A Note on Twisted Linear Actions on a Sphere

Tohl ASOH

Graduate School of Information Science, Tōhoku University, Katahira, Sendai, 980-77, Japan

Received May 13, 1996; final version accepted June 17, 1996

Let $G$ be a Lie group. A TC-pair $(\rho, M)$ of $G$ gives a twisted linear action on a sphere. Assume that $G/\ker \rho$ is non-compact. Then the TC-pairs are equivalent, if the corresponding twisted linear actions are coincides with each other. Hence also so, if the corresponding twisted linear actions are transitive and equivariantly diffeomorphic.

KEYWORDS: Non-compact Lie group action, Sphere, Twisted linear action, Transitive action

1. Introduction

A series of non-compact Lie group actions on a sphere is constructed and studied by F. Uchida [6-8]. Let $G$ be a Lie group. We say that $(\rho, M)$ is a TC-pair of degree $n + 1$ if $\rho: G \rightarrow GL(n + 1, R)$ is a representation and $M$ is a real square matrix of degree $n + 1$ satisfying

\begin{equation}
\forall xMx > 0 \text{ and } \rho(g)M = M\rho(g) \text{ for any } x \in R^{n+1} - \{0\}, \ g \in G,
\end{equation}

i.e., the transversality and the commutative conditions. Then we have a smooth $G$-action $\phi: G \times S^a \rightarrow S^a$.

\begin{equation}
\phi(g, x) = (\exp t(g, x)M)\rho(g)x \in S^a \text{ for } (g, x) \in G \times S^a,
\end{equation}

where $t: G \times S^a \rightarrow R$ is a smooth function. This action $\phi$ is called the twisted linear action of $G$ on $S^a$ determined by a TC-pair $(\rho, M)$ (cf. [6; §1.2]).

We say that TC-pairs $(\rho_i, M_i)$ ($i = 1, 2$) are equivalent if there exist $A \in GL(n + 1, R)$ and $c > 0$ such that $\rho_i(g) = A^{-1}\rho_2(g)A(g \in G)$ and $cM_i = A^{-1}M_2A$.

Thus the following holds (cf. [6; Lemma 1.2]).

\begin{equation}
\text{If TC-pairs } (\rho_i, M_i) \ (i = 1, 2) \text{ are equivalent, then the corresponding twisted linear actions } \phi_1 \text{ and } \phi_2 \text{ are equivariantly diffeomorphic to each other.}
\end{equation}

We consider the question whether the converse of this fact is true. In this note we state a partial affirmative answer to this question.

Assume that $G$ is a connected semi-simple Lie group. Then we have the following.

THEOREM 1.4. Let $\phi_i$ be the twisted linear action determined by a TC-pair $(\rho_i, M_i)$ of degree $n + 1 \geq 3$ ($i = 1, 2$). Assume that $G/\ker \rho_i$ is non-compact. Then $\phi_1 = \phi_2$ implies $(\rho_1, M_1) = (\rho_2, cM_2)$ for some $c > 0$.

As a corollary of this theorem, we obtain the following.

COROLLARY 1.5. Suppose that the twisted linear action $\phi_i$ of $(\rho_i, M_i)$ is transitive on $S^a$ ($n \geq 2$), and that $G/\ker \rho_i$ is non-compact ($i = 1, 2$). Then $(\rho_1, M_1)$ and $(\rho_2, M_2)$ are equivalent if $\phi_1$ is equivariantly diffeomorphic to $\phi_2$.

In §2 we state some results on twisted linear actions on a sphere. After preparing some lemmas in §3, we show Theorem 1.4 in §4 and Corollary 1.5 in §5.

2. Twisted linear actions

Let $G$ be a Lie group, and $\mathfrak{g}$ be the Lie algebra of $G$. Denote by $\mathfrak{x} (S^n)$ the Lie algebra of smooth vector fields on $S^n$. For a smooth $G$-action $\phi$ on $S^n$, the following is known (cf. [5; Ch. II, Theorem III]).

\begin{equation}
\text{The map } \phi^*: \mathfrak{g} \rightarrow \mathfrak{x} (S^n), \text{ given by }
\phi^*(X), h = \lim_{r \to 0} \{ h(\phi(\exp (-rX), x)) - h(x) \}/r \ (X \in \mathfrak{g})
\end{equation}

for any smooth function $h$ around $x \in S^n$, is a Lie algebra homomorphism.
In the rest of this section, assume that there is given a TC-pair \((\rho, M)\) of degree \(n + 1\), and let \(\phi\) be the twisted linear action determined by \((\rho, M)\).

