Higher-Order Spectra of Cluster Point Processes Generating 1/f Fluctuations

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1/f fluctuations have been extensively observed in various fields including both biology and physics. Recently, a cluster point process has been successfully applied to modeling the physically and the physiologically observed series of events which exhibit 1/f fluctuations. In this paper, the higher-order spectra up to the 3rd order are derived for the general cluster point processes whose primary and secondary processes are renewal according to the heuristic shot noise approach. As far as the cluster point process whose primary process is Poissonian, the results coincide with those obtained by the probability generating functional. In the context of 1/f fluctuations, the power spectral densities (PSD) and the bispectra are given for varied parameters, especially for the power law distribution of cluster size. In addition to the rising landscape in the lower frequency range, these 2nd- and 3rd-order spectra are shown to have undulating structures that reflect the periodicity of event occurrence within the clusters, which indicates that there are quadratic correlations between the dominant frequency components that construct the 1/f profile in the PSD. Although several point process models that can produce the 1/f PSD have been proposed, the bispectra could be a possible measure for selecting an appropriate model from these candidates and may contribute to identifying the generation mechanism of the 1/f fluctuation.

KEYWORDS: cluster point process, bispectrum, 1/f fluctuations, shot noise, probability generating functionals

1 Introduction

1/f fluctuations have been extensively observed in various fields including both biology and physics. These fluctuations are characterized by a power spectral density which is inversely proportional to frequency. The intriguing features of the 1/f fluctuations, including their fractal nature (Mandelbrot, 1982) and the slow diffusion between stationarity and nonstationarity (Ogura, 1994) have attracted many researchers. Modeling studies of the generation mechanisms have been carried out from both a phenomenological and a mechanistic point of view (Musha et al. eds., 1991). Among these models, stochastic point processes have been utilized to study the generation mechanisms of 1/f fluctuations observed in a series of temporarily concentrated events (Lowen and Teich, 1991; Meessmann et al., 1993). We have modeled the activities of the central single neurons during a dream sleep as a cluster point process (CPP) which shows the 1/f fluctuations. The counting statistics and power spectral density (PSD) structures of the neuronal activities were investigated in relation to those of the CPP (Grüneis et al., 1989; Grüneis et al., 1993).

Because a 1/f fluctuation can be characterized by the corresponding PSD, the correlation between the frequency components is not specified. In order to identify a generation mechanism for 1/f fluctuations, the correlation information is essential. In fact, the higher-order moments have been found to differentiate the 1/f fluctuations (Akabane and Agu, 1997). The conventional statistics for this purpose are a bispectrum which shows quadratic correlations between the frequency components (Brillinger, 1965; Nikias and Raghuveer, 1987). In our case, the bispectra were obligatory to clarify how faithfully the actual neuronal activities were modeled by the CPP (Takahashi et al., 1993). Apart from the context of the 1/f fluctuations, diverse signals also exist which are characterized by point processes. For more general point processes, the higher-order spectra or moments should be derived for identifying an appropriate model.

In this paper, we first derive the spectral structures of the CPP up to the 3rd order in a heuristic manner, following previous studies where the CPP has been regarded as a shot noise process (Baiter et al., 1982; Grüneis, 1984; Takahashi et al., 1993). These structures are then extended to a more general case in which both the primary and secondary processes are renewal. Secondly, some of the heuristic results are confirmed in a mathematical way based on a probability generating functional (p.g.f.) approach (Dayley and Vere-Jones, 1988). In addition, the PSD and bispectral structures are briefly summarized in the context of the 1/f fluctuations, which clarify correlation structures among frequency components exhibiting 1/f fluctuations. These studies could contribute to extending the applicability of CPP and to exploring statistical structures of point processes.
2 Higher-order spectra of cluster point processes: shot noise approach

The cluster point process (CPP) treated here is illustrated in Fig. 1. A series of primary events are assumed to form a renewal process. Each of the primary events triggers a secondary series of events ("cluster"). The secondary process consists of \( m \) random events, where \( m \) is the stochastic variable "cluster size." Namely, the cluster is a truncated renewal process. Such a cluster point process is characterized by the causal relation between the primary event and the cluster generation. A renewal point process is a point process whose inter-event intervals are identically and independently distributed (i.i.d.) (Cox and Lewis, 1966). A Poisson process is a special case of a renewal process. We assume orderliness for the point processes treated here: two or more events do not occur at the same time. In the following sections, P and R are used for indicating a Poisson process and a renewal process, respectively. For example, CPP whose primary process is Poissonian and whose secondary process is renewal is denoted by CPP (P-R). In this paper, a point process whose inter-event intervals are mutually dependent is not treated for both of the primary and secondary processes. Therefore, CPP (R-R) is the most general case in our framework.

2.1 Power spectral density of the cluster point process

We describe a heuristic procedure for deriving higher-order spectra of a cluster point process by regarding it as a shot noise process (Papoulis, 1984). As shown in Fig. 2, a shot noise process is formally described as a process superposing individual events \( x_n(t - \theta_n) \), each of which is initiated at \( t = \theta_n \). \( x_n(t) \) is assumed to be an appropriate random function decaying sufficiently fast as \( t \to \infty \) in its mean sense, i.e., \( x(t, \omega) (\omega \in \Omega) \), where from now on \( \omega \) is omitted without notification. Actually, \( x_n(t) \) consists of a cluster of random points as defined later. By regarding the CPP as the shot noise, its spectra could be obtained without considering the peculiarity of the point processes.

