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<td>&quot;THE P-HYPONORMALITY OF THE ALUTHGE TRANSFORM&quot;</td>
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The p-Hyponormality of The Aluthge Transform

Takashi YOSHINO

Mathematical Institute (Kawachi) Tohoku University, Kawachi, Aoba-ku, Sendai 980, Japan

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In this paper, we shall introduce the Aluthge transform of an operator and study what kind of p-hyponormality is preserved under the Aluthge transformation.

KEYWORDS: p-hyponormality, Aluthge transform

For a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ and for some $p$ such as $p > 0$, if $(T^*T)^p \geq (TT^*)^p$, then $T$ is called to be p-hyponormal.

In [1], A. Aluthge introduced the operator $\tilde{T} = |T|^{1/2}T/|T|^{1/2}$ for an operator $T$ with its polar decomposition $T = U|T| = |T^*|U$ and, in this paper, we say that $T(s, t) = |T|^{s}U|T|^{t}$ defined more generally for any $s$ and $t$ such as $s \geq 0$ and $t \geq 0$ is the Aluthge transform of $T$. About the $q$-hyponormality of the Aluthge transform $T(s, t)$ of a p-hyponormal operator $T$, we have the following.

**Theorem.** Let $T$ be p-hyponormal for some $p$ such as $p > 0$. Then, for any $s$ and $t$ such as $\max(s, t) \leq p$,

$$T(s, t)^*T(s, t) \geq |T|^{2(s+t)} \geq T(s, t)T(s, t)^*$$

(i.e., the Aluthge transform $T(s, t)$ of $T$ is 1-hyponormal) and, for any $s$ and $t$ such as $\max(s, t) > p$,

$$\{T(s, t)^*T(s, t)\}^{(p+\min(s, t))/(s+t)} \geq |T|^{2(p+\min(s, t))} \geq \{T(s, t)T(s, t)^*\}^{(p+\min(s, t))/s+t}$$

(i.e., the Aluthge transform $T(s, t)$ of $T$ is $(p+\min(s, t))/(s+t)$-hyponormal).

**Remark.** In the case where $s = t = \frac{1}{2}$, above Theorem is proved by A. Aluthge ([1]) under the condition that $U$ is unitary.

To prove our theorem we need the following lemmas which are the slight modifications of known results ([2], [3]) and state them without proof.

**Lemma 1.** If $A \geq B \geq 0$, then

1. \((B^*A^*B)^{\delta}\geq (B^*B)^{\delta}\) and
2. \((A^*A^*A)^{\delta}\geq (A^*B^*A)^{\delta}\)

for each $\alpha$, $\beta$, $\gamma$ and $\delta$ such as $1/\beta \leq \alpha$, $0 < \beta < 1$, $\gamma = (\alpha \beta - 1)/(2(1 - \beta))$, $0 \leq \delta \leq 1$. (In the case where $\gamma = 0$, it is well-known as Heinz's inequality).

**Lemma 2.** If $A \geq 0$ and $\|B\| \leq 1$, then

\((B^*AB)^{\delta} \geq B^*A^*B\) for each $\delta$ such as $0 \leq \delta \leq 1$.

**Proof of Theorem.** Since $T$ is p-hyponormal, $|T|^{2p} \geq |T^*|^{2p}$. Let $\max(s, t) \leq p$. Then we have

$$T(s, t)^*T(s, t) = |T|^{s}U^*|T|^sU|T|^t = |T|^{s}U^*\left(|T|^{2p}\right)^{s/p}U|T|^t$$

