On the States Having Pure-State Restrictions for a Pair of Regions with a Non-Trivial Intersection

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For given two regions of the lattice, assume that our state has a pure-state restriction for each of them. We discuss whether its restrictions to the intersection and the union regions are pure states or not. It is immediate to see that the answer is affirmative for any tensor-product systems. (We, however, include its proof for the completeness sake.) In this note, we consider mostly CAR systems. We show that the assertion above holds for any such a state of finite-dimensional CAR systems. For infinite-dimensional CAR systems, assuming additionally the product properties between each of the regions and its complement region on the states satisfying the standing assumption, we can show the purity on the intersection region for them. However, the latter part of the assertion will not always hold for the infinite-dimensional CAR systems unlike for the finite-dimensional CAR or any dimensional tensor-product systems. We establish the criterions for the purity and the non-purity on the union region for those states of the infinite-dimensional CAR systems.

KEYWORDS: CAR, three-composed systems, pure-state restriction

1. Introduction

In this note, we study the following question (A).

(A) If a state is pure on each of two regions $I$ and $J$, would it be pure on their intersection $I \cap J$ and on their union $I \cup J$?

The purity of a state refers to the non-existence of any non-trivial decomposition of the state into a convex sum of other states.

This work can be considered as a continuation of our previous paper [1] where we assumed that all subregions are mutually disjoint. Now we consider pair of regions with a non-trivial intersection; this setting produces several complications. We should remark that the problem (A) was taken up by A. R. Kay and B. S. Kay [4]. (See [4, §6].)

In Theorem 3.1, we show that (A) holds for any tensor-product and any finite-dimensional CAR systems without assuming any other conditions on the state. Here the “finite-dimension” means that the dimension of each of the subsystems on the given two regions is finite.

We will consider (A) for infinite-dimensional CAR systems as well. To deal with this problem for those systems, we need the additional technical assumption that the state satisfies the product property between $I$ and its complement region $I^c$ and also that between $J$ and $J^c$. (We, however, expect that our results about the infinite-dimensional CAR systems could be improved so as to cover more general situations.) Under this assumption, we are able to show the first part of (A), i.e. the purity for the intersection region in Theorem 4.1 (1).

However, the latter part of (A), i.e. the purity for the union region will not hold in general for the infinite-dimensional CAR systems. We establish the general criterions for the purity and the non-purity on the union region for the product states having pure-state restrictions on the pair of given regions in Theorem 4.1 (2) and (3).

The states violating the latter part of (A) can be constructed by using the states studied in [1]. (See Theorem 3 in [1], or Proposition 4.4 in this article.) More precisely, such a state should have a restricted state $\varphi_2$ on the intersection region $I \cap J$ such that the GNS representations of $\varphi_2$ and of $\varphi_2 \Theta$ are disjoint, and also should have the restricted states $\varphi_1$ and $\varphi_3$ on the regions $I \setminus J$ and on $J \setminus I$, respectively, which are even states such that $\varphi_i$ ($i = 1, 3$) is a pure on the even part $A(I)_+$ and is a mixed state on $A(I)$ whose any non-trivial decomposition is given by the following form:

$$\varphi_i = \frac{1}{2} (\hat{\varphi}_i + \hat{\varphi}_i \Theta),$$

where $\hat{\varphi}_i$ is a pure state of $A(I)$ and the GNS representations of $\hat{\varphi}_i$ and of $\hat{\varphi}_i \Theta$ are disjoint. For its details, see “Purity, Non-Purity of $\varphi_{1,3,+}$” in the proof of Theorem 4.1.

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2. Notation

We introduce CAR systems on a lattice. For each lattice site \(i\), there exist annihilation and creation operators \(a_i\) and \(a_i^*\). Denote the anti-commutator by \([A, B] = AB + BA\). The CAR (canonical anticommutation relations)

\[
\begin{align*}
[a_i^*, a_j] &= \delta_{ij} 1, \\
[a_i, a_j^*] &= [a_i, a_j] = 0,
\end{align*}
\]

are satisfied for any sites \(i, j\). We write the total system on the lattice by \(\mathcal{A}\). For each subset \(I\) of the lattice, there is a subsystem \(\mathcal{A}(I)\) generated by \(a_i, a_i^*\), \(i \in I\). (A generalization where there exist a finite number of Fermions on each lattice site can be treated similarly without any essential change.)

We denote the even–odd automorphism by \(\Theta\) determined by

\[
\Theta(a_i) = -a_i, \quad \Theta(a_i^*) = -a_i^*
\]

for all \(i\). The even and odd parts of \(\mathcal{A}\) and \(\mathcal{A}(I)\) are defined by

\[
\mathcal{A}_\pm \equiv \{A \in \mathcal{A} \mid \Theta(A) = \pm A\}, \quad \mathcal{A}(I)_\pm \equiv \mathcal{A}_\pm \cap \mathcal{A}(I).
\]

For any \(A \in \mathcal{A}\) (resp. \(A \in \mathcal{A}(I)\)), we have the following decomposition

\[
A = A_+ + A_-, \quad A_\pm = \frac{1}{2}(A \pm \Theta(A)) \in \mathcal{A}_\pm \quad \text{(resp. } \in \mathcal{A}(I)_\pm\text{)}.
\]

A state \(\varphi\) of \(\mathcal{A}\) (resp. \(\mathcal{A}(I)\)) is called even if it is \(\Theta\)-invariant:

\[
\varphi(\Theta(A)) = \varphi(A)
\]

for all \(A \in \mathcal{A}\) (resp. \(A \in \mathcal{A}(I)\)). Note that \(\varphi(A) = 0\) for all \(A \in \mathcal{A}_-\) (resp. \(\mathcal{A}(I)_-\)) is equivalent to the condition that \(\varphi\) is an even state of \(\mathcal{A}\) (resp. \(\mathcal{A}(I)\)).

On the other hand, tensor-product systems are given by

\[
\mathcal{A}(I) \otimes \mathcal{A}(J) = \mathcal{A}(I \cup J)
\]

for any disjoint regions \(I\) and \(J\).

We now consider a pair of disjoint regions \(I_1\) and \(I_2\). Let \(\varphi_1\) be a state of \(\mathcal{A}(I_1)\) and \(\varphi_2\) be that of \(\mathcal{A}(I_2)\). If a state \(\varphi\) of the joint system \(\mathcal{A}(I_1 \cup I_2)\) (which is the \(C^*\)-subalgebra of \(\mathcal{A}\) generated by \(\mathcal{A}(I_1)\) and \(\mathcal{A}(I_2)\)) coincides with \(\varphi_1\) on \(\mathcal{A}(I_1)\) and \(\varphi_2\) on \(\mathcal{A}(I_2)\), namely,

\[
\varphi(A_1) = \varphi_1(A_1), \quad A_1 \in \mathcal{A}(I_1),
\]

\[
\varphi(A_2) = \varphi_2(A_2), \quad A_2 \in \mathcal{A}(I_2),
\]

then \(\varphi\) is called a joint extension of \(\varphi_1\) and \(\varphi_2\).

