Self-Organized Criticality in a Random Network Model

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A new model of self-organized criticality is defined by incorporating a random network model in order to explain endogenous complex fluctuations of economic aggregates. The model can feature many globally interactive systems such as economies or societies.

KEYWORDS: Self-organized criticality, random network model, global interaction, economic fluctuations

1 Introduction

The concept of self-organized criticality has attracted some attention in the literature of economic dynamics. It is thought that small non-linearity in production behavior might cause large and complex endogenous fluctuations of aggregate supply/demand. Bak, Chen, Schelinkman and Woodford (1993) proposed this idea by using a sand-pile model for the first time in economic literature. A common feature of this stream of literature on self-organized criticality is that the system is assumed to be only locally interactive. However, one of the most remarkable features of economic and social phenomena is that they occur in a globally interactive network.

A new model of self-organized criticality is defined in this article by incorporating random allocation of bonds of percolation models. The main result of this model is that the system globally converges to a critical state at which the distribution of the accumulated amount of input/output flows caused by a single shock follows a power law. This implies that a large fluctuation endogenously occurs in the system. The model has the characteristic that it is globally interactive and is easily applicable to the chronic critical phenomena in economies or societies.

2 Model

Suppose that the system consists of \( N \) nodes. Node \( i, i = 1, \ldots, N \), has a bond parameter \( K_i \) which denotes the number of connecting bonds. Hence \( K_i \) determines the number of outputs received from node \( i \). This kind of system which consists of \( N \) nodes and \( K \) bonds is known as NK model. In our model, the bonds are randomly allocated to different nodes in each period. Such random allocation of bonds is used in Derrida and Pomeau (1986) to define their model of annealed approximation of NK models. Random bond models seem natural when global interactions in economies or societies are considered since such interactions among agents often occur through random matchings. Node \( i \) takes \( K_i \) possible states called \( s_0, s_1 = 0, 1, \ldots, K_i-1 \). Whenever an input comes into node \( i \), its state \( s_i \) decreases by one; namely, inputs change \( s_i \) as \( K_i - 1 \rightarrow K_i - 2 \rightarrow \cdots \rightarrow 0 \). If an input comes onto a node in state \( s_i = 0 \), the state of the node goes back to \( s_i = K_i - 1 \). At this point, node \( i \) also releases outputs onto \( K_i \) nodes which are randomly selected among the \( N \) nodes. Note that this is the only time when any output is discharged from node \( i \).

We place two assumptions on the dynamics of the system.

Assumption 1 Whenever there are no inputs in the system, an exogenous shock is introduced as an input into a randomly selected node to activate the system again.

Assumption 2 Each node can accept only one input in a period. If two or more inputs enter a node, they have no additional effect.

The first assumption is a simplification of the overall uncertainty of the system. The second assumption completes the function of nodes.

Let us define a unit of time, period, as the time it takes for an input coming into a node to turn into outputs, and call it \( t \). Let \( M(t) \) denote the number of aggregate outputs in \( t \). For convenience of later discussion, let us normalize this value as \( m(t) = M(t)/N \).

Assumption 3 (Initial condition) \( M(0) = 1 \). Initial arrangement of states is allowed to be taken arbitrarily.

An exogenous input at the first period initiates an input/output flow. The input could cause chain reactions of inputs/outputs (an avalanche) depending on states of the nodes which receive inputs.

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Let us call the group of nodes which have the same bond parameter \( j \), \( \text{"} j \text{-nodes}, \)" where \( j = 2, 3, \ldots, K \) and \( K \) is the maximum value of the bond parameter in the system. Let \( \tilde{N}_{i,j}(t) \) denote the number of \( j \)-nodes in state \( i \) at period \( t \), \( i = 0, 1, \ldots, j - 1 \). \( \tilde{N}_{i,j}(t) \) are normalized as \( \tilde{n}_{i,j}(t) = \frac{N_{i,j}(t)}{N} \). \( p_j \) denotes a ratio of the \( j \)-nodes to all the nodes, hence \( \sum_j p_j = 1 \). Each \( p_j \) is assumed to be a given constant.