The differential of \(\rho\) induces the representation \(d\rho: g \to g(n + 1, R)\). For each \(X \in g\) we set

\[
\gamma_X(x) = 'xd\rho(X)x \quad \text{and} \quad \eta(x) = 'xMx > 0 \quad (x \in R^{n+1} - \{0\}),
\]

where `'x` denotes the transposed of \(x\).

**Lemma 2.2.** For each \(X \in g\)

\[
\phi^+(X)_x = -d\rho(X)x + (\gamma_X(x)/\eta(x))Mx \in T_xS^* \subset R^{n+1} \quad (x \in S^n).
\]

**Proof:** For \(X \in g\) and \(x \in S^n\) we have \(\phi(\exp rX, x) = (\exp t_x(x)M)(\exp rX)x \in S^n(r \in R)\) by (1.2), where \(t_x(x) = t(\exp rX, x)\). Hence by (2.1)

\[
-\phi^+(X)_x = d\rho(X)x + (dt_x(x)/dr)(0)Mx \in T_xS^*,
\]

since \(d\rho(X) = (d\rho(\exp rX)/dr)(0)\). Therefore (2.3) follows immediately from `'\(\phi^+(X)_x = 0\). ■

Denote by \(\langle , \rangle\) the standard metric on \(R^{n+1}\). For a vector field \(X = \sum_{i=1}^{n+1} h_i(\partial/\partial x_i)\) on \(S^n(\subset R^{n+1})\), the divergence \(\text{div} X\) and the Riemannian connection \(\nabla_vX(v \in T_xS^n)\) of \(X\) are given by the following formulas.

\[
\text{div} X = \sum_{i=1}^{n+1} \partial h_i/\partial x_i \quad \text{and} \quad \nabla_vX = \sum_{i=1}^{n+1} v(h_i)(\partial/\partial x_i) + \langle X, v \rangle \sum_{i=1}^{n+1} x_i(\partial/\partial x_i).
\]

For each \(X \in g\) and \(x \in R^{n+1} - \{0\}, y \in R^{n+1},\) we put

\[
\alpha_X(x, y) = (1/\eta(x))\gamma(y(d\rho(X) + 'd\rho(X) - (\gamma_X(x)/\eta(x))(M + 'M))x.
\]

Obviously \(\alpha_X(x, x) = 0\) and \(\alpha_X(x, y)\) is linear on the second factor \(y\). Furthermore we have

\[
v(\gamma_X(\eta) = \alpha_X(x, v) \quad \text{for} \quad v \in T_xR^{n+1} = R^{n+1}.
\]

**Lemma 2.6.** For each \(X \in g\) and \(x \in S^n, v \in T_xS^n\)

\[
(\text{div } \phi^+(X))(x) = \langle \gamma_X(x)/\eta(x), X \rangle \quad \text{and} \quad \text{Trace } M = \text{Trace } d\rho(X) - (\phi^+(X)_x, \log \eta),
\]

\[
(\nabla_v\phi^+(X))(x) = -d\rho(X)v + (\gamma_X(x)/\eta(x))Mv + \alpha_X(x, v)Mx + \langle v, \phi^+(X), x \rangle x.
\]

**Proof:** By routine calculations, the lemma follows from Lemma 2.2 and (2.4–5). ■

**Lemma 2.9.** \(\phi^+(X) = 0\) if and only if \(d\rho(X) = cM\) for some \(c \in R\).

**Proof:** The sufficiency is clear. Suppose \(\phi^+(X) = 0\). By (2.7) we get \(\gamma_X(x)/\eta(x) = \text{Trace } d\rho(X)/\text{Trace } M = c,\) and the lemma follows immediately from (2.3). ■

3. Preliminaries

Let \((\rho, M)\) be a TC-pair of degree \(n + 1 \geq 3\). In this section, we assume that \(g\) is a non-compact semi-simple Lie algebra, and that \(d\rho: g \to \mathfrak{sl}(n + 1, R)\) is a faithful representation, i.e., \(\ker d\rho = \{0\}\).

We have a direct sum decomposition (a Cartan decomposition)

\[
g = \mathfrak{t} \oplus \mathfrak{p},
\]

where \(\mathfrak{t}\) is a maximal compact subalgebra and \(\mathfrak{p} \neq \{0\}\) is a vector subspace. The following is known (cf. [3; Ch. IX, Theorem 7.4]).