As a special case, if

\[
\varphi(A_1 A_2) = \varphi_1(A_1) \varphi_2(A_2)
\]

for all \(A_1 \in \mathcal{A}(I_1)\) and all \(A_2 \in \mathcal{A}(I_2)\), then \(\varphi\) is called a product-state extension of \(\varphi_1\) and \(\varphi_2\). The definition of product states can be extended to an arbitrary number of \(\mathcal{A}(I_1), \mathcal{A}(I_2), \ldots\) with mutually disjoint regions \(\{I_i\}\). Namely, for a set of given states \(\varphi_i\) of \(\mathcal{A}(I_i)\), a state \(\varphi\) of \(\mathcal{A}(I_1 \cup I_2)\) is called a product state extension of \(\{\varphi_i\}\) if it satisfies

\[
\varphi(A_1 A_2 \ldots A_k) = \prod_{i=1}^k \varphi_i(A_i)
\]

for any finite \(k\) and all \(A_i \in \mathcal{A}(I_i), i \in \{1, 2, \ldots, k\}\).

We will need the following lemma saying that any pair of odd elements of disjoint regions anticommutes, while the other pairs of even and odd elements commute. (It is a direct consequence of the CAR and we omit its proof.)

**Lemma 2.1.** Let \(I_1\) and \(I_2\) be mutually disjoint and \(A_\sigma \in \mathcal{A}(I_1)_\sigma, B_\sigma' \in \mathcal{A}(I_2)_\sigma'\) where \(\sigma, \sigma' = + or -\). Then

\[
A_\sigma B_{\sigma'} = \epsilon(\sigma, \sigma') B_{\sigma'} A_\sigma,
\]

where

\[
\epsilon(\sigma, \sigma') = \begin{cases} 
-1, & \text{if } \sigma = \sigma' = - \\
+1, & \text{otherwise.}
\end{cases}
\]
3. (A) for Tensor-Product Systems and Finite-Dimensional CAR Systems

As for both tensor-product and finite-dimensional CAR systems, we can have a complete affirmative answer to the question (A).

**Theorem 3.1.** Let I and J be two regions of the lattice with the non-trivial intersection $I \cap J$. Assume also that the region $I$ is not included by J and vice versa, namely, neither $I \setminus J$ nor $J \setminus I$ is empty. Let $\varphi$ be a state of a tensor-product system or any finite-dimensional CAR system. Suppose that the restrictions of $\varphi$ to $\mathcal{A}(I)$ and to $\mathcal{A}(J)$ are both pure states. Then the restrictions of $\varphi$ to the subsystem of the intersection region $\mathcal{A}(I \cap J)$ and that of the union region $\mathcal{A}(I \cup J)$ are both pure states.

The “finite-dimensionality” above means that both I and J are finite and hence both $\mathcal{A}(I)$ and $\mathcal{A}(J)$ are finite-dimensional.

We will denote $K_1 \equiv I \setminus J$, $K_2 \equiv I \cap J$ and $K_3 \equiv J \setminus I$. For simplicity, the numbers 1, 2 and 3 will indicate the corresponding regions and the subsystems.

**Proof of Theorem 3.1.**

- Finite-dimensional case (both for tensor-product and CAR systems). As was written in [4], (A) can be proved by using the strong subadditivity (SSA) of von Neumann entropy. Namely, the SSA with the basic fact that the entropy vanishes if and only if the state is pure immediately implies (A). Note that the SSA proved for tensor-product systems by Lieb–Ruskai [5] is also valid for CAR systems (without any assumptions on the states) as shown in [2]. In conclusion, (A) holds both for finite-dimensional tensor-product and finite-dimensional CAR systems.
- Infinite-dimensional case (only valid for tensor-product systems). For the infinite-dimensional case, the above entropy argument does not make sense. Although we should restrict ourselves to tensor-product systems due to some difficulty caused by CAR, we will provide a simple argument showing (A) irrespective of the dimension of subsystems.

First we notice the following well-known fact given e.g. as [7, Chapter IV, Lemma 4.11]:

**Lemma 3.2.** If $\mathcal{B}$ is a $C^*$-subalgebra of a $C^*$-algebra $\mathcal{C}$ and if the restriction $\omega_{\mathcal{B}}$ of a state $\omega$ of $\mathcal{C}$ to $\mathcal{B}$ is a pure state, then

$$\omega(xy) = \omega(x)\omega(y), \quad x \in \mathcal{B}, \ y \in \mathcal{B}' \cap \mathcal{C}. \quad (3.1)$$

Due to this lemma and the purity assumption of $\varphi_1$ and $\varphi_3$, we have

$$\varphi(AA_3) = \varphi(A)\varphi(A_3), \quad (3.2)$$

for all $A \in \mathcal{A}(I)$ and $A_3 \in \mathcal{A}(K_3)$, and

$$\varphi(A_1A) = \varphi(A_1)\varphi(A), \quad (3.3)$$

for all $A_1 \in \mathcal{A}(K_1)$ and $A \in \mathcal{A}(J)$. By (3.2), (3.3) and the following tensor-product structure of the system,

$$\mathcal{A}(I \cup J) = \mathcal{A}(K_1) \otimes \mathcal{A}(K_2) \otimes \mathcal{A}(K_3), \quad (3.4)$$

we obtain

$$\varphi(A_1A_2A_3) = \varphi(A_1)\varphi(A_2)\varphi(A_3) \quad (3.5)$$

for all $A_1 \in \mathcal{A}(K_1)$, $A_2 \in \mathcal{A}(K_2)$ and $A_3 \in \mathcal{A}(K_3)$. (Clearly, the order of $I, 2, 3$ is arbitrary.)

Since $\varphi_1$ is pure if and only if both $\varphi_1$ and $\varphi_2$ are pure (or $\varphi_3$ is pure if and only if both $\varphi_2$ and $\varphi_3$ are pure), $\varphi_1$, $\varphi_2$ and $\varphi_3$ are all pure and hence $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is pure. □

**Remark.** The above argument holds irrespective of any (cross) norm $\beta$ which specifies the tensor product $\mathcal{A}(K_i) \otimes_\beta \mathcal{A}(K_j)$ of a pair of $C^*$-algebras $\mathcal{A}(K_i)$ and $\mathcal{A}(K_j)$ ($i \neq j$). (See [3, p. 855, Lemma 11.3.6].)

In the next section, we consider our main concern, infinite-dimensional CAR systems.

