almost surely and such that $\{M^{t,x,y}_f(s \wedge \tau_n) \mid 0 \leq s < \infty\}$ is a $P_0$-martingale. For any $\tau \in \delta$, we have

$$f(t, x, y) = E_0 \left[ e^{-r(t \wedge \tau)} f(H^{t,x,y}(\tau \wedge \tau_n)) + \Lambda^{t,x,y}_d(\tau \wedge \tau_n) \right]$$

By (3.7), the dominated convergence theorem and monotone convergence theorem, we get

$$f(t, x, y) = E_0 \left[ e^{-r t} f(H^{t,x,y}(\tau)) + \Lambda^{t,x,y}_d(\tau) \right]$$

for any $\tau \in \delta$, and thus

$$f(t, x, y) \geq \sup_{\tau \in \delta} E_0 \left[ e^{-r t} f(H^{t,x,y}(\tau)) \right] \geq \sup_{\tau \in \delta} E_0 \left[ e^{-r t} \left( \frac{y + \int_0^t X^u(du)}{\tau + t} - k \right)^{+} \right] = g^*(t, x, y).$$

In order to show $f(t, x, y) \leq g^*(t, x, y)$, we set
\[ \tau^* = \inf \left\{ s \geq 0 \mid X^x(s) \leq d \left( t + s, y + \int_0^t X^x(u)du \right) \right\}. \]

Then from (3.22) and (viii) we have for every \( T > 0, \)
\[
\begin{align*}
f(t, x, y) &= \mathbb{E}_0 \left[ e^{-rt} f(H^{a, T}(\tau^* \wedge T)) \right] \\
&= \mathbb{E}_0 \left[ e^{-rt} \left( \frac{y + \int_0^t X^x(u)du}{t + \tau^*} - k \right) I_{[t^* \leq T]} + e^{-RT} f(H^{a, T}(T)) I_{[t^* > T]} \right] \\
&\leq g^*(t, x, y) + \mathbb{E}_0 \left[ e^{-RT} f(H^{a, T}(T)) \right].
\end{align*}
\]

Also, by (x) we get
\[
\begin{align*}
\mathbb{E}_0 \left[ e^{-RT} f(H^{a, T}(T)) \right] &\leq c_0 e^{-RT} \mathbb{E}_0 \left[ \frac{y}{T + t} + \frac{1}{T + t} \int_0^T X^x(u)du + X^x(T) + 1 \right] \\
&\leq c_0 \left\{ e^{-RT} \frac{y}{T} + \frac{x}{T} + xe^{-RT} + e^{-RT} \right\} \rightarrow 0 \quad \text{as } T \rightarrow \infty,
\end{align*}
\]

and hence \( f(t, x, y) \leq g^*(t, x, y) \) on \((0, \infty) \times (0, \infty)^2\). Thus \( f = g^* \) on \((0, \infty) \times (0, \infty)^2\).

To show that \( d = c \), it suffices to establish \( \mathcal{D} = \mathcal{C} \). It follows from (v) and (viii) that
\[
\mathcal{L}f(t, x, y) = \mathcal{L} \left( \frac{y}{T} - k \right) = \left\{ x - \frac{y}{T} - rt \right\} < 0
\]
for \( x < d(t, y) \leq \left( \frac{1}{T} + rt(z - k) \right) I_{[z, y)}. \) Therefore, for \((t, x, y) \in \mathcal{C}\) we have \( \mathcal{L}f(t, x, y) = \mathcal{L}g^*(t, x, y) = 0 \), which means that \((t, x, y) \notin (0, \infty)^3 \setminus \mathcal{D} \). Thus \( \mathcal{C} \subset \mathcal{D} \) since \( \mathcal{C} \) and \( \mathcal{D} \) are open. The roles of \( \mathcal{C} \) and \( \mathcal{D} \) in this argument may be reversed to obtain \( \mathcal{D} \subset \mathcal{C} \).

As a corollary, we note the following.

**Remark 3.10.** Instead of (ii) and (iii) in Theorem 3.9, if we suppose that

- (ii)’ The function \( t \mapsto rt(f(t, x, y) + k) \) is non-decreasing for each \( x, y \geq 0, \)
- (iii)’ For each \( t > 0 \) and \( x, y \geq 0, \)
\[
0 \leq \frac{f(t, x, y + \Delta y) - f(t, x, y)}{\Delta y} \leq \frac{1}{t},
\]

and define
\[
\mathcal{D} = \left\{ (t, x, y) \in (0, \infty)^3 \mid f(t, x, y) > \left( \frac{y}{T} - k \right)^+ \right\}
\]
\[
\mathcal{D}(t, y) = \left\{ x > 0 \mid (t, x, y) \in \mathcal{D} \right\} = (d(t, y), \infty),
\]

then it is easy to see that \( d(t, y) \) satisfies (ii) and (iii) in Theorem 3.9 and hence \( f = g^* \).