(3.1) \(g\) contains a subalgebra isomorphic to \(\mathfrak{sl}(2, R)\), and hence \(\dim \mathfrak{p} \geq 2\).

Since \((M + 'M)/2\) is symmetric and positive definite, we find a square matrix \(L\) satisfying \((M + 'M)/2 = LL^\top\). Take \(X \in g\). Let \(a_i \in R\) and \(L_{vi} (i = 1, 2, \ldots, n + 1)\) be the eigen-value and the eigen-vector of \(L^{-1}(d\rho(X) + 'd\rho(X))L^{-1}/2\), respectively, such that \(\langle L_{vi}, L_{vj} \rangle = \delta_{ij}\). Then

\[
(d\rho(X) + 'd\rho(X))_{vi} = a_i(M + 'M)v_i \quad (1 \leq i \leq n + 1).
\]

Clearly we have \(\eta(v_i) = 1, a_i = \gamma_X(v_i)\) and

\[
\alpha_X(v_i, x) = 0 \quad \text{for any} \quad x \in R^{n+1} - \{0\} \quad (1 \leq i \leq n + 1).
\]

Now we prepare some lemmas and a proposition.
Lemma 3.3. \[ a_1 r_1 + a_2 r_2 + \cdots + a_{n+1} r_{n+1} = 0 \] for \( r_i = \|(M + 'M) v_i/2\|_2 \quad (1 \leq i \leq n+1). \)

Proof: Set \( U = (L v_1 L v_2 \cdots L v_{n+1}) \in O(n + 1). \) Hence

\[ U^{-1} L (dp(X) + 'dp(X)) L^{-1} U / 2 = U^{-1} L' L U D, \]

where \( D = (a_1 a_2 \cdots a_{n+1}) \) is the diagonal matrix. The diagonal elements of \( U^{-1} L' L U \) is given by \( (L v_i) L' L (L v_i) = r_i \) \( (1 \leq i \leq n+1). \)

Therefore the lemma follows from (\( \ast \)), since Trace \( dp(X) = 0. \)

Lemma 3.4. Suppose \( 0 \neq X \in \mathfrak{p}. \) Then for any \( a_i \) there exists \( a_j \) such that \( a_i \neq a_j. \)

Proof: Assume \( a_1 = a_2 = \cdots = a_{n+1}. \) Thus Lemma 3.3 implies \( a_i = 0 \) \( (1 \leq i \leq n + 1), \) and hence \( dp(X) + 'dp(X) = 0. \) Then \( dp(X) = 0, \) because of \( X \in \mathfrak{p}. \) This is contrary to the conditions that \( dp \) is faithful and \( X \neq 0. \)

Lemma 3.5. There exists \( X \in \mathfrak{p} \) such that if \( a_i \neq a_j \) then \( a_k \neq a_j \) for some \( k \neq i. \)

Proof: Assume \( a_1 \neq a_2 = \cdots = a_{n+1} \) for each \( 0 \neq X \in \mathfrak{p}. \) By Lemma 3.3 we have \( a_i \neq 0 \) \( (1 \leq i \leq n + 1). \) Then the eigen-value of \( dp(X) + 'dp(X) \) is non-zero for each \( 0 \neq X \in \mathfrak{p}. \)

Let \( X \) and \( Y \) be linearly independent elements in \( \mathfrak{p} \) by (3.1). From the assumption, the corresponding eigenvalues \( a_i \) and \( b_j \) \( (1 \leq i \leq n + 1) \) for \( X \) and \( Y, \) respectively, satisfy

\[ a_1 \neq a_2 = \cdots = a_{n+1} \quad \text{and} \quad b_1 \neq b_2 = \cdots = b_{n+1}, \]

where \( n \geq 2. \) Thus we can choose \( 0 \neq v \in R^{n+1} \) such that

\[ (dp(X) + 'dp(X)) v = a_0 (M + 'M) v \quad \text{and} \quad (dp(Y) + 'dp(Y)) v = b_0 (M + 'M) v. \]

By setting \( Z = b_2 X - a_2 Y \in \mathfrak{p}, \) we obtain \((dp(Z) + 'dp(Z)) v = 0. \) This leads a contradiction.

The proof of the lemma is completed.

For each \( X \in \mathfrak{g} \) and \( x, y \in R^{n+1} - \{0\}, \) we set

\[ \beta_X(x, y) = \gamma_x(x) / \eta(x) - \gamma_x(y) / \eta(y). \]

Then \( \beta_x(v_i, v_j) = a_i - a_j \) \( (1 \leq i, j \leq n + 1), \) and we have the following.