This system has the following interpretation. Let us regard the nodes as firms which produce different goods. Firm technology shows increasing return to scale, so it is optimal for a \( j \)-firm to produce \( j \) goods at once, to put \( j - 1 \) goods aside for inventory, and to sell the inventory until exhausted. A \( j \)-firm pays wage of \( j \) to its employee when it produces \( j \) goods. The worker consumes his income in the next period after working, which creates new orders for goods. Hence the size of \( \text{"} \)avalanche\( \)\ means the accumulated amount of demand which is created in a process of chain reaction of production/demand. Here we assume that workers choose consumption goods randomly, although this model can incorporate a broader set of consumption schedules as long as those schedules are widely dispersed amongst the population.

3 Main results

3.1 Global stability

First, a deterministic relationship between \( m(t) \) and \( n_{i,j}(t) \) is derived as follows. Let \( Y_{i,j}(t) \) denote the number of \( j \)-nodes that receive an order at \( t \) while they are in state \( i \), and normalize them as \( y_{i,j}(t) = \frac{Y_{i,j}(t)}{N} \). \( Y_{i,j}(t) \) is a random variable which follows a binomial distribution with probability \( n_{i,j}(t) \) and parameter \( M(t) \), which is expressed in this article as \( Y_{i,j}(t) \sim \text{Bin}(M(t), n_{i,j}(t)) \). It is obvious by the definition of \( y_{i,j}(t) \) that

\[
m(t) = \sum_{j=1}^{K} \sum_{i=0}^{j-1} y_{i,j}(t). \tag{1}
\]

Since each \( Y_{i,j}(t) \) node produces \( j \) units of goods respectively,

\[
m(t + 1) = \sum_{j=1}^{K} jy_{0,j}(t). \tag{2}
\]

On the other hand, we observe that for any \( j \),

\[
n_{i,j}(t + 1) - n_{i,j}(t) = -y_{i,j}(t) + y_{i+1,j}(t) \quad \text{for} \quad 0 \leq i \leq j - 2
\]

\[
n_{j-1,j}(t + 1) - n_{j-1,j}(t) = -y_{j-1,j}(t) + y_{0,j}(t). \tag{4}
\]

After some calculations, a relationship

\[
m(t + 1) - m(t) + \sum_{j=1}^{K} \sum_{i=0}^{j-1} (j - i)(n_{i,j}(t + 1) - n_{i,j}(t)) = 0 \tag{5}
\]

is obtained. Thus we have a deterministic relationship,

\[
m(t) + \sum_{j=1}^{K} \sum_{i=0}^{j-1} (j - i)n_{i,j}(t) = C. \tag{6}
\]

for any \( t \) for a constant \( C \) which is determined by initial condition.

Next we discuss the dynamics of the distribution of states \( n_{i,j}(t) \). Our aim is to prove the following proposition.

**Proposition 1** The distribution of states of nodes \( \{n_{i,j}\} \) converges to the stationary state \( \{p_j/j\} \) from any initial state.

**Proof:** Let \( \Delta x(t) \) denote an increment of the variable \( x \) from \( t \) to \( t + 1 \). \( n_{i,j}(t) \) is regarded as the probability for nodes to be in the state \( i, j \) conditional to the initial state. Thus Chapman-Kolmogorov equations are written as follows.

\[
\Delta n_{0,2}(t) = (n_{1,2}(t) - n_{0,2}(t))m(t), \quad \Delta n_{1,2}(t) = (n_{2,2}(t) - n_{1,2}(t))m(t)
\]

\[
\vdots
\]

\[
\Delta n_{0,K}(t) = (n_{1,K}(t) - n_{0,K}(t))m(t), \quad \ldots, \quad \Delta n_{K-1,K}(t) = (n_{0,K}(t) - n_{K-1,K}(t))m(t)
\]

The \( i \)-th element of \( j \)-th law of motion denote the dynamics of the probability of \( j \)-nodes in a state \( i \). \( m(t) \) is induced from \( n_{i,j}(t) \) by the deterministic relationship (6). It is obvious from (7) that the state satisfying \( n_{i,j} = p_j/j \) is a stationary state when \( m(t) > 0 \), \( \forall t \), which is satisfied by Assumption 1.

