On the Iterated Martingale Transforms

Litan YAN

Department of Mathematics, Toyama University, Toyama 930-8555, Japan
E-mail address: yan@math.sci.toyama-u.ac.jp

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Let $f = (f_n, \mathcal{F}_n)_{n \geq 0}$ be a martingale on some filtered complete probability space $(\Omega, \mathcal{F}, P)$ with the usual conditions. We define the iterated martingale transforms $I_n^{(m)}(f) = (I_n^{(m)}, (\mathcal{F}_n)) (m \geq 1)$ with respect to $f$, the discrete analogues of the iterated stochastic integrals. We obtain the $L^p (1 \leq p)$-estimates of $I_n^{(m)}(f)$

$$\|I_n^{(m)}f\|_p \leq (4mp)^n \|S^n(f)\|_p$$

and we also characterize a continuous martingale by the limit of the iterated martingale transforms.

KEYWORDS: Martingale Transforms, $L^p$-estimate and the Hermite polynomials

1. Introduction

Let $f = (f_n, \mathcal{F}_n)_{n \geq 0}$ be a martingale defined on some filtered complete probability space $(\Omega, \mathcal{F}, P)$ with the usual conditions. It is well-known that Burkholder’s martingale transform $g$ is defined as

$$g_n = \sum_{k=1}^{n} v_{k-1} d_k, \quad n \geq 1, \quad g_0 = 0,$$

where $v = (v_n)_{n \geq 0}$ is an adapted sequence with $v_0 = 0$, $d_0 = 0$ and $d_k = f_k - f_{k-1}$ ($k = 1, 2, 3, \ldots$) is the difference sequence of $f$. The operator $T_2 : f \rightarrow g$ has become a powerful tool on the study of martingales.

In this note, we consider $I_n^{(m)}(f) = (I_n^{(m)}, (\mathcal{F}_n)) (m \geq 0)$, the particular types of the martingale transforms, defined by inductively

$$I_n^{(m)} = \sum_{j=0}^{n} I_{n-j}^{(m-1)} d_j \quad \text{and} \quad I_0^{(m)} = 0 \quad (m \geq 0)$$

with

$$I_n^{(0)} = 1 \quad \text{and} \quad I_n^{(1)} = f_n \quad \text{for} \; n = 0, 1, 2, \ldots,$$

which are the discrete analogues of the iterated stochastic integrals. Clearly, $I_n^{(m)}(f)$ is a local martingale for every $m$, and we shall call it an iterated martingale transform of degree $m$ (with respect to martingale $f = (f_n, \mathcal{F}_n)$) from now on.

As usual, we set $f_0 = 0$ and

$$f^*_n = \sup_{j \leq n} |f_j|, \quad f^*_n = \sup_{n \geq 0} |f_n|;$$

$$S_n(f) = \left( \sum_{k=0}^{n} d_k^2 \right)^{1/2}, \quad S(f) = S_n(f),$$

where $d = (d_k)$ is the difference sequence of $f$.

2. Two properties for $I_n^{(m)}(f)$

In this section, we give two simple properties for $I_n^{(m)}(f)$.

**Proposition 1.** Let $f = (f_n)_{n \geq 0}$ be a martingale with the difference sequence $d = (d_n)_{n \geq 0}$. Then we have

$$\sum_{m=0}^{n} \frac{\lambda^m}{m!} I_n^{(m)} = \prod_{j=0}^{n} (1 + \lambda d_j) \quad (n \geq 0).$$

**Proof.** By using induction on $n$, we can prove the proposition.

Indeed, clearly, (2) is true if $n = 0$ and $n = 1$. We now suppose that (2) is established for $n - 1$ in place of $n$. Since

Dedicated to Professor Norihiko Kazamaki on his sixtieth birthday.
\[ I^{(m)}_n - I^{(m)}_{n-1} = mI^{(m-1)}_{n-1} d_n, \]  

we have
\[
\prod_{j=0}^{n} (1 + \lambda d_j) = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m)}_{n-1} + \lambda d_n \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m)}_{n-1} \\
= 1 + \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} (I^{(m)}_{n-1} + mI^{(m-1)}_{n-1} d_n) = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m)}_n,
\]

which means that (2) is valid for every \( n \geq 0 \).

It follows that if \( X = (X_n, \mathcal{F}_n) \) is a solution of the following stochastic difference equation
\[ X_n = 1 + \lambda \sum_{j=1}^{n} X_{j-1} d_j,
\]

then \( X \) can be expressed as
\[ X_n = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m)}_n.
\]

On the other hand, from (2) we see that \( I^{(m)}(f) \) is similar to an orthogonal polynomial of degree \( m \) with the "generating function" \( \prod_{j=0}^{n} (1 + \lambda d_j) \). By using induction on \( m \) we can obtain the following recursion relation.