4. (A) for Infinite-Dimensional CAR Systems

We have seen that for tensor-product systems the state $\varphi$ in Theorem 3.1 satisfies the product property (3.5). On the other hand, the product property (3.5) does not hold in general for the states satisfying the assumption of Theorem 3.1 for CAR systems (irrespective of the dimension). See the remark below. This is one of obstacles to invent a method for the CAR case similar to that for the tensor-product systems given in the preceding section.

In this note, we shall assume additionally the product property specified in Theorem 4.1 below for infinite-dimensional CAR systems, although this input narrows the range of our study for those systems, and may be possibly weakened or removed.
Remark. The examples which satisfy the assumption of Theorem 3.1 but do not satisfy (3.5) can be made of the states of finite-dimensional CAR systems considered in [6] where they are named half-sided entangled states.

Our result is as follows.

Theorem 4.1. For an infinite-dimensional CAR system, suppose the same assumption as Theorem 3.1, namely, the regions I and J are so and the restrictions of ϕ to A(I) and to A(J) are both pure states. Assume in addition that

$$\varphi(AA_3) = \varphi(A)\varphi(A_3)$$

(4.1)

for all $A \in A(I)$ and $A_3 \in A(K_3)$ and

$$\varphi(A_1A) = \varphi(A_1)\varphi(A)$$

(4.2)

for all $A_1 \in A(K_1)$ and $A \in A(J)$.

(1) The first part of (A) holds, namely, the restriction of ϕ to the subsystem of the intersection region $A(I \cap J)(= A(K_2))$ is pure.

(2) The second part of (A) holds, namely, the restriction of ϕ to the subsystem of the union region $A(I \cup J)$ is pure, unless the GNS representations $\pi_{\varphi_2}$ of $\varphi_2$ and $\pi_{\varphi_3}$ of $\varphi_3$ are disjoint and at the same time both $\varphi_1$ and $\varphi_3$ are non-pure states.

(3) If $\pi_{\varphi_1}$ and $\pi_{\varphi_3}$ are disjoint, and both $\varphi_1$ and $\varphi_3$ are non-pure states, then the restriction of ϕ to $A(I \cup J)$ is not pure.

Based on its proof, we collect some results from [1] about the product-state extension of the states prepared on disjoint regions.

Proposition 4.2. Let $I_1, I_2, \ldots$ be an arbitrary (finite or infinite) number of mutually disjoint regions and $\varphi_i$ be a given state of $A(I_i)$ for each $i$.

(1) A product-state extension of $\varphi_i$, $i = 1, 2, \ldots$, exists if and only if all states $\varphi_i$ except at most one are even. It is unique if it exists. It exists if and only if all $\varphi_i$ are even.

(2) Suppose that all $\varphi_i$ are pure. If there exists a joint extension of $\varphi_i$, $i = 1, 2, \ldots$, then all states $\varphi_i$ except at most one have to be even. If this is the case, the joint extension is uniquely given by the product-state extension of $\varphi_i$, $i = 1, 2, \ldots$, and is a pure state of $A(\bigcup I_i)$.

Proposition 4.3. Let $\varphi$ be the product-state extension of states $\varphi_i$ with disjoint $I_i$. Assume that all $\varphi_i$ are even possibly except $\varphi_1$.

(1) $\varphi_1$ is pure if $\varphi$ is pure.

(2) Assume that the GNS representations $\pi_{\varphi_1}$ of $\varphi_1$ and $\pi_{\varphi_1} \Theta$ of $\varphi_1 \Theta$ are not disjoint. Then $\varphi$ is pure if and only if all $\varphi_i$ are pure. In particular, this is the case if $\varphi$ is even.

Proposition 4.4. Let $\varphi$ be the product-state extension of states $\varphi_1$ of $A(I_1)$ and $\varphi_2$ of $A(I_2)$ with disjoint $I_1$ and $I_2$ such that $\varphi_1$ is even and the GNS representations $\pi_{\varphi_2}$ and $\pi_{\varphi_2} \Theta$ are disjoint.

(1) $\varphi$ is pure if and only if $\varphi_2$ and the restriction of $\varphi_1$ to $A(I_1)$ are both pure.

(2) Suppose that $\varphi$ is pure. Then $\varphi_1$ is a pure state or a mixed state such that any non-trivial decomposition of $\varphi_1$ is given by the following form:

$$\varphi_1 = \frac{1}{2} (\hat{\varphi}_1 + \hat{\varphi}_1 \Theta),$$

(4.3)

where $\hat{\varphi}_1$ is a pure state of $A(I_1)$ and the GNS representations $\pi_{\varphi_1}$ of $\hat{\varphi}_1$ and $\pi_{\varphi_1} \Theta$ of $\hat{\varphi}_1 \Theta$ are disjoint.

Propositions 4.2, 4.3 and 4.4 (1) are exactly same as Theorems 1, 2 and 3 (1) in [1], respectively. Also Proposition 4.4 (2) has been shown essentially as Theorem 3 (2) in [1], but the statement here is a little more sophisticated than in [1]. For this reason and the later use, we will include the proof of Proposition 4.4 (2). First we show the following lemma.

Lemma 4.5. Let $I$ be an arbitrary set and $\omega_+ \in A(I)_+$. Then there exists a unique even state $\omega$ of $A(I)$ whose restriction to the even part $A(I)_+$ coincides with $\omega_+$.

Proof. Because any element $A \in A(I)$ can be decomposed as a summation of its even and odd parts as $A = A_+ + A_-$, $A_+ \in A(I)_+$, and any such an $\omega$ satisfies $\omega(A_-) = 0$ and $\omega(A_+) = \omega(A_+)$, $\omega$ should be unique (if it exists). We are going to show the existence of $\omega$. Let $v$ be a self-adjoint unitary element of $A(I)_+$, take for example $v = a_i + a_i^*$ for a fixed $i \in I$. Let $(H_{\omega_+}, \pi_{\omega_+}, \Omega_{\omega_+})$ be the GNS triplet for $\omega_+$. We define

$$H_\omega \equiv H_{\omega_+} \oplus H_{\omega_-},$$

$$\Omega_\omega \equiv \Omega_{\omega_+} \oplus 0.$$

Let $K$ be the unitary operator on $H_\omega$ determined by
Define

\[ \pi_{\omega}(A_+) \equiv \pi_{\omega_0}(A_+ \oplus \pi_{\nu_0}(vA_+, v) \quad \text{for} \quad A_+ \in \mathcal{A}(I)_+, \]
\[ \pi_{\omega}(A_+ + B_+ v) \equiv \pi_{\omega}(A_+) + \pi_{\omega}(B_+)K \quad \text{for} \quad A_+, B_+ \in \mathcal{A}(I)_+. \]

By a straightforward computation, we see that \( \pi_{\omega} \) is a representation of \( \mathcal{A}(I) \) satisfying

\[ (\Omega_{\omega}, \pi_{\omega}(A_+)\Omega_{\omega}) = \omega_+(A_+), \quad A_+ \in \mathcal{A}(I)_+, \]
\[ (\Omega_{\omega}, \pi_{\omega}(A_-)\Omega_{\omega}) = 0, \quad A_- \in \mathcal{A}(I)_-. \]