$$\geq |T|^{t}U^*\left(|T|^s\right)^{t/p}|T|^t$$

by Heinz's inequality

$$= |T|^{s}U^*|T|^s|U|T|^t = |T|^s|T|^sT^s$$

$$= |T|^{2(s+t)} = |T|^{s}\left(|T|^{2p}\right)^{s/p}|T|^t$$

by Heinz's inequality

$$\geq |T|^{t}\left(|T|^s\right)^{t/p}|T|^t = T(s, t)T(s, t)^*.$$ 

Since, in the case where $p < s$,

$$\{U^*|T|^{2p}U\}^{s/p} \geq U^*\left(|T|^{2p}\right)^{s/p}U$$

by Lemma 2

$$= U^*|T|^{2p}U$$

$$\geq U^*|T|^s|U|T|^t = |T|^{2p},$$

we have

$$\{T(s, t)^*T(s, t)\}^{(p+\min(s, t))/(s+t)} = \left(|T|^s|U|T|^sU\right)^{(p+\min(s, t))/(s+t)}$$

$$= |T|^{s}|U|T|^sU\left(|T|^s\right)^{t/p}|T|^t$$

$$= |T|^{2(s+t)} = |T|^s\left(|T|^{2p}\right)^{s/p}|T|^t$$

$$\geq |T|^{t}\left(|T|^s\right)^{t/p}|T|^t = T(s, t)T(s, t)^*.$$ 

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by putting $A$, $B$, $\alpha$, $\beta$, $\delta$ in Lemma 1 (1) as follows

$$A = (U^* | T|^{2p})^{1/2p}, \quad B = | T|^{2p},$$

$$\alpha = \frac{s}{p}, \quad \beta = \frac{p + t}{s + t}, \quad \delta = 1 \quad \text{(in this case } \gamma = \frac{t}{2p})$$

because

$$\alpha \beta - 1 = \frac{s(p + t) - p(s + t)}{p(s + t)} = \frac{t(s - p)}{p(s + t)} > 0.$$

Next, let $\max (s, t) > p$. Then, in the case where $\max (s, t) = s$, we have

$$\{T(s, t)^* T(s, t)\}^{(p + \min (s, t))/((s + t))} = \{T(s, t)^* T(s, t)\}^{(p + s)/((s + t))}$$

$$\geq | T|^{2(p + s)} \quad \text{by (i)}$$

$$= | T|^{2(p + \min (s, t))}.$$  

(ii)

If $p \geq t$, then

$$| T|^{2(p + s)} = \{T^* | T|^{2p} | T|^{2p} \}^{(p + s)/((s + t))}$$

$$\geq \{T^* | T|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} \quad \text{by Heinz's inequality}$$

$$= \{T^* | T|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} = \{T^* | T|^{2p} | U|^{2p} \}^{(p + s)/((s + t))}$$

$$= \{T(s, t)^* T(s, t)^* \}^{(p + s)/((s + t))} = \{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t))}/((s + t))$$

and if $p < t$, then

$$| T|^{2(p + s)} = \{((T^* | T|^{2p} | T|^{2p} | T|^{2p} | T|^{2p})^{(p + s)/((s + t))})^{(p + s)/((s + t))}\}^{(p + s)/((s + t))}$$

$$\geq \{((T^* | T|^{2p} | T|^{2p} | T|^{2p} | T|^{2p})^{(p + s)/((s + t))})^{(p + s)/((s + t))}\}^{(p + s)/((s + t))}$$

$$= \{T^* | T|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} = \{T^* | T|^{2p} | U|^{2p} \}^{(p + s)/((s + t))}$$

$$= \{T(s, t)^* T(s, t)^* \}^{(p + s)/((s + t))} = \{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t)))/((s + t))}$$

by putting $A$, $B$, $\alpha$, $\beta$, $\delta$ in Lemma 1 (2) as follows

$$A = | T|^{2p}, \quad B = | T^* |^{2p},$$

$$\alpha = \frac{t}{p}, \quad \beta = \frac{p + s}{t + s}, \quad \delta = \frac{p + t}{p + s} \quad \text{(in this case } \gamma = \frac{t}{2p})$$

because $0 < (p + t)/(p + s) \leq 1$ by the assumption and because

$$\alpha \beta - 1 = \frac{t(p + s) - p(t + s)}{p(t + s)} = \frac{s(t - p)}{p(t + s)} > 0.$$