The global stability of the stationary state is proved by Liapunov's direct method. Let us define a Liapunov function \( V(\cdot) \) as
Let a function \( f(\cdot) \) denote the difference equations of \( n_{ij} \),
\[
(n_{0,i}(t+1), \ldots, n_{i,j}(t+1), \ldots, n_{K-1,k}(t+1)) = f(n_{0,i}(t), \ldots, n_{i,j}(t), \ldots, n_{K-1,k}(t)).
\]
(9)

Define \( \Delta V(\cdot) \) as,
\[
\Delta V(n_{0,i}, \ldots, n_{i,j}, \ldots, n_{K-1,k}) = V(f(n_{0,i}, \ldots, n_{i,j}, \ldots, n_{K-1,k})) - V(n_{0,i}, \ldots, n_{i,j}, \ldots, n_{K-1,k}).
\]
(10)

It is obvious that \( V(\cdot) \) has a unique minimum at the stationary state. The sufficient condition for the global stability of the stationary state is \( \Delta V(\cdot) \leq 0 \) at any point in the domain \( 0 \leq n_{ij} \leq p_j \), \( \forall i, j \). In calculations of \( \Delta V \) the notation \( m \) instead of \( C - \sum_j \sum_i (j-i)n_{ij} \) is employed for conciseness. Also we define \( n_{ij} \) as \( n_{0,j} \). The expression of \( \Delta V \) is simplified as,
\[
\Delta V(n_{0,i}, \ldots, n_{i,j}, \ldots, n_{K-1,k}) = \sum_{j=2}^{K} \sum_{i=0}^{j-1} \left( (n_{i+1,j} - n_{i,j})m + n_{i,j} - \frac{p_j}{j} \right)^2 - \left( n_{i,j} - \frac{p_j}{j} \right)^2.
\]
(11)

Hence the stability condition is,
\[
\Delta V \leq 0 \Leftrightarrow m(m-1) \leq 0 \Leftrightarrow 0 \leq m \leq 1.
\]
(12)

As is mentioned, \( m(t) > 0, \forall t \), holds by Assumption 1. \( m(t) \leq 1 \) is satisfied by Assumption 2, since when \( m(t) > 1 \), the probability that more than one inputs enter a node is one, so \( m(t) \) will be reduced until \( m(t) \leq 1 \) is satisfied. Hence the global stability of the stationary state is proved.

3.2 Endogenous fluctuations

Now we investigate the magnitude of fluctuations caused by a micro-shock at the stable stationary state. Our aim is to show that the distribution of the avalanche size follows a power law, which shows us that the steady state of this model is a critical state.

Let \( T \) denote a stopping time of a chain reaction of input/output.
\[
\begin{cases}
M(t) > 0, & t < T \\
M(t) = 0
\end{cases}
\]
(13)

\( T \) is a random variable. Size of an avalanche \( W(T) \) is defined as
\[
W(T) = \sum_{t=0}^{T} M(t).
\]
(14)
The power law of the avalanche size is defined as \( \Pr(W = w) \propto w^{-\alpha} \), where \( \alpha \) is a constant.