Therefore \( \omega(A) = (\Omega_{\omega}, \pi_{\omega}(A)\Omega_{\omega}) \) gives the desired state of \( \mathcal{A}(I) \).

\[ \square \]

**Proof of Proposition 4.4 (2).** Applying Lemma 4.5 to the state \( \varphi_{1+} \) of \( \mathcal{A}(I)_+ \) with its GNS triplet \( ( \mathcal{H}_{\varphi_1}, \pi_{\varphi_1}, \Omega_{\varphi_1} ) \) and a self-adjoint unitary \( v_1 \in \mathcal{A}(I)_- \), we find that \( \mathcal{H}_1 \equiv \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_2}, \Omega_1 \equiv \Omega_{\varphi_1} \oplus 0 \) and

\[ \pi_1(A_{1+} + B_{1+} v_1) \equiv \begin{pmatrix} \pi_{\varphi_1}(A_{1+}) & \pi_{\varphi_1}(B_{1+}) \\ \pi_{\varphi_1}(v_1 B_{1+} v_1) & \pi_{\varphi_1}(v_1 A_{1+} v_1) \end{pmatrix} \quad (4.4) \]

for \( A_{1+}, B_{1+} \in \mathcal{A}(I)_+ \) will give the GNS triplet of the even state \( \varphi_1 \) on \( \mathcal{A}(I)_+ \). In (4.4) we have identified \( \mathcal{B}(\mathcal{H}_1) \) with \( \mathcal{M}_2(\mathcal{B}(\mathcal{H}_{\varphi_1})) \). Hence the purity (the non-purity) of \( \varphi_1 \) is equivalent to the irreducibility (reducibility) of the representation \( \pi_1 \) of \( \mathcal{A}(I)_+ \).

To determine \( \pi_1(\mathcal{A}(I)_+) \)'s, let

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \pi(\mathcal{A}(I)_+)'. \quad (4.5) \]

By direct computation, we see that the commutativity of (4.4) and (4.5) is equivalent to the condition that \( a = d \) is a scalar and \( b = c \) satisfies

\[ \pi_{\varphi_1}(v_1 A_{1+} v_1)c = \pi_{\varphi_1}(A_{1+}). \quad (4.6) \]

From (4.6), we see that \( c^2 \in \pi_{\varphi_1}(\mathcal{A}(I)_+) \) due to \( v_1^2 = 1 \). Therefore \( c^2 \) is a scalar, because of the purity of \( \varphi_{1+} \).

If there does not exist any non-zero \( c \) satisfying (4.6), then \( \varphi_1 \) does not have any non-trivial decompositions and hence it is a pure state. If there exists non-zero \( c \), we may assume \( c^2 = 1 \) by multiplication of a scalar. If \( c_1 \) and \( c_2 \) satisfy (4.6), then \( c_1 c_2 \) must be a scalar by the same reason. Hence \( c \) is unique up to a multiplication of a scalar. If \( c \) satisfies (4.6), \( c^* \) also satisfies (4.6). By the uniqueness, we have \( c^* = e^{i\theta} c \) for some \( \theta \in \mathbb{R} \). Since \( c^* c = e^{i\theta} c^2 = c^2 \) is positive real, we have \( c^* c = 1 \) and hence \( c = c^* \). Namely \( c \) is a self-adjoint unitary. For \( \varphi_1 \) to be a non-pure state, \( \pi_1(\mathcal{A}(I)_+) \)'s has to be non-trivial and such a self-adjoint unitary \( c \) must exist, and vice versa.

We are going to specify the (non-trivial) decomposition of \( \varphi_1 \) corresponding to non-zero \( c \).

Since \( c \) belongs to \( \pi_{\varphi_1}(\mathcal{A}(I)_+) \)'s, \( \mathcal{B}(\mathcal{H}_{\varphi_1}) \),

\[ \tilde{\pi}_1(A_{1+} + B_{1+} v_1) \equiv \pi_{\varphi_1}(A_{1+}) + \pi_{\varphi_1}(B_{1+}) c, \quad A_{1+}, B_{1+} \in \mathcal{A}(I)_+ \quad (4.7) \]

becomes a representation of \( \mathcal{A}(I)_+ \) on \( \mathcal{H}_{\varphi_1} \), because of (4.6). Since \( \pi_{\varphi_1}(\mathcal{A}(I)_+) \) is already irreducible on \( \mathcal{H}_{\varphi_1} \), so is \( \tilde{\pi}_1(\mathcal{A}(I)_+) \).

Thus

\[ \tilde{\varphi}_1(A_{1+}) \equiv (\Omega_{\varphi_1}, \tilde{\pi}_1(A_{1+})\Omega_{\varphi_1}), \quad A_{1+} \in \mathcal{A}(I)_+, \quad (4.8) \]

is a pure state of \( \mathcal{A}(I)_+ \).

Since \( \tilde{\varphi}_1 \) and \( \varphi_1 \) coincide on \( \mathcal{A}(I)_+ \), we have

\[ \frac{1}{2} (\tilde{\varphi}_1 + \tilde{\varphi}_1 \Theta) = \varphi_1 \]

due to the evenness of \( \varphi_1 \).

To prove that \( \pi_{\varphi_1} \) and \( \pi_{\varphi_1 \Theta} \) are disjoint, assume the contrary. Since \( \pi_{\varphi_1} \) is irreducible, \( \pi_{\varphi_1} \) and \( \pi_{\varphi_1 \Theta} \) are unitarily equivalent. So there exists a unitary \( u_1 \) on \( \mathcal{H}_{\varphi_1} \) implementing \( \Theta \) on the representation \( \pi_{\varphi_1} \) by Lemma 4.7 below. However, it has to commute with \( \pi_{\varphi_1}(\mathcal{A}(I)_+) = \pi_{\varphi_1}(\mathcal{A}(I)_+) \Theta \). Thus \( u_1 \) has to be trivial due to \( \pi_{\varphi_1}(\mathcal{A}(I)_+) \)'s \( \equiv \mathcal{B}(\mathcal{H}_{\varphi_1}) \). Hence \( u_1 \) cannot implement a non-trivial automorphism \( \Theta \). Hence \( \pi_{\varphi_1} \) and \( \pi_{\varphi_1 \Theta} \) are disjoint.

We will sum the necessary results obtained (or implicitly obtained) in the course of the above proof as the following lemma.