Proposition 3.6. There exists \( X \in \mathfrak{p} \) satisfying the following two conditions: (i) For any \( i \) there exists \( j \) such that \( \beta_X(v_i, v_j) \neq 0, \) and (ii) If \( \beta_X(v_i, v_j) \neq 0, \) then \( \beta_X(v_k, v_j) \neq 0 \) for some \( k \neq i \) \( (1 \leq i, j, k \leq n + 1). \)

Proof: The proposition follows immediately from Lemmas 3.4 and 3.5.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4. Assume that \( G \) is a connected non-compact semi-simple Lie group. Then the Lie algebra \( \mathfrak{g} \) of \( G \) is also non-compact and semi-simple.

Let \( (\rho, M_i) \) be a TC-pair of degree \( n + 1 \geq 3 \) with Trace \( M_i = 1 \) \( (i = 1, 2), \) and \( \phi_i \) be the twisted linear action determined by \( (\rho, M_i) \) \( (i = 1, 2). \) We use the notations \( \gamma_X, \eta_i, \alpha_X, \alpha_X^{(i)} \) and \( \beta_X^{(i)} \) for \( X \in \mathfrak{g} \) and \( (\rho, M_i) \) as in the previous sections.

Assume \( \phi_1 = \phi_2. \) Then we have the following lemmas.

Lemma 4.1. For \( X \in \mathfrak{g} \) and \( x, y \in R^{n+1} - \{0\}, \)

\[ \alpha_X^{(i)}(x, y) M_i x + \beta_X^{(i)}(x, y) M_i y = \alpha_X^{(i)}(x, y) M_2 x + \beta_X^{(i)}(x, y) M_2 y. \]

Proof: From Trace \( M_i = 1 \) and \( \phi_1 = \phi_2, \) (2.8) implies

\[ -dp_1(X) v + (\gamma_X^{(i)}(x) / \eta_1(x)) M_i v + \alpha_X^{(i)}(x, v) M_i x \]

\[ = -dp_1(X) v + (\gamma_X^{(i)}(x) / \eta_2(x)) M_2 v + \alpha_X^{(i)}(x, v) M_2 x \]

for \( X \in \mathfrak{g}, x \in S^n \) and \( v \in T_x S^n. \) Since both sides of (\( \ast \)) is linear on \( v \) and \( \alpha_X^{(i)}(x, x) = 0, \) we see that (\( \ast \)) also holds for any \( X \in R^{n+1} - \{0\} \) and \( v \in R^{n+1}. \) Therefore the lemma follows from the definition of \( \beta_X^{(i)}. \)

As in §3, let \( L v_i \) be the eigen-vector of \( 'L^{-1}(dp_1(X) + 'dp_1(X))L^{-1}/2 \) such that \( \langle L v_i, L v_j \rangle = \delta_{ij} \) \( (1 \leq i, j \leq n + 1) \) for \( X \in \mathfrak{g}, \) where \( (M_1 + 'M_1)/2 = LL \). By (3.2) and Lemma 4.1 we get

\[ \beta_X^{(i)}(v_i, v_j) M_i v_j = \alpha_X^{(i)}(v_i, v_j) M_2 v_i + \beta_X^{(i)}(v_i, v_j) M_2 v_i. \]

Consider the following condition for \( X \in \mathfrak{g}. \)

(4.3) (i) For any \( i \) there exists \( j \) such that \( \beta_X^{(i)}(v_i, v_j) \neq 0, \) and (ii) If \( \beta_X^{(i)}(v_i, v_j) \neq 0, \) then \( \beta_X^{(i)}(v_k, v_j) \neq 0 \) for
some $k \neq i$ ($1 \leq i, j, k \leq n + 1$).

**Lemma 4.4.** Let $X \in p$ with (4.3). Then $\alpha_X^{ij}(v_i, v_j) = 0$ for any $1 \leq i, j \leq n + 1$.

**Proof:** (i) If $\beta_X^{ij}(v_i, v_j) = 0$, then (4.2) shows $\alpha_X^{ij}(v_i, v_j) = 0$. (ii) Suppose $\beta_X^{ij}(v_i, v_j) \neq 0$. By (4.3) we have $\beta_X^{ij}(v_k, v_j) \neq 0$ for some $k \neq i$. The equations (4.2) for $(v_i, v_j)$ and $(v_k, v_j)$ imply $\beta_X^{ij}(v_k, v_j)\alpha_X^{ij}(v_i, v_j) = 0$. Therefore $\alpha_X^{ij}(v_i, v_j) = 0$, as desired. 