Let us consider the system after the convergence to the stationary state occurs. Let us reset the time and consider that the system is at the stable stationary state at \( t = 0 \), i.e. \( n_{ij}(0) = p_j/j, \forall i, j, \) and \( M(0) = 1 \). Let \( q_j \) denote the probability of \( M(1) = j \), and let \( g(s) \) denote a probability generating function of \( M(1) \), i.e.,
\[
g(s) = \sum_{j=0}^{\infty} p_j q_j s^j.
\]
The probability for an input to hit \( j \)-node in state \( j-1 \), and then to produce new \( j \) inputs at \( t = 1 \), is \( n_{j-1,j}(0) = p_j/j \). If the input does not hit such nodes, no new input is produced at \( t = 1 \). So \( q_j = p_j/j \) for \( j = 2, 3, \ldots, K, q_0 = 1 - \sum_{j=2}^{K} p_j/j, \) and \( q_j = 0 \) for \( j = K+1, K+2, \ldots \). Hence we obtain the probability generating function,
\[
g(s) = 1 - \sum_{j=2}^{K} \frac{p_j}{j} + \sum_{j=2}^{K} \frac{p_j}{j} s^j.
\]
(15)

By the nature of generating function, \( E(M(1)) = g'(1) \), the first moment of \( M(1) \) is obtained to be one.

First, we want to show the following proposition.

**Proposition 2** When \( N \) increases to infinity, the stochastic process of \( M(t) \) asymptotically follows a branching process which is governed by the probability generating function \( g \).

Proof: The proposition results from two asymptotics. One is that the number of children sprung from one node is asymptotically independent over periods. This holds if the cross section distribution of nodes \( \{n_{ij}\} \) is asymptotically constant. If \( M(t) \) is finite, \( Y_{1,i}(t) \) is finite and then \( y_{1,i}(t) \) goes to zero when \( N \) increases to infinity. It is known that \( M(t), \forall t \), is finite with probability one when \( M(t) \) is a branching process and \( g'(1) \leq 1 \) holds (see Feller (1957)). Also, the deviation of \( M(t) \) from the conjectured limiting branching process is finite, since \( \{n_{ij}\} \) started at the stable stationary state. Hence from equation (3), \( n_{ij} \) is asymptotically constant at the stationary state when \( N \) is very large. The other asymptotic prerequisite is that the draw of nodes by inputs at
one period is mutually independent. This holds when $N$ goes to infinity, too. □

Now we can apply some known properties on branching processes to $M(t)$. If the initial input $M(0)$ is any finite number different to one, $M(t)$ in the following theorems is interpreted as additional inputs/outputs flow caused by a shock, and $W(T)$ is an accumulated amount of the flow. This interpretation is safe because setting $M(0) = 1$ is just a simplification without loss of generality. The first theorem says that the probability of the stopping time follows a power law with a characteristic exponent one (see Harris (1989), p. 21 (10.8)).

**Theorem 1 (Kolmogorov)**  
If a branching process $M(t)$ satisfies $g'(1) = 1$ and $g''(1) < \infty$, then

$$\Pr (M(t) > 0) \sim \frac{2}{g''(1)}.$$

In our case, $g''(1) = \sum_{j=1}^{K} (j-1)(j-2)p_j < \infty$ since $K < \infty$, so the prerequisite of the theorem is satisfied.

Furthermore, the following theorem is known about the distribution of the avalanche size $W(T)$ (Harris (1989), p. 32 Theorem 13.1).

**Theorem 2 (Otter)**  
Suppose that $q_0 > 0$ and that there is a point $a > 0$ in the interior of the circle of convergence of $g$ for which $g'(a) = g(a)/a$. Let $\alpha = a/g(a)$ and let $d$ be the largest integer such that $q_d \neq 0$ implies that $r$ is a multiple of $d$, $r = 1, 2, \ldots$. If $r - 1$ is not divisible by $d$, then we have $\Pr (W(T) = r) = 0$, while if $r - 1$ is divisible by $d$, then $\Pr (W(T) = r) = d(\alpha/(2\pi g''(\alpha)))^{1/2}r^{-3/2} + O(\alpha^{-r^{-1/2}})$, $r \rightarrow \infty$.

Our main proposition is straightforwardly derived from this theorem.

**Proposition 3**  
The probability of the size of the accumulated demand caused by a shock follows a power law. Explicitly,

$$\Pr (W(T) = w) = (2\pi g''(1))^{-1/2}w^{-3/2} + O(w^{-5/2}), \ w \rightarrow \infty$$

Proof:  
$g$ is a polynomial function so that such $a$ exists in our model, and $a = \alpha = 1$. Also $d = 1$. Apply these values to Theorem 2. □

Since the exponent of this power law is $3/2$, the right tail of the probability density can be approximated by a density function of a stable distribution with characteristic exponent $1/2$. This implies that the first and second moments of $W$ do not exist.