For a sufficiently long observation time \([0, T]\), the shot noise process \( y(t, T) \), which consists of \( N \) events, is described as

\[
y(t, T) = \sum_{n=1}^{N} x_n(t - \theta_n).
\]

(1)

The PSD of \( y(t) \) can be obtained as follows.

\[
P(f) = \lim_{T \to \infty} \left\{ \frac{1}{T} |Y(f, T)|^2 \right\},
\]

(2)

where \( Y(f, T) \) is

\[
Y(f, T) = \sum_{n=1}^{N} x_n(t - \theta_n)e^{-i2\pi ft}dt = \sum_{n=1}^{N} X_n(f)e^{-i2\pi ft}.
\]

(3)

\( X_n(f) \) is the following transformation of \( x_n(t) \)

\[
X_n(f) = \int_{-T}^{T} x_n(t')e^{-i2\pi ft'}dt'.
\]

(4)

\( \langle Y \rangle \) represents the ensemble average for the possible realizations of \( Y \). From now on, we use this notation for the average regardless of concerned ensembles. \( P(f) \) can be calculated further:

\[
P(f) = \lim_{T \to \infty} \left\{ \frac{1}{T} \left| \sum_{n=1}^{N} X_n(f)e^{-i2\pi ft} \right|^2 \right\} = \mu_p \langle |X(f)|^2 \rangle + \langle X(f) \rangle^2 Q(f),
\]

(5)

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![Fig. 1](image1.png)  
Schematic representation of cluster point process.

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![Fig. 2](image2.png)  
Shot noise process.
where

\[ Q_A(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_2} \langle e^{-i2\pi f(\theta_{n_1}-\theta_{n_2})} \rangle. \] (6)

\[ Q_A(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_2} \langle e^{-i2\pi f(\zeta_{n_1}-\zeta_{n_2})} \rangle. \] (8)

\[ Q_A(f) \] and \[ Q_A(f) \] represent the quadratic correlations between events in the primary and secondary processes, respectively. \( \{P_m\} \) denotes the probability distribution of \( m \): the number of events contained in each cluster. Throughout this paper, we treat the cluster which does not contain the primary process event. Because the possibility of an empty cluster is not considered here, \( m \) starts from 1. Of course, the results obtained here can also be applied to a case in which the primary event is included in the cluster without serious modifications.

Apparently, both terms \( Q_A \) and \( Q_B \) contain the following common structure.

\[ q^{(l)}(f) = \sum_{n_1 \neq n_2} \sum_{i=1}^{l-1} e^{-i2\pi f(\zeta_{n_1}-\zeta_{n_2})}. \] (9)

In order to prepare for calculating this term, we introduce the following variables: \( t_i \) and \( \tau_i \) denote the occurrence time of the \( i \)-th event and the \( i \)-th inter-event interval \( (\tau_i = t_i - t_{i-1}) \), respectively. \( \tau_i \) is i.i.d., and its characteristic function is denoted by \( u_{\tau_i}(f) \). We then have

\[ q^{(l)}(f) = 2 \text{Re} \sum_{\tau_1=1}^{l-1} \sum_{k=1}^{l-\tau_1} \langle e^{-i2\pi f(\tau_{\tau_1+k})} \rangle = 2 \text{Re} \sum_{\tau_1=1}^{l-1} \sum_{k=1}^{l-\tau_1} u_{\tau_1}^k \tau_1^k \left\{ \frac{u_{\tau_1}}{1-u_{\tau_1}} \left( \frac{u_{\tau_1}}{(1-u_{\tau_1})^2} \sum_{m=1}^{N_{\tau_1}} P_m u_m \right) \right\}, \] (10)

where \( \text{Re}\{x\} \) denotes a real part of \( x \). Based on this result, both terms are calculated as follows.

\[ Q_A(f) = 2 \text{Re} \left[ \frac{\sum_{i=1}^{N_{\tau_1}} u_{\tau_1}^i}{1-u_{\tau_1}} \right], \] (11)

\[ Q_B(f) = 2 \langle m-1 \rangle \text{Re} \left[ \frac{u_{\tau_1}}{1-u_{\tau_1}} \right] + 2 \text{Re} \left[ \frac{u_{\tau_1}}{(1-u_{\tau_1})^2} \sum_{m=1}^{N_{\tau_1}} P_m u_m \right] \] (12)

where \( u_{\tau_1} \) and \( u_\tau \) denote the characteristic functions of the inter-event interval in the primary process \( \{A_i\} \) and in the secondary process \( \{\lambda_i\} \), respectively. Although \( Q_A \) produces another term concentrated at \( f = 0 \) responsible for a dc component of \( x_\tau \), it is omitted throughout this paper because we are principally concerned with the overall structure of higher-order spectra. In addition,

\[ |\langle X(f) \rangle|^2 = \left( \frac{\sum_{i=1}^{N_{\tau_1}} u_{\tau_1}^i}{1-u_{\tau_1}} \right) \left( 1 - \frac{\sum_{i=1}^{N_{\tau_1}} P_m u_m}{1-u_{\tau_1}} \right). \] (13)

Finally, we reach the PSD of the CPP (R-R),

\[ P(f) = \mu \langle m \rangle + Q_A(f) + 2 \mu \tau_1(u_{\tau_1}) T_1(u_{\tau_1}) \text{Re} \left[ \frac{u_{\tau_1}}{1-u_{\tau_1}} \right] \] (14)

where

\[ T_1(\bar{a}) = \frac{\bar{a}}{1-\bar{a}} - \frac{\bar{a}}{1-\bar{a}} \sum_{m=1}^{N_{\tau_1}} P_m u_m. \] (15)

2.2 Bispectrum of the cluster point process

We derive a bispectrum of the shot noise process. The bispectrum is generally defined as follows (Nikias and Raghuveer, 1987).

\[ B(f_1, f_2) = \lim_{T \to \infty} \frac{1}{T} \left< Y(f_1, T) Y(f_2, T) \overline{Y}(f_1 + f_2, T) \right>. \] (16)