**Lemma 4.6.** On the same notation as Lemma 4.5, the commutant of \( \pi_{\omega}(\mathcal{A}(I)_+) \) and that of \( \pi_{\omega}(\mathcal{A}(I)_-) \) in \( \mathcal{B}(\mathcal{H}_{\omega}) \) are given by
Lemma 4.7. If \( \omega \) is a pure state of \( \mathcal{A}(I) \) and \( \pi_{\omega} \) and \( \pi_{\omega \Theta} \) are unitarily equivalent, then there exists a self-adjoint unitary \( u \in \pi_{\omega}(\mathcal{A}(I)_\omega)' \) satisfying
\[
\pi_{\omega}(A) = u \pi_{\omega}(\Theta(A))u^* \quad A \in \mathcal{A}(I).
\]
The following is given in [1] as Lemma 3.1.

Lemma 4.8. The state \( \psi \) in Theorem 4.1 satisfies
\[
\psi(A_1A_2A_3) = \psi(A_1)\psi(A_2)\psi(A_3)
\]
for all \( A_i \in \mathcal{A}(K_i) \), \( A_j \in \mathcal{A}(K_j) \) and \( A_k \in \mathcal{A}(K_k) \), where \( \{i, j, k\} \) is any order of \( \{1, 2, 3\} \).

Proof. By Proposition 4.2 (1), the assumed product property (4.2) between \( J \) and \( F \) implies that at least \( \varphi_1 \) or \( \varphi_{2,3} \) \((= \varphi_1)\) should be even. Similarly it implies that at least \( \varphi_{1,2} \) \((= \varphi_2)\) or \( \varphi_3 \) should be even. Therefore all \( \varphi_i \) with one possible exception are even.

It is enough to check (4.12) for \( A_i \in \mathcal{A}(K_i) \), \( A_j \in \mathcal{A}(K_j) \) and \( A_k \in \mathcal{A}(K_k) \) for all choices \( \sigma_i = \pm \), \( \sigma_j = \pm \) and \( \sigma_k = \pm \). By using (4.1) and then (4.2), we obtain
\[
\varphi(A_1A_2A_3) = \varphi(A_1A_2\varphi(A_3) = \varphi(A_1)\varphi(A_2)\varphi(A_3).
\]
If more than one of \( A_i, A_j, A_k \) are odd, then this (4.13) with the relation (2.10) in Lemma 2.1 implies that \( \varphi(A_1A_2A_3) = 0 \). Hence (4.12) is satisfied for this case. If one of \( A_i, A_j, A_k \) is odd, or all \( A_i, A_j, A_k \) are even, then \( A_1A_2A_3 = A_1A_2A_3 \). For these cases, (4.12) is satisfied by (4.13). Thus by linear combination of odd and even elements, we obtain (4.12). □

Proof of Theorem 4.1. We treat the following conditions individually which cover all the cases.

(a) \( \varphi_\Theta \) and \( \varphi_{\Theta(i)} \) are not disjoint for all \( i \in \{1, 2, 3\} \).

(b-i) \( \varphi_1 \) is not even.

(b-ii) \( \varphi_{\Theta(i)} \) and \( \varphi_{\Theta(i)} \) are disjoint.

(b-iii) \( \varphi_3 \) is not even.

Note that if \( \varphi \) is even, then all \( \varphi_i \) are even by Proposition 4.2 (1), and this is a special case of (\( a \)).

(a): First we consider (a). The assumed purity of \( \varphi_{1,2} \) \((= \varphi_1)\) and the product property between \( K_1 \) and \( K_2 \) imply that both \( \varphi_1 \) and \( \varphi_2 \) are pure states by Proposition 4.3 (2). Similarly, the purity of \( \varphi_{2,3} \) \((= \varphi_1)\) and the product property between \( K_2 \) and \( K_3 \) imply that both \( \varphi_2 \) and \( \varphi_3 \) are pure states. Since \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) are all pure states and \( \varphi \) satisfies the product property (4.12), \( \varphi \) is a pure state by Proposition 4.3 (2). We have now shown the assertion for (a).

(b-i) and (b-ii): Since \( \varphi_1 \) is non-even by the assumption and \( \varphi \) satisfies the product property (4.12), both \( \varphi_2 \) and \( \varphi_3 \) are even states by Proposition 4.2 (1). Hence by the purity of \( \varphi_{2,3} \) \((= \varphi_1)\) and the product property between \( K_2 \) and \( K_3 \), Proposition 4.3 (2) implies that both \( \varphi_2 \) and \( \varphi_3 \) are pure.

Because of the non-evenness of \( \varphi_1 \) and the product property of the pure state \( \varphi_{1,2} \) between \( K_1 \) and \( K_2 \), \( \varphi_1 \) should be pure by Proposition 4.3 (1).

By Proposition 4.2 (2), \( \varphi \) is pure because it is a product of pure states \( \varphi_1, \varphi_2 \) and \( \varphi_3 \). We have shown the assertion for (b-i). The proof for (b-ii) goes in parallel as (b-i).

(b-ii): We consider the remaining case (b-ii). The product property (4.12) of \( \varphi \) and the assumed non-evenness of \( \varphi_2 \) imply that \( \varphi_1, \varphi_3 \) and \( \varphi_{1,3} \) are all pure by Proposition 4.2 (1).

By the product property of \( \varphi_{1,2} \) between \( \mathcal{A}(K_1) \) and \( \mathcal{A}(K_2) \), its assumed purity on \( \mathcal{A}(K_{1,2}) \) and the condition (b-ii), Proposition 4.4 (1) implies that both \( \varphi_2 \) and \( \varphi_{1,3} \) \((= \varphi_{1,3}(\mathcal{A}(K_{1,2})))\) are pure. Similarly, we have that \( \varphi_2 \) and \( \varphi_{1,3} \) \((= \varphi_{1,3}(\mathcal{A}(K_{1,2})))\) are pure. Here we have shown that \( \varphi_2 \) is pure. (We have now completed the proof of Theorem 4.1-(1) for all the cases.)

Purity, Non-Purity of \( \varphi_{1,3,4} \). In what follows, we are going to establish a criterion whether \( \varphi_{1,3,4} \) \((= \varphi_{1,3,4}(\mathcal{A}(K_{1,3,4})))\) is a
pure state of $\mathcal{A}(K_1 \cup K_3)_+$ or not.

First we give a concrete construction of the GNS representation of $\varphi_{1,3}$. Since $\varphi_1$ and $\varphi_3$ are even, their GNS representations can be given in terms of $\varphi_{1,+}$ and $\varphi_{3,+}$ as shown in Lemma 4.5. Namely for $i = 1$ and $i = 3$, let

\[
\mathcal{H}_i = \mathcal{K}_{\varphi_i}, \quad \mathcal{H}_{\varphi_i},
\]

(4.14)

and

\[
\pi_i(A_{i,+} + B_{i,+}v_i) = \begin{pmatrix}
\pi_{\varphi_i}(A_{i,+}) & \pi_{\varphi_i}(B_{i,+}) \\
\pi_{\varphi_i}(v_iB_{i,+}v_i) & \pi_{\varphi_i}(v_iA_{i,+}v_i)
\end{pmatrix}
\]

(4.16)

for $A_{i,+}, B_{i,+} \in \mathcal{A}(K_1)_+$, where $(\mathcal{K}_{\varphi_i}, \pi_{\varphi_i}, \Omega_{\varphi_i})$ denotes the GNS triplet for the state $\varphi_{i,+}$ of $\mathcal{A}(I)_+$ and $v_i$ is a self-adjoint unitary in $\mathcal{A}(I)_+$. This $(\mathcal{H}_i, \pi_i, \Omega_i)$ gives the GNS triplet for the even state $\varphi_i$ on $\mathcal{A}(I)_+$.