4 Simulation when $K = 2$

Through analytical methods we can only go for the asymptotic nature of the process. It turns out that $M(t)$ is not strictly a branching process when $N$ is finite, and one might doubt that this finiteness affects the proposition. We have to resort to simulation in order to test if the proposition is robust with finite $N$. A simple case with $K = 2$ is investigated below.

In order to compare the simulated frequency to the theoretical counterpart, we introduce new notation.

$$r(t) = \Pr (M(t) > 0), \quad (16)$$

$$p(t) = \Pr (T = t). \quad (17)$$

Explicit expressions for $r(t)$ and $p(t)$ is obtained in Appendix A. These sequences are plotted in Figure 1.

As shown in Figure 1, it is clear that $p(t)$ also follows a power law with an exponent two when $N$ is large enough. It is learned that an exponent of the power law of $r(t)$ is equal to one from the equation $\sum_{r=1}^{\infty} p(r) = r(t)$.

We simulate a finite system with $10^3$ nodes and count the frequency of duration of chain reactions out of the trial number of periods $10^6$. The result of simulation is definitely affirmative to our proposition as seen in Figure 1. Appendix B shows that $W(T)$ obeys a power law with an exponent $3/2$ if the power law of $r(t)$ with an exponent one is established.

5 Discussion and Conclusion

The structure of our model is very close to Kauffman's "NK models," especially to its "annealed approximation" by Derrida and Pomeau. In NK models, it is said that the system has a critical value $K$ between 2 and 3 because the behavior of a network is very different for $K > 2$ and $K \leq 2$. However, in the analytical part of this model, we find that the nature of self-organized criticality in our model is not affected by the number of connecting bonds $K$.

We observe that there is a kind of a conservation law in our model, which is the relationship $m(t) + n_0(t) = \text{constant}$ during a period. This law could be a necessary condition in our framework to satisfy the zero trend property of $M(t)$ near the steady state, because if the value in the right hand side of the equation dissipates or accumulates over time, it gives rise to a non-zero trend of the stochastic process of $M(t)$.

In this paper we study self-organized criticality in a random network model. The stationary state has global
stability, but the point exhibits characteristic instability to a micro-shock, which is expressed by a power law. We show that the model of self-organized criticality could explain how a small exogenous shock is amplified through heterogeneous structure of economies and how large endogenous fluctuations occur.

In economic sense, it is inferred in this paper that the level of aggregate inventory is a crucial variable for fluctuations of aggregate product. When the inventory level is high, production is restrained and the inventory will be decreased. When the inventory level is low, production occurs and the inventory will be accumulated. In both ways the inventory level converges to a critical level, where it turns out that large fluctuations could take place.

We propose a global interaction model of self-organized criticality, while previous studies have assumed local interaction. We believe that the global interactive model would be much better suited for the research of economic or social phenomena. We hope that research along the lines of complex systems might shed some light on economic dynamics.

Appendix

A Explicit derivation of \( r(t) \) when \( K = 2 \)

We define \( u_{z,t} \) as the probability that the stopping time is less than or equal to \( t \) when the process started from the initial state \( z \).

\[
    u_{z,t} = \Pr (T \leq t | M(0) = z)
\]  

(18)

We find that the stochastic process of \( M(t) \) follows a branching process where the expected number of children sprung from a parent equals to one at the steady state, because the probability that one input provokes a node to throw outputs is 1/2 at the steady state. Transition probability of \( M(t) \) follows a binomial distribution with probability 1/2; \( M(t + 1)/2 \sim \text{Bin}(M(t), 1/2) \). It is obvious that \( u_{z,1} = 1/2^z \) for any non-negative integer \( z \). We naturally define that \( u_{0,t} = 1 \) for all \( t \).