4. **Value of the Perpetual American Call**

According to [9, Definition 1.7.7, p. 31 and Definition 2.6.2, p. 61], for each \( s \in [0, \infty), V^{AC}(s; \infty) \) is the value at time \( s \) of the perpetual American call on the time-average of the stock if and only if \( V^{AC}(s; \infty) \) is the minimal random variable \( \gamma(s) \) that is \( \mathcal{F}_s^{(T)} \)-measurable for some \( T \in [0, \infty) \)
\[
\tilde{Y}(t) \leq e^{-rt} \gamma(s) + \sigma \int_t^T e^{-rs} \pi(u) dW_0(u), \quad t \geq s
\]
for some martingale-generating portfolio process \( \pi(t) \) which means that \( \pi(t) \) is restrictedly progressively measurable, \( \int_0^T \pi(s)^2 ds < \infty \) for every finite \( T \) and \( [\sigma \int_0^T e^{-rs} \pi(u) dW_0(u); 0 \leq t < \infty] \) is a \( \mathbb{P}_0 \)-martingale.

**Theorem 4.1.** The value process for the perpetual American call on the time-average of the stock is given by \( [g^*(H^{0, 0}(s)); 0 \leq s < \infty] \) for each initial price \( x > 0 \) of the stock.

**Proof.** Fix \( x > 0 \) and define

\[ f(t, x, y) = \mathbb{E}_0 \left[ e^{-RT} f(H^{a, T}(T)) \right]. \]
\[
\pi^*(s) = X^s(s)g^*(H^{0,1,0}(s))1_{\{\bar{X}(s) > \varepsilon \}}(s, \int_0^s \bar{X}(u) du), \quad s > 0.
\]

Then from continuity of \(c(.)\) we know that \(\pi^*(.)\) is restrictedly progressively measurable. It follows from Theorem 3.6 that \([M^{(0,1,0)}(t) - M^{(0,1,0)}(0)] = \sigma \int_0^t e^{-\sigma u} \pi^*(u) dW_0(u); 0 \leq t < \infty\) is a \(P_0\)-martingale. And also we see that \(\int_0^s \pi^*(s)^2 ds < \infty\) for every finite \(T\) because 0 \(\leq g^*_s \leq 1 + \frac{\sigma^2}{2\sigma}\). Thus \(\pi^*(.)\) is a martingale-generating portfolio process. Moreover from (3.13) we have for 0 \(\leq s \leq t < \infty\),
\[
\begin{align*}
\hat{Y}(t) &= e^{-\sigma t} \left( \frac{1}{t} \int_0^t \bar{X}(u) du - k \right) \leq e^{-\sigma t} g^*(H^{0,1,0}(t)) \\
&= e^{-\sigma t} g^*(H^{0,1,0}(s)) + [M^{(0,1,0)}(t) - M^{(0,1,0)}(s)] - [\Lambda^{0,1,0}(t) - \Lambda^{0,1,0}(s)] \\
&\leq e^{-\sigma t} g^*(H^{0,1,0}(s)) + \sigma \int_t^s e^{-\sigma u} \pi^*(u) dW_0(u).
\end{align*}
\]

In order to establish this theorem, it suffices to show that \(g^*(H^{0,1,0}(s))\) is minimal. Fix \(s \in [0, \infty)\). Whenever (4.1) holds for some random variable \(\gamma(s)\) and some martingale-generating portfolio process \(\pi(.)\), we obtain for every \(\tau \in \delta\) and sufficiently large \(T \in [s, \infty)\),
\[
e^{-\sigma \tau} \gamma(s) \geq E_0 \left[ \hat{Y}(\tau \wedge T + s) \mid F_s^{(T)} \right] = e^{-\sigma \tau} E_0 \left[ e^{-\sigma (\tau \wedge T)} \left( \hat{y} + \int_0^{\tau \wedge T} \bar{X}(u) du \right) - k \right] \hat{x} = X^\tau(s), \quad \hat{y} = \int_0^\tau \bar{X}(u) du.
\]

Thus by the dominated convergence theorem, we have
\[
\gamma(s) \geq \lim_{\tau \to \infty} E_0 \left[ e^{-\sigma (\tau \wedge T)} \left( \hat{y} + \int_0^{\tau \wedge T} \bar{X}(u) du \right) - k \right] \hat{x} = X^\tau(s), \quad \hat{y} = \int_0^\tau \bar{X}(u) du
\]
for every \(\tau \in \delta\), and thus,
\[
\gamma(s) \geq \sup_{\tau \in \delta} E_0 \left[ e^{-\sigma \tau} \left( \hat{y} + \int_0^\tau \bar{X}(u) du \right) - k \right] \hat{x} = X^\tau(s), \quad \hat{y} = \int_0^\tau \bar{X}(u) du = g^*(H^{0,1,0}(s)).
\]

5. Conclusion

In this paper we have studied the problem of pricing the perpetual American call on the time-average of the stock. We have shown that the value of this American contingent claim becomes the optimal expected payoff function \(g^*\). We have identified the initial-boundary value problem satisfied by \(g^*\) and the integral equation satisfied by the stopping boundary. We also have verified that the pair of \(g^*\) and the stopping boundary is a unique solution to the associated free-boundary problem. Moreover we obtain the Doob–Meyer decomposition of the Snell envelope of the discounted payoff for this American contingent claim.

By the same lines as this paper, we can obtain the analogous results for the value of the American call on the time-average of the stock with finite expiration date (See [1]). We are convinced that the results shown in this study is very useful for computing the numerical value of the path-dependent American contingent claims. Future studies will focus on this issue.

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REFERENCES