We define the following operator on $\mathcal{H}_i$

\[
u_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(4.17)

where $I$ is the identity operator on $\mathcal{H}_{\varphi_i}$. We easily see

\[
u_i \pi_i(A_i) \nu_i^* = \pi_i(\Theta(A_i)) \quad \text{for} \quad A_i \in \mathcal{A}(I)_+,
\]

(4.18)

\[u_i \Omega_i = \Omega_i,
\]

(4.19)

\[u_i^* = 1, \quad u_i = u_i
\]

(4.20)

where $I_i$ denotes the identity operator on $\mathcal{H}_i$. Define

\[
\mathcal{H}_{1,3} = \mathcal{H}_1 \otimes \mathcal{H}_3,
\]

\[
\Omega_{1,3} = \Omega_1 \otimes \Omega_3,
\]

\[
U_{1,3} = u_1 \otimes u_3.
\]

There exists a unique representation $\Pi_{1,3}$ of $\mathcal{A}(K_1 \cup K_3)$ in $\mathcal{H}_{1,3}$ satisfying

\[
\Pi_{1,3}(A_1) = \pi_1(A_1) \otimes 1_3, \quad A_1 \in \mathcal{A}(K_1),
\]

(4.21)

and

\[
\Pi_{1,3}(A_{3,+}) = 1_1 \otimes \pi_3(A_{3,+}), \quad A_{3,+} \in \mathcal{A}(K_3)_+,
\]

(4.22)

\[
\Pi_{1,3}(A_{3,-}) = u_1 \otimes \pi_3(A_{3,-}), \quad A_{3,-} \in \mathcal{A}(K_3)_-.
\]

(4.23)

because $\Pi_{1,3}(a_1), \Pi_{1,3}(a_2^+), \Pi_{1,3}(a_2^-)$ and $\Pi_{1,3}(a_2^0)$ for $i \in K_1, k \in K_3$ satisfy the CAR due to (4.18) for $i = 1$. For $A_1 \in \mathcal{A}(K_1)$ and $A_3 \in \mathcal{A}(K_3)$, we have

\[
\Pi_{1,3}(A_1A_3) = \Pi_{1,3}(A_1A_{3,+} + A_{3,-})
\]

\[
= \Pi_{1,3}(A_1) \Pi_{1,3}(A_{3,+}) + \Pi_{1,3}(A_1) \Pi_{1,3}(A_{3,-})
\]

\[
= \pi_1(A_1) \otimes \pi_3(A_{3,+}) + \pi_1(A_1) \otimes \pi_3(A_{3,-})
\]

(4.24)

where $A_3 = A_{3,+} + A_{3,-}, A_{3,k} \in \mathcal{A}(K_3)_k$. By (4.24) and $\Omega_{1,3} = \Omega_1 \otimes \Omega_3$, we have

\[
\Pi_{1,3}(A_1A_3) \Omega_{1,3} = \left[\pi_1(A_1) \otimes \pi_3(A_{3,+})\right] \Omega_1 \otimes \Omega_3 + \left[\pi_1(A_1) \otimes \pi_3(A_{3,-})\right] \Omega_3
\]

\[
= \pi_1(A_1) \Omega_1 \otimes \pi_3(A_{3,+}) \Omega_3 + \pi_1(A_1) \otimes \pi_3(A_{3,-}) \Omega_3
\]

\[
= \pi_1(A_1) \Omega_1 \otimes \pi_3(A_{3,+}) \Omega_3 + \pi_1(A_1) \Omega_1 \otimes \pi_3(A_{3,-}) \Omega_3
\]

\[
= \pi_1(A_1) \Omega_1 \otimes \pi_3(A_{3,+}) \Omega_3
\]

\[
= \pi_1(A_1) \Omega_1 \otimes \pi_3(A_{3,+}) \Omega_3
\]

(4.25)

where (4.15), (4.16) and (4.19) are used. From (4.25) it follows that

\[
(\Omega_{1,3}, \Pi_{1,3}(A_1A_3) \Omega_{1,3}) = (\Omega_1 \otimes \Omega_3, \varphi_1(A_1) \Omega_1 \otimes \pi_3(A_3) \Omega_3)
\]

\[
= (\Omega_1, \varphi_1(A_1) \Omega_1) \cdot (\Omega_3, \pi_3(A_3) \Omega_3)
\]

\[
= \varphi_1(A_1) \varphi_3(A_3)
\]

\[
= \varphi(A_1A_3)
\]

(4.26)

since $\varphi_{1,3}$ is a product state. Therefore, for any $A \in \mathcal{A}(K_{1,3})$, we obtain

\[
(\Omega_{1,3}, \Pi_{1,3}(A) \Omega_{1,3}) = \varphi_{1,3}(A).
\]

(4.27)

By (4.25), $\Omega_{1,3}(\in \mathcal{H}_{1,3})$ is cyclic for $\Pi_{1,3}(\mathcal{A}(K_{1,3}))$, because $\Omega_1(\in \mathcal{H}_1)$ is cyclic for $\pi_1(\mathcal{A}(K_1))$ and so is $\Omega_3(\in \mathcal{H}_3)$ for
Therefore we have shown that \( (H_{1,3}, \Pi_{1,3}, \Omega_{1,3}) \) gives a GNS triplet for the state \( \varphi_{1,3} \) on \( \mathcal{A}(K_{1,3}) \).