Substituting the limit operation with \( T \) for that with \( N \) results in
\[ B(f_1, f_2) = \mu_p \lim_{N \to \infty} \frac{1}{N} \left[ \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} X_n(f_1)X_n(f_2)X_n(f_1 + f_2) \times e^{-i2\pi(f_1(\theta_{n_1} - \theta_{n_3}) + f_2(\theta_{n_2} - \theta_{n_3}))} \right]. \tag{17} \]

After some manipulations, we get
\begin{align*}
B(f_1, f_2) &= \mu_p \langle X(f_1)X(f_2)\bar{X}(f_1 + f_2) \rangle + \mu_p \langle X(f_1)X(f_2)\bar{X}(f_1 + f_2) \rangle \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_3} \sum_{n_2} \langle e^{-i2\pi f_1(\theta_{n_1} - \theta_{n_3})} \rangle \\
&+ \mu_p \langle X(f_1)\bar{X}(f_1 + f_2) \rangle \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_3} \sum_{n_2} \langle e^{-i2\pi f_2(\theta_{n_2} - \theta_{n_3})} \rangle \\
&+ \mu_p \langle X(f_1)X(f_2)\bar{X}(f_1 + f_2) \rangle \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_3} \sum_{n_2 \neq n_3} \langle e^{-i2\pi f_1(\theta_{n_1} - \theta_{n_3}) + f_2(\theta_{n_2} - \theta_{n_3})} \rangle. \tag{18} \end{align*}

Shortly, the triplet correlation of \( X(f) \) in eq. (18) reflects the stochastic relationship among events within a cluster. The triplet correlation is calculated further as
\begin{align*}
\langle X(f_1)X(f_2)\bar{X}(f_1 + f_2) \rangle &= \langle m \rangle + \left( \sum_{n_1 \neq n_3} \sum_{n_2} e^{-i2\pi f_1(\theta_{n_1} - \theta_{n_3})} \right) + \left( \sum_{n_1 \neq n_2} e^{-i2\pi f_2(\theta_{n_2} - \theta_{n_3})} \right) \\
&+ \left( \sum_{n_1 \neq n_2 \neq n_3} e^{-i2\pi f_1(\theta_{n_1} - \theta_{n_3}) + f_2(\theta_{n_2} - \theta_{n_3})} \right), \tag{19} \end{align*}
where \( \zeta_n \) denotes occurrence time within a cluster. There are some common structures in (18) and (19). In addition to the previously defined \( Q_i \) and \( Q_1, Q_{12}(f_1, f_2) \) and \( Q_{12}(f_1, f_2) \) are introduced for the following calculations.
\begin{align*}
Q_{12}(f_1, f_2) &= \lim_{N \to \infty} \frac{1}{N} \sum_{n_1 \neq n_3} \sum_{n_2} \sum_{n_3} \langle e^{-i2\pi f_1(\theta_{n_1} - \theta_{n_3}) + f_2(\theta_{n_2} - \theta_{n_3})} \rangle = \lim_{N \to \infty} \frac{1}{N} Q^N(f_1, f_2) \tag{20} \\
Q_{12}(f_1, f_2) &= \left( \sum_{n_1} \sum_{n_2} \sum_{n_3} e^{-i2\pi f_1(\theta_{n_2} - \theta_{n_3})} \right) \left( \sum_{n_1} \sum_{n_2} \sum_{n_3} e^{-i2\pi f_2(\theta_{n_3} - \theta_{n_1})} \right) = \sum_{m=1}^{N_\infty} P_m q^m_q(f_1, f_2). \tag{21} \end{align*}

Here,
\begin{align*}
q^0_q(f_1, f_2) &= \left( \sum_{n_1} \sum_{n_2} \sum_{n_3} e^{-i2\pi f_1(\theta_{n_2} - \theta_{n_3})} \right) \left( \sum_{n_1} \sum_{n_2} \sum_{n_3} e^{-i2\pi f_2(\theta_{n_3} - \theta_{n_1})} \right) \\
&= 2 \text{Re} \left[ S^{(0)}(u(f_1), u(f_1 + f_2)) + S^{(0)}(u(f_1), u(f_2)) + S^{(0)}(u(f_2), u(f_1 + f_2)) \right], \tag{22} \\
S^{(0)}(a, b) &= \sum_{k_1=1}^{l-a} \sum_{k_2=1}^{l-b} a^{k_1} b^{k_2} \left[ \frac{1}{(1-a)(1-b)} \right] \\
&= \frac{ab}{(1-a)(1-b)} \left[ l - \frac{1-b}{(1-a)(b-a)} d' + \frac{1-a}{(1-b)(b-a)} b' - \frac{1-a}{1-b} - \frac{1}{1-b} \right]. \tag{23} \end{align*}