And hence, for any $t \geq 0$, we have

$$| T|^{2(p + s)} \geq \{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t))/((s + t))}.$$  

(iii)

Therefore, in the case where $\max (s, t) = s$, we have, by (ii) and (iii),

$$\{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t))/((s + t))} \geq | T|^{2(p + \min (s, t))} \geq \{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t))/((s + t))}.$$  

In the case where $\max (s, t) = t$, we have

$$\{T(s, t)^* T(s, t)^* \}^{(p + \min (s, t))/((s + t))} = \{T(s, t)^* T(s, t)^* \}^{(p + s)/((s + t))}$$

$$= \{| T|^{2p} | U|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} \quad \text{(in this case } \gamma = \frac{s}{2p})$$

$$\leq \{| T|^{2p} | U|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} \quad \text{(in this case } \gamma = \frac{s}{2p})$$

$$= \{| T|^{2p} | T|^{2p} \}^{(p + s)/((s + t))} = | T|^{2(p + s)} \quad \text{(in this case } \gamma = \frac{s}{2p})$$

by putting $A$, $B$, $\alpha$, $\beta$, $\delta$ in Lemma 1 (2) as follows

$$A = | T|^{2p}, \quad B = | T^* |^{2p},$$

(iv)
\[ \alpha = \frac{t}{p}, \quad \beta = \frac{p+s}{t+s}, \quad \delta = 1 \quad \text{(in this case \quad } \gamma = \frac{s}{2p} \text{)} \]

because

\[ \alpha \beta - 1 = \frac{t(p+s) - p(t+s)}{p(t+s)} = \frac{s(t-p)}{p(t+s)} > 0. \]

If \( p \geq s \), then

\[ |T|^{2(p+s)} = \{ |T|^4 |T^2 |T^1 |(p+s)/(t+s) \} \leq \{ |T|^1 |U^* |T^* |T^2 |U |T^1 |(p+s)/(t+s) \}
\]

\[ = \{ |T|^1 |U^* |T^* |T^1 |(p+s)/(t+s) \}
\]

\[ \leq \{ |T|^1 |U^* |T^* |T^1 |(p+s)/(t+s) \} \quad \text{by Heinz's inequality}
\]

\[ = \{ |T|^1 |U^* |T^* |T^1 |(p+s)/(t+s) \}
\]

\[ = \{ T(s, t)^* T(s, t) \}^{(p+s)/(t+s)} = \{ T(s, t)^* T(s, t) \}^{(p \min(s, t))/(t+s)} \]

and if \( p < s \), then

\[ |T|^{2(p+s)} = \{ |T|^{2(p+t)} \}^{(p+s)/(p+t)} \leq \{ |T|^{2(p+t)} \}^{(p+s)/(p+t)} \quad \text{by (i) and by Heinz's inequality}
\]

\[ = \{ T(s, t)^* T(s, t) \}^{(p+s)/(p+t)} = \{ T(s, t)^* T(s, t) \}^{(p \min(s, t))/(t+s)} \]

because \( 0 < (p+s)/(p+t) \leq 1 \) by the assumption.

And hence, for any \( s \geq 0 \), we have

\[ |T|^{2(p+s)} \leq \{ T(s, t)^* T(s, t) \}^{(p \min(s, t))/(t+s)}. \quad \text{(v)} \]

Therefore, in the case where \( \max(s, t) = t \), we have also, by (iv) and (v),

\[ \{ T(s, t)^* T(s, t) \}^{(p \min(s, t))/(t+s)} \leq |T|^{2(p+s)} \leq \{ T(s, t)^* T(s, t) \}^{(p \ min(s, t))/(t+s)}. \]

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[3] T. Furuta, \( A \geq B \geq 0 \) assures \( (B^r A^r B^r)^{1/r} \geq B^{(r+2r)/q} \) for \( r \geq 0, p \geq 0, q \geq 1 \) with \( (1+2r)q \geq p+2r \), Proc. Amer. Math. Soc. 101 (1987), 85–88.