We can see that the (self-adjoint) unitary \( U_{1,3} \) implements the automorphism \( \Theta|_{\mathcal{A}(K_{1,3})} \) on this GNS space:

\[
U_{1,3}^* \Pi_{1,3}(A_1) U_{1,3} = u_1^* \otimes u_3^*(\pi_1(A_1) \otimes 1_1) u_1 \otimes u_3
= u_1^* \pi_1(A_1) u_1 \otimes 1_1
= \Theta(A_1) \otimes 1_3
= \Pi_{1,3}(\Theta(A_1)),
\]

\[
U_{1,3}^* \Pi_{1,3}(A_{3\pm}) U_{1,3} = u_1^* \otimes u_3^*(1_1 \otimes \pi_3(A_{3\pm})) u_1 \otimes u_3
= 1_1 \otimes u_3^* \pi_3(A_{3\pm}) u_3
= 1_1 \otimes \pi_3(\Theta(A_{3\pm}))
= \Pi_{1,3}(\Theta(A_{3\pm})),
\]

and by \( \Theta(A_{3-}) = -\Theta(A_{3-}) \in \mathcal{A}(K_3) \),

\[
U_{1,3}^* \Pi_{1,3}(A_{3-}) U_{1,3} = u_1^* \otimes u_3^* (1_1 \otimes \pi_3(A_{3-})) u_1 \otimes u_3
= u_1^* u_1 u_1 \otimes u_3^* \pi_3(A_{3-}) u_3
= u_1 \otimes \pi_3(\Theta(A_{3-}))
= \Pi_{1,3}(\Theta(A_{3-})).
\]

The purity of \( \varphi_{1,3+} \) is equivalent to the non-existence of non-trivial decompositions of \( \varphi_{1,3} \) among the even states of \( \mathcal{A}(K_1 \cup K_3) \) by Lemma 4.5. Therefore it is enough to know \( \left[ \Pi_{1,3}(\mathcal{A}(K_{1,3})) \right]' \cap U_{1,3} \) to determine whether \( \varphi_{1,3+} \) is pure or not on \( \mathcal{A}(K_1 \cup K_3) \), because \( U_{1,3} \) implements the automorphism \( \Theta|_{\mathcal{A}(K_1 \cup K_3)} \) with respect to \( \varphi_{1,3} \).

Take an element \( p' \in \left\{ \Pi_{1,3}(\mathcal{A}(K_{1,3})) \right\}' \). Noting that

\[
\left\{ \Pi_{1,3}(\mathcal{A}(K_1) \cup \mathcal{A}(K_3)) \right\}' \supset \left\{ \Pi_{1,3}(\mathcal{A}(K_{1,3})) \right\}'
\]

we have

\[
p' \in \left\{ \Pi_{1,3}(\mathcal{A}(K_1) \cup \mathcal{A}(K_3)) \right\}' = (\pi_1(\mathcal{A}(K_1)) \otimes \pi_3(\mathcal{A}(K_3)))'
= (\pi_1(\mathcal{A}(K_1))' \otimes \pi_3(\mathcal{A}(K_3))').
\]

Hence Lemma 4.6 implies that it has the following form

\[
p' = k1_1 \otimes s + l\tilde{c}_1 \otimes t,
\]

where \( k, l \in \mathbb{C} \) and

\[
\tilde{c}_1 = \begin{pmatrix}
0 & c_1 \\
c_1 & 0
\end{pmatrix} \in \mathcal{B}(H_1),
\]

\[
c_1 \in \mathcal{B}(\mathcal{H}_{\varphi_{1,3}}),
\]

such that

\[
\pi_{\varphi_{1,3}}(v_1 A_{1+} v_1) c_1 = c_1 \pi_{\varphi_{1,3}}(A_{1+})
\]

for every \( A_{1+} \in \mathcal{A}(K_{1+}) \) and

\[
s = \begin{pmatrix}
e_1 & m \cdot c_3 \\
n \cdot c_3 & f_1
\end{pmatrix},
\]

\[
t = \begin{pmatrix}
e'_{1,\varphi_{1,3}} & m' \cdot c_3 \\
n' \cdot c_3 & f'_{1,\varphi_{1,3}}
\end{pmatrix} \in \mathcal{B}(\mathcal{H}_1),
\]

\[
c_3 \in \mathcal{B}(\mathcal{H}_{\varphi_{1,3}}),
\]

satisfying

\[
\pi_{\varphi_{1,3}}(v_3 A_{3+} v_3) c_3 = c_3 \pi_{\varphi_{1,3}}(A_{3+})
\]

for every \( A_{3+} \in \mathcal{A}(K_{3+}) \).

Write \( p' \) of (4.28) in the matrix form

\[
p' = k1_1 \otimes s + l\tilde{c}_1 \otimes t
\]

\[
= \begin{pmatrix}
k1_{\varphi_{1,3}} & 0 \\
0 & k1_{\varphi_{1,3}}
\end{pmatrix} \otimes \begin{pmatrix}
e_1 & m \cdot c_3 \\
n \cdot c_3 & f_1
\end{pmatrix} + \begin{pmatrix}
0 & lc_1 \\
lc_1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
e'_{1,\varphi_{1,3}} & m' \cdot c_3 \\
n' \cdot c_3 & f'_{1,\varphi_{1,3}}
\end{pmatrix}.
\]

Suppose that this \( p' \) commutes with \( U_{1,3} \), we compute

\[
\pi_3(\mathcal{A}(K_3)).
Therefore we obtain

\[ kmc_3 = knc_3 = lc_1 e' = lc_1 f' = 0. \]  

Substituting this (4.35) into (4.33), we specify the form of \( p' \) as

\[
p' = \left( \begin{array}{cc} k_{1_{\phi_1}} & 0 \\ 0 & k_{1_{\phi_1}}'^{'} \end{array} \right) \otimes \left( \begin{array}{cc} e_{1_{\phi_1}} & 0 \\ 0 & f_{1_{\phi_1}} \end{array} \right) + \left( \begin{array}{cc} 0 \\ lc_1 \end{array} \right) \otimes \left( \begin{array}{cc} 0 \\ n' \cdot c_3 \end{array} \right).
\]

We then check the condition that \( p' \) commutes with \( \Pi_{1,3}(A(K_3)_-) \), which has not yet been used so far. For \( B_{3-} = A_{3+} v_3 \in A(K_3)_- \) such that \( A_{3+} \in A(K_3)_+ \),

\[
\Pi_{1,3}(B_{3-}) = \left( \begin{array}{cc} 1_{\phi_1} & 0 \\ 0 & -1_{\phi_1} \end{array} \right) \otimes \left( \begin{array}{cc} 0 & \pi_{\phi_1}(A_{3+}) \\ \pi_{\phi_1}(v_3 A_{3+} v_3) & 0 \end{array} \right).
\]

Suppose the commutativity of \( p' \) in the form (4.36) with this \( \Pi_{1,3}(B_{3-}) \), we compute

\[
[U_{1,3}, p']
\]

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\[
\begin{align*}
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= 0.
\end{align*}
\]

Therefore we obtain

\[ 3/k_{c_0} = 3/k_{c_0} = 3/l_{c_1} e' = 3/l_{c_1} f' = 0. \]  

Substituting this (4.35) into (4.33), we specify the form of \( p' \) as

\[
p' = \left( \begin{array}{cc} k_{1_{\phi_1}} & 0 \\ 0 & k_{1_{\phi_1}}'^{'} \end{array} \right) \otimes \left( \begin{array}{cc} e_{1_{\phi_1}} & 0 \\ 0 & f_{1_{\phi_1}} \end{array} \right) + \left( \begin{array}{cc} 0 \\ lc_1 \end{array} \right) \otimes \left( \begin{array}{cc} 0 \\ n' \cdot c_3 \end{array} \right).
\]