The bispectral structure of the cluster point process is obtained as follows. For simplicity, the abbreviations \( u_{a_1} = u_s(f_1), u_{a_2} = u_s(f_2), \) and \( u_{a_3} = u_s(f_1 + f_2) \) are used, which similarly apply to \( u_s \).
\begin{align*}
B(f_1, f_2) &= \mu_p \langle m \rangle + Q_s(f_1) + Q_s(f_2) + Q_s(f_1 + f_2) + Q_{12}(f_1, f_2) \\
&+ \mu_p \left( T_1(u_{a_1} + u_{a_2})T_2(u_{a_3}, u_{a_2}, u_{a_1}) \right) Q_s(f_1 + f_2) \\
&+ \mu_p \left( T_1(u_{a_1})T_2(u_{a_1}, u_{a_2}, u_{a_2}) \right) Q_{12}(f_1, f_2) \\
&+ \mu_p \left( T_1(u_{a_2})T_2(u_{a_1}, u_{a_2}, u_{a_1}) \right) Q_{12}(f_1, f_2) \\
&+ \mu_p \left( T_1(u_{a_3})T_1(u_{a_2})T_2(u_{a_1}, u_{a_2}) \right) Q_{12}(f_1, f_2), \tag{24} \end{align*}
where
\begin{align*}
Q_{12}(f_1, f_2) &= 2 \text{Re} \left[ \frac{u_{a_1} u_{a_3}}{(1-u_{a_1})(1-u_{a_3})} + \frac{u_{a_1} u_{a_2}}{(1-u_{a_1})(1-u_{a_2})} + \frac{u_{a_2} u_{a_3}}{(1-u_{a_2})(1-u_{a_3})} \right] \tag{25} \\
T_2(a, b, c) &= T_1(a) + U(a, b) + U(a, c) \tag{26} \\
U(a, b) &= \frac{ab}{(1-a)(1-b)} \left[ 1 - \frac{1}{b-a} P_m b^m + \frac{1-b}{b-a} \sum_{m=1}^{N_\infty} P_mb^m \right] \tag{27}. \end{align*}
In the case that \( \Lambda \) obeys the exponential distribution with its mean \( \langle \Lambda \rangle \), i.e. Poissonian, \( u_{\Lambda}(f) \) is given by
\[
    u_{\Lambda} = u_{\Lambda}(f) = \frac{1}{1 - i2\pi f \langle \Lambda \rangle}.
\] (28)

Because the following identity is satisfied,
\[
    Q_{\Lambda}(f) = \text{Re} \left[ \frac{u_{\Lambda}}{1 - u_{\Lambda}} \right] = 0,
\] (29)
we reach the well-known result concerning the shot noise process: the two-sided PSD is described by
\[
    P(f) = \mu_{\nu} \langle |X(f)|^2 \rangle = \mu_{\nu} (\langle m \rangle + Q_{\Lambda}(f)).
\] (30)

This is the PSD for the CPP (P-R), which can be regarded as the Bartlett-Lewis process (Bartlett, 1966). Since additionally \( Q_{\Lambda}(f_1, f_2) = 0 \) for this special case, the bispectrum can be reduced as follows.
\[
    B(f_1, f_2) = \mu_{\nu} \langle X(f_1)X(f_2)X(f_1 + f_2) \rangle
    = \mu_{\nu} (\langle m \rangle + Q_{\Lambda}(f_1) + Q_{\Lambda}(f_2) + Q_{\Lambda}(f_1 + f_2) + Q_{\Lambda}(f_1, f_2)).
\] (31)

### 3 Higher-order spectra of the cluster point process: probability generating functional approach

In the previous section, the shot noise approach for deriving the spectra was taken, where each cluster was regarded as an appropriate random function formally consisting of temporally separated \( \delta \) functions. Namely, the point process was not treated in a mathematically strict sense. In this section, the higher-order spectra are instead derived based on a probability generating function for the CPP as a mathematical treatment of a stochastic point process.

Here, we write \( M \) as the generic symbol of a point process. We assume orderliness for \( M \): two or more events do not occur at the same time. A complete description of \( M \) is given by its probability generating functionals \( G[\xi] \) defined by
\[
    G[\xi] = E \left\{ \exp \int \log \xi(t) dM(t) \right\}
\] (33)
where \( \xi \) is a suitable function \( 1 \geq \xi > 0 \) (Westcott, 1971), and \( M(t) \) represents the number of points in \((0, t]\).

When \( M \) is a cluster process, it has two components: the primary process \( M_{\nu} \) and the process of cluster members \( M_{\xi} \). Each point of \( M_{\nu} \) initiates \( M_{\xi} \). If \( M_{\nu} \) has p.g.f. \( G_{\nu}[\xi] \) and p.g.f. of a cluster initiated at \( t \) is \( G_{\xi}[\xi | t] \), then the p.g.f. of \( M \) constructed by the independent development of clusters is \( \Pi_{\nu} G_{\nu}[\xi | t] \) on a realization \{\( t_i \)\} of \( M_{\nu} \). Thus the p.g.f. of \( M \) is given by the fundamental relation
\[
    G[\xi] = G_{\nu}[G_{\xi}[\xi | t]]
\] (34)
which is from Moyal (1962). Here, according to this formulation, the higher-order cumulants of \( M \) are derived. The corresponding higher-order spectra are obtained by their Fourier transformations.

#### 3.1 Higher-order cumulants of the cluster point process

Here, we derive the higher-order spectra for the cluster point process in which \( M_{\nu} \) and \( M_{\xi} \) are homogeneous Poisson and renewal processes, respectively, i.e. CPP (P-R). The p.g.f. of \( M_{\nu} \) with the Poisson parameter \( \mu_{\nu} \) has the form,
\[
    G_{\nu}[\xi] = \exp \mu_{\nu} \left[ - \int (1 - \xi(t)) dt \right],
\] (35)

Because \( M_{\xi} \) is assumed to be renewal, its p.g.f. is given as
\[
    G_{\xi}[\xi | t] = P_1 \int \xi(t + v) g(v) dv + P_2 \int \xi(t + v_1) \xi(t + v_1 + v_2) g(v_1) g(v_2) dv_1 dv_2 + \cdots
\] (36)
where \( g(x) \) is the probability density function (p.d.f.) of the inter-event interval \( x \) in \( M_{\xi} \). By substituting (36) into (35), we obtain the p.d.f. of \( M \).
\[
    G[\xi] = \exp \mu_{\nu} \left[ \left\{ P_1 \int \xi(t + v) g(v) dv + P_2 \int \xi(t + v_1) \xi(t + v_1 + v_2) g(v_1) g(v_2) dv_1 dv_2 + \cdots \right\} - 1 \right] dt.
\] (37)
The factorial cumulant generating functional (c.g.f.l.) is derived from (37) as follows.