**Proposition 2.** For \( m \geq 1 \),
\[ I^{(m)}_n = I^{(m-1)}_n f_n - (m - 1)I^{(m-2)}_n S^2_1(f) + A_{m,n}, \]

where
\[ A_{m,n} = \sum_{j=3}^{m} (-1)^{j-1} \frac{(m-1)!}{(m-j)!} \sum_{k=0}^{n} \frac{\lambda^m}{m!} I^{(m-j)}_n d_k.
\]

**Proof.** Fix \( \omega \in \Omega \). Let \( \lambda \) be in a neighborhood of 0 such that for every \( k \geq 1 \)
\[ \frac{1}{1 + \lambda d_k} = \sum_{j=0}^{\infty} (-1)^j \lambda^j d_k^n \]

holds. Differentiating both sides of (2) with respect to \( \lambda \), we get
\[ \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m+1)}_n = \sum_{k=0}^{n} \frac{d_k}{1 + \lambda d_k} \prod_{j=0}^{n} (1 + \lambda d_j) \quad (n \geq 0).
\]

On the other hand, we have
\[
\sum_{k=0}^{n} \frac{d_k}{1 + \lambda d_k} \prod_{j=0}^{n} (1 + \lambda d_j) = \sum_{k=0}^{n} d_k \sum_{j=0}^{\infty} (-1)^j \lambda^j \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m)}_n \\
= \sum_{k=0}^{n} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \lambda^j \frac{\lambda^m}{m!} I^{(m)}_n \\
= \sum_{k=0}^{n} \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} (-1)^j \lambda^j \frac{\lambda^p}{(p-l)!} I^{(p-l)}_n \\
= \sum_{p=0}^{\infty} \left( \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j}{(p-l)!} I^{(p-l)}_n \sum_{k=0}^{n} d_k^{l+1} \right) \lambda^p
\]
and so
\[ \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} I^{(m+1)}_n = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} (-1)^j \frac{\lambda^j}{(m-j)!} I^{(m-j)}_n \sum_{k=0}^{n} d_k^{l+1} \right) \lambda^p,
\]

which gives
\[ I^{(m+1)}_n = \sum_{j=0}^{m} (-1)^j \frac{m!}{(m-j)!} I^{(m-j)}_n \sum_{k=0}^{n} d_k^{l+1} \\
= I^{(m)}_n f_n - mI^{(m-1)}_n S^2_1(f) + \sum_{j=3}^{m} (-1)^{j-1} \frac{m!}{(m+1-j)!} I^{(m+1-j)}_n \sum_{k=0}^{n} d_k.
\]

This completes the proof of the proposition.
3. $L^p$-estimates of $I^{(m)}(f)$

In this section, we shall give a sufficient condition for the iterated stochastic integrals $I^{(m)}(f) = (I^{(m)}_n, (\mathcal{F}_n))$ to be a true martingale for every $m \geq 2$.

The martingale inequalities

$$c_p \|S(f)\|_p \leq \|f^*\|_p \leq c_p \|S(f)\|_p \quad (1 \leq p < \infty) \tag{5}$$

are celebrated as the Burkholder-Davis-Gundy inequalities. We need here the inequalities

$$\|f^*\|_p \leq 4p \|S(f)\|_p \quad (1 \leq p < \infty). \tag{6}$$

For this proof, see [4].

**Theorem 1.** For $1 \leq p < \infty$ and $m \geq 1$, the inequality

$$\|I^{(m)}\|_p \leq (4mp)^m \|S^m(f)\|_p \tag{7}$$

holds for every martingale $f$.

**Proof.** Assume $1 \leq p < \infty$.

The inequality (7) is verified by induction.

Clearly, (7) is true for $m = 1$ by (6). Let (7) be true for $1, 2, \ldots, m - 1$.

From (3) we have

$$S^2_2(I^{(m)}) = m^2 \sum_{j=1}^{n} (I^{(m-1)}(f))^2 \, d_j^2$$

and so

$$S(I^{(m)}(f)) \leq mI^{(m-1)}S(f). \tag{8}$$

Applying the Hölder inequality with exponents $r = m/(m-1)$ and $s = m$ to (8), we get

$$\|S(I^{(m)})\|_p \leq m \|I^{(m-1)}\|_{mp/m-1} \|S(f)\|_{mp}.$$ 

Combining this with (6), we get

$$\|I^{(m)}\|_p \leq 4p \|S(I^{(m)})\|_p \leq 4mp \|I^{(m-1)}\|_{mp/(m-1)} \|S(f)\|_{mp}.$$ 

and so by the induction hypothesis

$$\|I^{(m)}\|_p \leq (4mp)^m \|S(f)\|_{mp}^m = (4mp)^m \|S^m(f)\|_p.$$ 

This completes the proof of the theorem. \hfill \Box

As a corollary, it follows at once that, if $\|S(f)\|_{\infty} < \infty$, then every $I^{(m)}(f)$ is a martingale bounded in $L^p$ for all $p \geq 1$. To be exact, we obtain the following.