**Lemma 4.5.** Let $X \in p$ with (4.3). Then

$$\eta_i(v_i)/\eta_2(v_i) = \eta_i(v_j)/\eta_2(v_j) \quad (1 \leq i, j \leq n + 1).$$

**Proof:** By (4.2) and Lemma 4.4 we have

$$\beta_X^{ij}(v_i, v_j)M_{ij} = \beta_X^{ij}(v_i, v_j)M_{ij}.$$ 

Hence $\beta_X^{ij}(v_i, v_j)\eta_i(v_i) = \beta_X^{ij}(v_i, v_j)\eta_2(v_i)$.

(i) If $\beta_X^{ij}(v_i, v_j) \neq 0$, then we have $\eta_i(v_i)/\eta_2(v_i) = \beta_X^{ij}(v_i, v_j)/\beta_X^{ij}(v_i, v_j) = \eta_i(v_i)/\eta_2(v_i)$.

(ii) Suppose $\beta_X^{ij}(v_i, v_j) = 0$. By (4.3) we get $\beta_X^{ij}(v_k, v_j) = \beta_X^{ij}(v_k, v_j) \neq 0$ for some $k$. Clearly $\beta_X^{ij}(v_k, v_j) = \beta_X^{ij}(v_k, v_j)$ follows from $\beta_X^{ij}(v_k, v_j) = 0$. By using (i) we have

$$\eta_i(v_i)/\eta_2(v_i) = \beta_X^{ij}(v_k, v_i)/\beta_X^{ij}(v_k, v_i) = \beta_X^{ij}(v_k, v_i)/\beta_X^{ij}(v_k, v_i) = \eta_i(v_i)/\eta_2(v_i).$$

Then the proof of the lemma is completed. 

**Proof of Theorem 1.4:** (i) By Lemma 2.9, we see that $\phi^+_i(X) = 0$ if and only if $d\rho_i(X) = 0$ for each $X \in \mathfrak{g}$ ($i = 1, 2$). Hence we get $d\rho_1 = d\rho_2$, since $\phi^+_i = \phi^+_2$. Then $g = g' \oplus \ker d\rho_i$ for some semi-simple ideal $g'$ of $g$. To show our theorem we may assume that $d\rho_i$ is faithful ($i = 1, 2$). From Proposition 3.6 we choose $X \in p$ satisfying (4.3). By (4.3) (i) and Lemma 4.5 we have

$$c = \eta_i(v_i)/\eta_2(v_i) = \beta_X^{ij}(v_i, v_j)/\beta_X^{ij}(v_i, v_j) \quad (1 \leq j \leq n + 1)$$

for some $i$. Then $M_{ij} = \beta_X^{ij}(v_j, v_j)$ by (4.6), and hence $M_i = cM_2$. Since $\text{Trace} M_i = 1$, we obtain $M_i = M_2$.

(ii) From (2.7) and $\phi^+_i = \phi^+_2$, we get

$$\gamma_X^{ij}(x)/\eta_i(x) - \gamma_X^{ij}(x)/\eta_2(x) = \phi^+_i(X)(\log (\eta_2/\eta_i)) = 0 \quad (x \in S^n),$$

for each $X \in \mathfrak{g}$. Then $d\rho_1(X) = d\rho_2(X) (X \in \mathfrak{g})$ follows from (2.3) and (i). Thus $\rho_1 = \rho_2$ holds, since $G$ is connected.

Therefore $(\rho_1, M_i) = (\rho_2, M_2)$, and the proof of Theorem 1.4 is completed. 

**5. Transitive actions on a sphere**

Let $K$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{f}$. Then $K$ is a maximal compact subgroup of $G$. First we notice the result due to F. Uchida [6; Theorem 3.3].

(5.1) For each $TC$-pair $(\rho, M)$ of degree $n + 1$, there exists a $TC$-pair $(\rho', M')$ which is equivalent to $(\rho, M)$ and $\rho'(K) \subset O(n + 1)$.

Then, by (1.3), the restricted $K$-action of a twisted linear action is equivariantly diffeomorphic to a linear action.