\[
H[1 + \xi] = \log G[1 + \xi] = \mu_p \int \left\{ P_1 (1 + \xi(t + v)) g(v) dv \right. \\
+ P_2 \int (1 + \xi(t + v_1))(1 + \xi(t + v_1 + v_2)) g(v_1) g(v_2) dv_1 dv_2 \\
+ P_3 \int (1 + \xi(t + v_1))(1 + \xi(t + v_1 + v_2))(1 + \xi(t + v_1 + v_2 + v_3)) \\
\times g(v_1) g(v_2) g(v_3) dv_1 dv_2 dv_3 + \cdots \right\} - 1 \ dt.
\]

(38)

Generally, the c.g.f.l. is expanded with the factorial cumulant densities (Daley and Vere-Jones, 1988).

\[
H[1 + \xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int \xi(t_1) \cdots \xi(t_k) c_{t_1} c_{t_2} \cdots c_{t_k} dt_1 \cdots dt_k.
\]

(39)

Comparing the same \( k \)-th order coefficients in (38) with (39), one can obtain \( k \)-th order factorial cumulant density \( c_{t_k} \). Collecting the 1st order terms of \( \xi \) in (38) makes

\[
\mu_p \int \left\{ \sum_{m=1}^{N_p} P_m \xi(t) + \sum_{m=1}^{N_p} P_m \int \xi(t + v) f(v) dv + \sum_{m=1}^{N_p} P_m \int \int \xi(t + v_1 + v_2) g(v_1) g(v_2) dv_1 dv_2 \\
+ \sum_{m=1}^{N_p} P_m \int \int \int \xi(t + v_1 + v_2 + v_3) g(v_1) g(v_2) g(v_3) dv_1 dv_2 dv_3 + \cdots \right\} dt.
\]

(40)

Finally, this is reduced into

\[
\mu_p \int \sum_{m=1}^{N_p} P_m \xi(t) m dt.
\]

(41)

Because the term with \( k = 1 \) in (39) is

\[
\int \xi(t) c_{t_1} dt,
\]

(42)

the 1st order factorial cumulant density can be given as follows.

\[
c_{t_1} = \mu_p \sum_{m=1}^{N_p} m P_m.
\]

(43)

Similarly, the 2nd-order terms in (38) are

\[
\frac{1}{2!} \int \int \xi(t) \xi(t + w) \left[ \mu_p \sum_{m=1}^{N_p} P_m \sum_{k=1}^{m} (m - k)(g^{k*}(w) + g^{k*}(- w)) \right] dw dt
\]

(44)

where \( g^{k*} \) indicates the \( k \)-th-order convolution of \( g \).

The term with \( k = 2 \) in (39) is

\[
\frac{1}{2!} \int \int \xi(t) \xi(t + w) c_{t_2}(t, t + w) dw dt.
\]

(45)

Using the time-invariant relation \( c_{t_2}(t, t+w) = c_{t_2}(w) \) because of the stationarity, one can obtain the 2nd-order factorial cumulant density

\[
c_{t_2}(w) = \mu_p \sum_{m=1}^{N_p} P_m \sum_{k=1}^{m} (m - k) \{ g^{k*}(w) + g^{k*}(- w) \}.
\]

(46)

By comparing the 3rd-order terms in (38) and (39),

\[
\frac{1}{3!} \int \int \int \xi(t) \xi(t + w_1) \xi(t + w_2) c_{t_3}(t, t + w_1, t + w_2) dw_1 dw_2 dt = \frac{1}{3!} \int \int \int \xi(t) \xi(t + w_1) \xi(t + w_2) \\
\times \left[ \mu_p \sum_{m=1}^{N_p} P_m \sum_{n_1=1}^{m} \sum_{n_2=1}^{m} \sum_{n_3=1}^{m} f_{n_1-n_2}^{(-1)*}(w_1) f_{n_2-n_3}^{(-1)*}(w_2 - w_1) \right] dw_1 dw_2 dt,
\]

(47)

the 3rd-order factorial cumulant density is derived as follows.
\[ c_{[1]}(w_1, w_2) = \mu_p \sum_{m=1}^{N_p} P_m \sum_{n_1=1}^{m} \sum_{n_2=1}^{m} \sum_{n_3=1}^{m} f^{-n_1-n_2-n_3}(w_1) f^{n_1-n_2-n_3}(w_2-w_1), \]  

where the time-invariant expression, \( c_{[2]}(t, t + w_1, t + w_2) = c_{[2]}(w_1, w_2) \), is used. \( n_1, n_2, \) and \( n_3 \) are the number of events within the cluster. Consequently, the cumulants up to the 3rd order are given based on their relation to the cumulant densities (Ogura et al., 1986).

\[ c_1 = c_{[1]} \]  
\[ c_2(w) = \delta(w)c_{[1]} + c_{[2]}(w) \]  
\[ c_1(w_1, w_2) = D_1(w_1, w_2) + D_2(w_1, w_2) + D_3(w_1, w_2) \]  
\[ D_1(w_1, w_2) = \delta(w_1)\delta(w_2)c_{[1]} \]  
\[ D_2(w_1, w_2) = \delta(w_2 - w_1)c_{[2]}(w_1) + \delta(w_1)c_{[2]}(w_2) + \delta(w_2)c_{[2]}(w_1) \]  
\[ D_3(w_1, w_2) = c_{[3]}(w_1, w_2), \]

### 3.2 Higher order spectra of the cluster point process

The PSD of a stochastic point process is defined by a Fourier transformation of the 2nd-order cumulant (Bartlett, 1963).

\[ P(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_2(w)e^{-2\pi i fu} dw. \]  

Substituting (43) and (50) into (52), one can obtain

\[ P(f) = \mu_p \{ \langle m \rangle + Q^{(m)}_{\phi}(f) \} \]  
\[ Q^{(m)}_{\phi}(f) = 2\text{Re} \left[ \frac{u}{(1-u)^2} \sum_{m=1}^{N_p} u^n (1-u)\langle m \rangle - 1 \right] \]

where \( u(f) \) is a characteristic function of the p.d.f. of inter-event intervals within a cluster.