Let us study a propagation process initiated by a perturbation at \((n_0, m) = (1/2, 0)\), because at this point the effect of a shock can be conveniently measured by the duration for which a chain reaction continues. We ask the probability of the stopping time to be less than or equal to \( t \) when one exogenous input is added to the system at the steady state, i.e., \( u_{1,t} \). We can see a relationship,

\[
    u_{1,t} = \frac{1}{2} \sum_{k_1=0}^{1} \left( \frac{1}{k_1} \right) u_{2k_1,1-t}.
\]  

(19)

Apply the similar expression for \( u_{1,s} \), recursively.

\[
    u_{1,t} = \frac{1}{2} \sum_{k_1=0}^{1} \left( \frac{1}{k_1} \right) \frac{1}{2^{2k_1}} \sum_{k_1=0}^{2k_1} \left( \frac{2k_1}{k_1} \right) \cdots \frac{1}{2^{2k_{t-2}}} \sum_{k_{t-2}=0}^{2k_{t-2}} \left( \frac{2k_{t-2}}{k_{t-2}} \right) \cdots \frac{1}{2^{2k_{t-1}}} \sum_{k_{t-1}=0}^{2k_{t-1}} \left( \frac{2k_{t-1}}{k_{t-1}} \right) u_{2k_{t-1},1-t}.
\]  

(20)

By utilizing the nature of the binomial, we can simplify this expression as,
\[ u_{t,r} = \frac{1}{2q_{t-1}} \]  
(21)  
\[ q_{t+1} = \frac{4q_t^2}{1 + 4q_t^2} \]  
(22)  
\[ q_0 = 1. \]  
(23)  

Since the sequence \( q_t \) starts from 1 and monotonically declines to 1/2, we verify that the nature of binomials used in the derivation of this expression can be actually used properly.

Now by the definition of \( r(t) \),

\[ r(t) = 1 - u_{t,r}. \]  
(24)  

So, \( r(t) \) is sequentially determined as,

\[ r(t+1) - r(t) = -\frac{r(t)^2}{2} \]  
(25)  
\[ r(0) = 1. \]  
(26)  

\( p(t) \) is derived from \( r(t) \) by a relationship,

\[ p(t) = r(t) - r(t + 1). \]  
(27)  

**B Power law of \( W(T) \) inferred from power law of \( r(t) \)**

In this section it is shown that \( W(T) \) follows a power law when \( r(t) \) is known to follow a power law. We define \( x(t) \) to be the expected value of the flow of the avalanche at \( t \) conditional on \( M(t) > 0 \),

\[ x(t) = E [M(t) | M(t) > 0]. \]  
(28)  

\( M(t) \) follows a branching process when \( N \to \infty \) and \( M(t) > 0 \),

\[ E [M(t+1) | M(t) > 0] = M(t). \]  
(29)  

Taking the unconditional expectation of the both sides,

\[ E [M(t+1)] = E [M(t)]. \]  
(30)  

Note that,

\[ E [M(t)] = r(t) E [M(t) | M(t) > 0] + (1 - r(t)) E [M(t) | M(t) = 0] = r(t)x(t). \]  
(31)  

Then we get,

\[ r(t)x(t) = r(t+1)x(t+1). \]  
(32)  

Since we know that \( r(t) \sim t^{-1} \) from the simulation and Theorem 2, it is obvious that

\[ x(t) \sim t. \]  
(33)  

For any given \( T \), the expected size of the avalanche is expressed as follows.

\[ E [W(T)] = \sum_{t=0}^{T} x(t) \sim T^2. \]  
(34)  

Now taking into consideration that \( T \) is a random variable following that

\[ Pr (T \geq t) = r(t) \sim t^{-1}, \]  
(35)  

we finally obtain that,

\[ Pr (W(T) \geq W) \sim W^{-1/2} \]  
for large \( W \).

Thus we find that the density function of the avalanche generated in one process caused by a micro-shock follows a power law with an exponent 3/2.

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