Let $U$ be a compact connected Lie group which acts effectively and transitively on $S^n$ ($n \geq 1$), and $H$ be its isotropy subgroup. It is known that such $U$-action is equivariantly diffeomorphic to a linear action via a representation $v: U \to O(n + 1)$, and further $v$ is unique up to equivalence in $O(n + 1)$. Denote by $N(H, U)$ and $Z(v(U), O(n + 1))$ the normalizer of $H$ in $U$ and the centralizer of $v(U)$ in $O(n + 1)$, respectively. Then the following is known (cf. [1; Lemmas 2.2-3]).

(5.2) $Z(v(U), O(n + 1)) \cong N(H, U)/H \cong S^m$ ($m = 0, 1, 3$).

Let $\text{Homeo}^0(S^n)$ denote the group of $U$-equivariant homeomorphisms of $S^n$, which is naturally isomorphic to $N(H, U)/H$. Then we see the following lemma.

**Lemma 5.3.** The mapping

$$L: Z(v(U), O(n + 1)) \to \text{Homeo}^0(S^n),$$

given by $L(A)(x) = Ax$ ($x \in S^n$) for $A \in Z(v(U), O(n + 1))$, is isomorphic.

**Proof:** Clearly $L$ is monomorphic, and a monomorphism between $S^m$ ($m = 0, 1, 3$) is isomorphic. Then the lemma follows from (5.2).
Let $\phi_i$ be the twisted linear action determined by a TC-pair $(\rho_i, M_i)$ of degree $n + 1 \geq 3$ ($i = 1, 2$). Assume that $\phi_i$ is transitive and that $\phi_i$ is equivariantly diffeomorphic to $\phi_2$. Then we have the following lemmas.

**Lemma 5.4.** There exists a TC-pair $(\rho'_i, M'_i)$, which is equivalent to $(\rho_i, M_i)$ ($i = 1, 2$), satisfying $\rho'_i | K = \rho'_2 | K: K \to O(n + 1)$.

**Proof:** By (5.1) we may assume $\rho_i(K) \subset O(n + 1)$ ($i = 1, 2$). From the result of D. Montgomery [4], the restricted $K$-action $\phi_i | K \times S^n$ is also transitive on $S^n$, and this is a linear action via $\rho_i | K$. Then $\rho_i | K$ is equivalent to $\rho_2 | K$ in $O(n + 1)$, i.e., $\rho_i(k) = A^{-1} \rho_2(k) A$ ($k \in K$) for some $A \in O(n + 1)$.

Set

$$\rho'_2(g) = A^{-1} \rho_2(g) A \quad (g \in G) \quad \text{and} \quad M'_2 = A^{-1} M_2 A.$$ 

Therefore $(\rho'_2, M'_2)$ is equivalent to $(\rho_2, M_2)$, and $\rho'_i | K = \rho_i | K$.

The proof of the lemma is completed. ■

**Lemma 5.5.** Suppose $\rho_i | K = \rho_2 | K: K \to O(n + 1)$. Then there exists $(\rho'_i, M'_i)$, which is equivalent to $(\rho_i, M_i)$, such that the twisted linear action $\phi'_i$ of $(\rho'_i, M'_i)$ coincides with $\phi_2$.

**Proof:** Let $\Phi: S^n \to S^n$ be an equivariant diffeomorphism from $\phi_i$ to $\phi_2$. Because the $K$-action on $S^n$ via $\rho_i | K$ is transitive, there exists a connected normal subgroup $U$ of $\rho_i(K)$, which is effective and transitive on $S^n$. Since $\Phi \in \text{Homeo}^U(S^n)$, we get $\Phi(x) = Ax$ ($x \in S^n$) for some $A \in Z(U, O(n + 1))$ by Lemma 5.3. Set

$$\rho'_i(g) = A \rho_i(g) A^{-1} \quad (g \in G) \quad \text{and} \quad M'_i = AM_i A^{-1}.$$ 

Thus $(\rho'_i, M'_i)$ is a TC-pair, and let $\phi'_i$ be the twisted linear action determined by $(\rho'_i, M'_i)$. Then, for $g \in G$ and $x \in S^n$,

$$\phi'_2(g, \Phi(x)) = \Phi \phi_2(g, x) = A(\exp t M_i) \rho_i(g) x = (\exp t M'_i) \rho_i(g) \Phi(x) = \phi'_i(g, \Phi(x)).$$

This shows $\phi_2 = \phi'_i$ as desired. ■

**Proof of Corollary 1.5:** The corollary follows from Lemmas 5.4–5 and Theorem 1.4.

Therefore the proof of the corollary is completed. ■

**References**