A bispectrum of a stochastic point process is defined as the two-dimensional Fourier transformation of the 3rd-order cumulant (Brillinger, 1978),

\[ B_2(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_1(w_1, w_2)e^{-2\pi i (f_1 w_1 + f_2 w_2)} dw_1 dw_2. \]

The Fourier transformation is performed for the terms in (51) separately. The transformation for \( D_1 \) gives

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_1(w_1, w_2)e^{-2\pi i (f_1 w_1 + f_2 w_2)} dw_1 dw_2 = \mu_p \langle m \rangle. \]

For \( D_2 \),

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_2(w_1, w_2)e^{-2\pi i (f_1 w_1 + f_2 w_2)} dw_1 dw_2 = \mu_p \{ Q^{(m)}_{\phi}(f_1) + Q^{(m)}_{\phi}(f_2) + Q^{(m)}_{\phi}(f_1 + f_2) \}. \]

In addition, \( D_3 \) is transformed into

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_3(w_1, w_2)e^{-2\pi i (f_1 w_1 + f_2 w_2)} dw_1 dw_2 = \mu_p Q_3(f_1, f_2), \]

where \( f_3 = f_1 + f_2 \), and

\[ Q_3(f_1, f_2) = \sum_{m=1}^{N_p} P_m 2\text{Re} \left[ S^{(m)}_{\phi}(u(f_1), u(f_2)) + S^{(m)}_{\phi}(u(f_1), u(-f_2)) + S^{(m)}_{\phi}(u(f_2), u(f_1)) \right]. \]

\[ S^{(m)}_{\phi}(a, b) = \sum_{n_1=1}^{m-2} \sum_{n_2=1}^{m-n_1-1} \sum_{n_3=1}^{m-n_1-n_2} a^n b^m \]

\[ = \frac{ab}{(1-a)(1-b)} \left[ \frac{1}{(1-a)(b-a)} a^n + \frac{1}{(1-b)(b-a)} b^m + \frac{1}{1-a} - \frac{1}{1-b} \right]. \]

Finally, the bispectrum of the CPP (P-R) is given by

\[ B(f_1, f_2) = \mu_p \langle m \rangle + Q_2(f_1) + Q_2(f_2) + Q_2(f_1 + f_2) + Q_2(f_1, f_2). \]

Considering the correspondences \( Q_2 \leftrightarrow Q_1 \) in (12), \( Q_{12} \leftrightarrow Q_{21} \) in (21), and \( S^{(m)}_{\phi} \leftrightarrow S^{(m)} \) in (23), the PSD and the bispectra of the CPP (P-R) obtained by the shot noise and the p.g.f. approaches lead to the same representa-
tion. So far, we have used the former approach for deriving both the PSD and the bispectra (Grüneis et al., 1989; Grüneis et al., 1993; Takahashi et al., 1993). We have been confined to the CPP whose primary process is Poissonian. This condition simplifies the practical representation for the higher-order statistics. If this condition is not applied, it is remarkably difficult to carry out the derivation, because the p.g.fl. does not have any closed forms in cases where both $\mathcal{M}_b$ and $\mathcal{M}_c$ are not Poissonian.

4 Structural properties of higher-order spectra of the cluster point process

4.1 Spectral structure of the CPP (P-P) model

The CPP (P-P) model has been of interest in cases where the cluster size $m$ obeys the power law distribution in the context of $1/f$ fluctuations.

$$ P_m = \frac{m^z}{\sum_{n=1}^{N_m} n^z} $$

(62)

Here, we summarize the structural properties of the PSD and the bispectra of CPP (P-P) for the reader’s convenience (Grüneis and Musha, 1986; Takahashi et al., 1993). The basic structure of the PSD is determined by the model parameters $N_m$ and $z$: the distribution of cluster size ($P_m$) is primarily responsible for the overall structure (eq. (14)). Figure 3 shows the PSD for varied model parameters $N_m$ and $z$ (see the case $v_p = v_c = 1$). The PSDs are shown to approach respective levels asymptotically in the low- and high-frequency limits, where the asymptotic level in the low frequency limit is well above that in the high. In addition, a $1/f$ region exists in an intermediate frequency range. The higher and lower corner frequencies also depend on the model parameters (Grüneis and Musha, 1986). The asymptotic level of the one-sided PSD in the high-frequency limit is given by

$$ P(f) \to 2\mu_c \langle m \rangle, \quad f \to \infty $$

and that in the low limit is given by

$$ P(f) \to 2\mu_b \langle m^2 \rangle, \quad f \to 0 $$

Actually, as shown above, the CPP model has its $1/f$ profile only in the limited frequency range. In this sense, the model generates the stationary process, which is different from those exhibiting a complete power law PSD such as a fractional Brownian motion and an intermittent chaos (Mandelbrot, 1982; Aizawa and Kohyama, 1984). However, in practice, due to the finite observation time, $1/f$ fluctuations could be found only in the limited frequency range. Therefore, the CPP model could be applied to the $1/f$ fluctuations by modifying the parameters $N_m$ and $z$ so as to fit the $1/f$ slope region of the model to the concerned frequency range. In addition, the result that the PSD approaches the mean occurrence rate in the high frequency limit is not due to the CPP but its structure as a point process.