We then check the condition that \( p' \) commutes with \( \Pi_{1,3}(A(K_3)_-) \), which has not yet been used so far. For \( B_{3-} = A_{3+} v_3 \in A(K_3)_- \) such that \( A_{3+} \in A(K_3)_+ \),

\[
\Pi_{1,3}(B_{3-}) = \left( \begin{array}{cc} 1_{\phi_1} & 0 \\ 0 & -1_{\phi_1} \end{array} \right) \otimes \left( \begin{array}{cc} 0 & \pi_{\phi_1}(A_{3+}) \\ \pi_{\phi_1}(v_3 A_{3+} v_3) & 0 \end{array} \right).
\]

Suppose the commutativity of \( p' \) in the form (4.36) with this \( \Pi_{1,3}(B_{3-}) \), we compute

\[
[U_{1,3}, p']
\]

39

\[
\begin{align*}
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= 0.
\end{align*}
\]

Therefore we obtain

\[ kmc_3 = knc_3 = lc_1 e' = lc_1 f' = 0. \]  

Substituting this (4.35) into (4.33), we specify the form of \( p' \) as

\[
p' = \left( \begin{array}{cc} k_{1_{\phi_1}} & 0 \\ 0 & k_{1_{\phi_1}}'^{'} \end{array} \right) \otimes \left( \begin{array}{cc} e_{1_{\phi_1}} & 0 \\ 0 & f_{1_{\phi_1}} \end{array} \right) + \left( \begin{array}{cc} 0 \\ lc_1 \end{array} \right) \otimes \left( \begin{array}{cc} 0 \\ n' \cdot c_3 \end{array} \right).
\]

We then check the condition that \( p' \) commutes with \( \Pi_{1,3}(A(K_3)_-) \), which has not yet been used so far. For \( B_{3-} = A_{3+} v_3 \in A(K_3)_- \) such that \( A_{3+} \in A(K_3)_+ \),

\[
\Pi_{1,3}(B_{3-}) = \left( \begin{array}{cc} 1_{\phi_1} & 0 \\ 0 & -1_{\phi_1} \end{array} \right) \otimes \left( \begin{array}{cc} 0 & \pi_{\phi_1}(A_{3+}) \\ \pi_{\phi_1}(v_3 A_{3+} v_3) & 0 \end{array} \right).
\]

Suppose the commutativity of \( p' \) in the form (4.36) with this \( \Pi_{1,3}(B_{3-}) \), we compute

\[
[U_{1,3}, p']
\]

39

\[
\begin{align*}
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= \left( k_{1_{\phi_1}} e_{1_{\phi_1}} + m \cdot c_3 \right) \otimes e_{1_{\phi_1}} + (k_{1_{\phi_1}} e_{1_{\phi_1}} - m \cdot c_3) \\
&= 0.
\end{align*}
\]

where we have used (4.32) to derive the second last equality.

Assume that this (4.38) holds for all \( B_{3-} (= A_{3+} v_3) \in A(K_3)_- \). Then we obtain the following relation:

\[ k(e - f) = lc_1 (m' + n') c_3 = 0, \]  

since \( \pi_{\phi_1}(A(K_3)_+) \) commutes with \( \mathcal{A}(\mathfrak{p}_{\phi_1}) \),

Substituting this (4.39) into (4.36), we obtain the general form of \( p' \)

\[
p' = g_1 1_1 \otimes 1_3 + g_2 0 \left( \begin{array}{cc} c_1 \\ c_1 \end{array} \right) \otimes \left( \begin{array}{cc} 0 \\ -c_3 \end{array} \right).
\]

By using (4.30) and (4.32), we can verify that \( p' \) in the above form (4.40) commutes with
Theorem 4.1 (3) claims. Presently, we cannot make such a state concretely, because we have no example of mixed state of some results of [1]. Therefore at least for the case (ii), namely, we actually do not need this additional input.

Hence \( \Pi_{1,3}(A(K_{1,3})) \cap U_{1,3}^\prime \) is two-dimensional if there exist non-zero \( c_1 \) satisfying (4.30) and \( c_3 \) satisfying (4.32). Otherwise, it is a multiple of identity. Since \( \varphi \) is a product state of \( \varphi_{1,3} \) and the pure state \( \varphi_{2} \) satisfying the condition (\( \beta \)-ii), namely \( \pi_{\varphi_{2}} \) and \( \pi_{\varphi_{\theta}} \) are disjoint, \( \varphi \) is pure if and only if \( \varphi_{1,3,2} \) is pure by Proposition 4.4 (1). Therefore combining those results, we obtain Theorem 4.1 (2) and Theorem 4.1 (3).

5. Discussion

(I) The readers will be wondering whether there really exist the states which do not satisfy the latter part of (A) as Theorem 4.1 (3) claims. Presently, we cannot make such a state concretely, because we have no example of mixed states \( \varphi_{1} \) in Proposition 4.4 (2) (Theorem 3 (2) in [1]) on which our somehow abstract discussion is based on.

We, however, note that all our assumptions can be compatible; they do not cause any inconsistency for the existence of the desired state \( \varphi \). In fact, if we are given an example of the non-pure case of Proposition 4.4 (2), namely, if we are given two states \( \varphi_{1} \) on \( A(K_{1}) \) and \( \varphi_{3} \) on \( A(K_{3}) \) such that each \( \varphi_{i} \) \( (i = 1, 3) \) is a pure state on the even part \( A(K_{e}) \) and is a mixed state on \( A(K_{o}) \) whose any non-trivial decomposition is given by \( g \) \( \varphi_{i} = \frac{1}{2} (\varphi_{i} + \varphi_{i}) \), where \( \varphi_{i} \) is a pure state of \( A(I_{i}) \) and the GNS representations \( \pi_{\varphi_{i}} \) and \( \pi_{\varphi_{i}} \) are disjoint, we can produce our desired state \( \varphi \) as a product-state extension of those \( \varphi_{1}, \varphi_{3} \), and some \( \varphi_{2} \) on \( A(K_{2}) \) satisfying (\( \beta \)-ii).

(II) We have required additionally the product property of \( \varphi \) for infinite-dimensional CAR systems to show our main result, Theorem 4.1. We announce that the above product property can be derived from the condition (\( \beta \)-ii) by using some results of [1]. Therefore at least for the case (\( \beta \)-ii), we actually do not need this additional input.

(III) Finally, we may be allowed to mention some possible study beyond this work. As we have seen, it is almost trivial to show (A) for tensor-product systems. However, in view of the original motivation of [4], we should not be content with those simplest independent systems. To study (A) in the context of quantum field theory seems to be challenging, where the three-composed systems may be formulated in terms of some notions of independence (or their modifications) developed in local quantum physics.

REFERENCES