Next, the bispectral structure of CPP (P-P) is summarized based on our previous results (Takahashi et al., 1993). The bispectra of CPP are shown for some $N_m$ and $z$ in Fig. 5. They are characterized by the monotonic rising landscape in the lower frequency range and the almost flat structure in the higher frequency range. Since for the high-frequency limit $Q_1(f), Q_{12}(f_1, f_2) \to 0$ in eq. (32), the asymptotic level becomes

$$ B(f_1, f_2) \to \mu_b \langle m \rangle, \quad f_1, f_2 \to \infty. $$

For another limit $f_1, f_2 \to 0$, we have

$$ Q_1(f) \to \langle m^2 \rangle - \langle m \rangle, \quad f_1, f_2 \to 0 $$

and

$$ Q_{12}(f_1, f_2) \to \langle m^3 \rangle - 3\langle m^2 \rangle + 2\langle m \rangle, \quad f_1, f_2 \to 0 $$

From the above results and eq. (32), the asymptotic level in the low frequency limit becomes

$$ B(f_1, f_2) \to \mu_c \langle m^2 \rangle, \quad f_1, f_2 \to 0. $$

(63)

Characteristically, the bispectrum of the CPP whose primary process is Poissonian has no imaginary part.

4.2 Spectral structure of CPP (R-R)

We deal with the CPP (R-R) model whose inter-event intervals in both the primary and secondary processes obey Gamma distributions. The Gamma p.d.f. of inter-event intervals is given by

$$ p(x, v) = (v/\langle x \rangle)(vx/\langle x \rangle)^{-1} \exp (-vx/\langle x \rangle)\Gamma(v), $$

(64)
where $x$ and $v$ denote an inter-event interval and the Gamma parameter, respectively. Notice that $x$ is normalized by its mean $\langle x \rangle$. $v$ is represented by variance $\sigma_x^2$ and $\langle x \rangle$ as follows,

$$v = \frac{\langle x \rangle}{\sigma_x^2}.$$  

(65)

As $v$ increases, intervals tend to be distributed more concentrically around the mean, which indicates that each event tends to occur periodically.

Throughout this paper, the PSD as well as the following bispectra are shown with the normalized frequency axis $f(\lambda)$, where $\langle \lambda \rangle$ is the mean interval for the renewal process with no clusters and that of the secondary process for the CPPs. Figure 3A shows the PSDs of the renewal process whose inter-event intervals obey the Gamma distribution. The PSD is no more flat in spite of the independency of the inter-event intervals. The peak in the PSD becomes distinct around $\log f(\lambda) = 0 (\langle \lambda \rangle = 1)$, which is due to the tendency of the possible intervals to be concentrated around the mean $\langle \lambda \rangle$. The PSD asymptotically approaches $2\mu/\langle \lambda \rangle = 2\mu$ in the high-frequency limit (one-sided PSD), where $\mu$ is the mean occurrence rate. The asymptotic level in the low-frequency limit, $2\mu/v (v > 1)$, is lower than in the high-frequency limit. Figure 4 shows the corresponding bispectra, which are pictured only in the triangle frequency domain surrounded by $f_1(\lambda) < f_2(\lambda)$ and $f_1(\lambda) + f_2(\lambda) < f_{\max}(\lambda)$ due
$\nu = 25$

$\nu = 5$

Fig. 4 Bispectra (absolute value) of the renewal processes whose inter-event intervals obey the Gamma distribution of varied $\nu$'s. The normalized frequency axes $f_s(\lambda)$ and $f_s(\lambda)$ are used, where $\langle \lambda \rangle$ is the mean inter-event interval in the secondary process. Due to the symmetricity of the bispectrum concerning $f_s(\lambda)$ and $f_s(\lambda)$, the results are shown only in the triangular frequency domain surrounded by $f_s(\lambda) < f_s(\lambda)$ and $f_s(\lambda) + f_s(\lambda) < f_{\text{max}}(\lambda)$, where $f_{\text{max}}(\lambda) = 10$. This applies for all the following bispectra shown here.

$z = 0$

$z = -2$

$N_m = 1000$

$N_m = 100$

$N_m = 1$

Fig. 5 Bispectra (absolute value) of the CPP whose inter-event intervals in the primary and the secondary processes are Poissonian, i.e., $\nu_p = 1$ and $\nu_c = 1$.

to their symmetricity concerning $f_1(\lambda)$ and $f_2(\lambda)$ (e.g., Helland and Itsweire, 1985). Naturally, the bispectrum is flat with the level of the mean occurrence rate $\mu$ for $\nu = 1$, i.e., Poissonian (not shown here). For $\nu = 5$, it is reduced in the frequency range lower than $f(\lambda) = 1$. Characteristically, for $\nu = 25$, it has a distinct peak around $f_1(\lambda) = f_2(\lambda) = 1$, and there is a ridge at $f_1(\lambda) = 1$. The respective bispectra approach the asymptotic level $1/\langle \lambda \rangle = \mu$ in the high-frequency limit.

Figures 3B to 3D show the PSDs of CPP (R-R) for varied parameter values together with those for the CPP (P-P), where $\nu_p$ and $\nu_c$ denote the Gamma parameters of the primary and the secondary processes, respectively.
The PSDs are characterized by a deep dip in the frequency range where \( \log f \langle \lambda \rangle = -1 \sim 0 \) and a peak around \( \log f \langle \lambda \rangle = 0 \), both of which are distinct from the CPP (P-P). This dip structure already appears in the PSD of CPP (P-R) \( (\nu_p = 1) \) (Grüneis et al., 1989). The asymptotic level in the low-frequency limit is obtained from (14) as

\[
2\mu_p \langle m^2 \rangle = \langle m \rangle^2 (1 - (\sigma_\lambda / \langle \lambda \rangle)^2), \quad f \to 0
\]

and that in the high is

\[
2\mu_p \langle m \rangle, \quad f \to \infty
\]

for the one-sided PSD. The parameter dependency of the PSD structure is summarized as follows. An increase in \( N_m \) lowers the corner frequency, i.e., the lower edge of the slope. \( z \) controls the PSD slope, e.g., \( z = -2 \) is responsible for a 1/f profile. An increase of \( \nu_c \) deepens the dip and raises the peak; \( \nu_p \) contributes only to the depth of the dip.

Figures 6 to 8 show the bispectra of CPP (R-R). For comparison, those of CPP (P-P) are presented in Fig. 5. Provided that the values of \( N_m \) and \( z \) are the same, roughly speaking, the bispectra of CPP (R-R) share the same overall structure as those of CPP (P-P): they have an asymptotic level in the high-frequency limit and a rising structure in the lower frequency range. The bispectra with \( \nu_c = 25 \) are characterized by a deep valley around \( f_s \langle \lambda \rangle = 0.5 \) and a ridge around \( f_s \langle \lambda \rangle = 1 \) except for those with \( N_m = 1 \). In contrast, even a larger \( \nu_p \) does not produce any distinct structures in the bispectra in comparison with those of the CPP (P-P) shown in Fig. 5, except for the case with \( N_m = 1 \) in which a small ridge appears, as shown in Fig. 7. Considering the characteristic peak shown in the PSD and the bispectrum of the renewal (Gamma) process with no clusters, including the cluster structures makes a remarkable difference. This may possibly be explained by the randomness of the Poissonian cluster processes masking the periodicity of the primary process. The other parameter dependencies are summarized as follows. Because the asymptotic level in the high frequency limit is the mean occurrence rate \( \mu_p \langle m \rangle \), increases in \( N_m \) and \( z \) raise the overall level of the bispectra and enhances their undulations. \( \nu_c \) primarily shapes the undulations in the bispectra.

![Fig. 6 Bispectra (absolute value) of the CPP whose inter-event intervals obey the Gamma distributions with \( \nu_p = 1 \) and \( \nu_c = 25 \).](image)
Because a bispectrum represents a quadratic interaction between frequency components, the above results indicate that large interferences exist between dominant frequency components in the corresponding PSD which construct 1/f structure and peaks.

5 Discussion

The higher-order spectra up to the 3rd order have been derived for the general cluster point processes (CPP) whose primary and secondary processes are renewal according to the heuristic shot noise approach. The results concerning CPP (P-R) coincide with those obtained by the probability generating functional (p.g.f.l.). However, for the most general case in which both the primary and secondary processes are renewal, we did not give the bispectrum by the p.g.f.l. due to the p.g.f.l. not having any closed forms. Therefore, the p.g.f.l. approach might be effective only when the primary process is Poissonian. On the other hand, the shot noise approach gives the spectra of CPP (R-R) by making use of characteristic functions of the inter-event intervals. In the context of 1/f fluctuations, the PSD and the bispectra are shown for varied parameters especially for the power law distribution of cluster size. In addition to the rising landscape in the lower frequency range, these 2nd- and 3rd-order spectra are shown to have undulating structures which reflect the periodicity of event occurrences within the clusters. The obtained bispectra indicate that there are correlations between the dominant frequency components which construct the 1/f PSD profile. Although this correlation structure represents the statistical properties of the CPP, its mechanistic interpretation is not yet given here. This will be subject of a future report.

Because this paper is confined to the theoretical derivations of the higher-order spectra of the CPP, applications of the CPP are not described. Nevertheless, the CPP was applied to modeling the actual neuronal spike trains which exhibit 1/f fluctuations (Grüneis et al., 1989; Grüneis et al., 1993). The bispectra of the neuronal spike trains were also estimated for a variety of physiological states (Takahashi et al., 1993). Similarities between the bispectra of the neuronal spike trains and the CPP (P-P) suggest that the neuronal spike trains exhibiting 1/f fluctuations could be characterized by the cluster-structured point process. So far, the neuronal activities we have been concerned with have not needed the most general CPP (R-R) for their modeling. Thalamic
neuronal activities, which are known to exhibit periodic bursting, might, however, be a possible candidate (Steriade and Deschénes, 1984).

Diverse signals in various kinds of systems which can be regarded as point processes have been found to exhibit 1/f fluctuations. Teich and his colleagues have proposed several models of 1/f fluctuations generated by stochastic point processes: the fractal shot noise-driven, the fractal Gaussian noise-driven doubly stochastic point processes (Lowen and Teich, 1991), and the fractal renewal processes (Lowen and Teich, 1993). Although their and our point process models can similarly produce the 1/f PSD, practically, one should be selected as a suitable model in order to identify a generation mechanism relevant to the concerned phenomena. The resulting bispectral information is one of possible measures for making this selection. For example, the bispectrum of the fractal Gaussian noise-driven process is supposed to be small due to the Gaussianity in its intensity. The fractal renewal process is supposed to have a different bispectrum from CPP because there are no clusters in the renewal process. Although the bispectra have not been given for the above point processes, the differentiation from the CPP which exhibit the same 1/f PSD could be done. Apart from the context of 1/f fluctuations, the cluster point processes have so far provided an established framework for analyzing a series of events with non-homogeneous occurrence intensity, e.g., traffic flow (Bartlett, 1963), computer failure patterns (Lewis, 1964), earthquakes (Vere-Jones, 1970), and the distribution of galaxies (Neyman and Scott, 1958). Our results presented here could be applied to examining the validity of the respective point process models.

REFERENCES