**Corollary 1.** If $\|S(f)\|_{\infty} < \infty$, then for $1 \leq p < \infty$ and $m \geq 1$

$$\|I^{(m)}\|_p \leq m!(C_p)^m \|S(f)\|_{\infty}^m$$

where $C_p = \sqrt{10p}$ if $1 \leq p < 2$ and $C_p = p$ if $2 \leq p < \infty$.

**Proof.** It is well-known that the right hand side inequality in (5) is valid with $C_p = p$ if $2 \leq p < \infty$ and $C_p = \sqrt{10p}$ if $1 \leq p < 2$ (see (3.4) in [2, p. 87] and II 2.8 in [4, p. 37]). Thus, in the same way as in the proof of Theorem 1, we may obtain the corollary. \hfill \Box

Let now

$$s_n(f) = \left( \sum_{k=0}^{n} E[d^2_k | \mathcal{F}_{k-1}] \right)^{\frac{1}{2}}$$

and

$$s(f) = s_\infty(f).$$

**Corollary 2.** Let $m \geq 2$ and $1 \leq p < \infty$. Then

$$\|s(I^{(m)})\|_p \leq \frac{(4mp)^{m+1}}{32p} \|S^m(f)\|_p. \tag{9}$$

**Proof.** Since $I^{(m)}_j$ is $\mathcal{F}_j$-measurable for $j \geq 1$, we have

$$S^2_n(I^{(m)}) = \sum_{j=0}^{n} E[(I^{(m)}_j - I^{(m)}_{j-1})^2 | \mathcal{F}_{j-1}]$$

$$= m^2 \sum_{j=0}^{n} (I^{(m-1)}_j)^2 E[d^2_j | \mathcal{F}_{j-1}]$$

The proof is then straightforward. \hfill \Box
\[ \leq m^2 I^{(m-1)\ast 2} S_n^2(f). \]

It follows from the Hölder inequality with exponents \( r = m/(m - 1) \) and \( s = m \) that
\[ \|s(I^{(m)})\|_p \leq m\|I^{(m-1)\ast}\|_{mp/(m - 1)}\|s(f)\|_{mp}, \]
and therefore by (7)
\[ \|s(I^{(m)})\|_p \leq m4mp^{m-1}\|S(f)\|_{mp}^{m-1}\|s(f)\|_{mp}. \]

Recall that the inequality
\[ \|s(f)\|_p \leq \sqrt{\frac{1}{2}} \|S(f)\|_p \quad (2 \leq p < \infty) \]
holds (see IV.1.4 in [4, p. 126]). Thus (9) is obtained.

\[ \square \]

4. **Approximation to a continuous martingale**

Let \( M = (M_t, \mathcal{F}_t)_{t \geq 0} \) be a continuous martingale with the quadratic variation process \( \langle M \rangle \), and let \( H_m(x, y) \) \((m \geq 0)\) be the Hermite polynomials with parametric variable \( y > 0 \), that is,
\[ H_m(x, y) = (-y)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-(x^2/2)} \quad (m \geq 0). \]

It is well-known that \( H_m(M, \langle M \rangle) = (H_m(M_t, \langle M \rangle_t), \mathcal{F}_t) \) is a continuous local martingale for any \( m \geq 1 \) (see Proposition 3.8 in [6, p. 151]), and for \( 1 < p < \infty \), furthermore, \( H_m(M, \langle M \rangle) \) is an \( L^p \)-bounded martingale if and only if \( M \) is an \( L^\infty \)-bounded martingale (see [3]).

In this section, we shall approximate \( H_m(M, \langle M \rangle) \) by the iterated martingale transforms.

Define the martingale \( f(n) = (f_{n,j}, \mathcal{F}_{n,j})_{0 \leq j \leq n} \) for every \( n \geq 1 \) by
\[ f_{n,j} = M_{j/n}, \quad \mathcal{F}_{n,j} = \mathcal{F}_{j/n} \quad (j = 0, 1, 2, \ldots, n). \]

Then it is well-known that for every \( t \in [0, 1] \)
\[ S_{[nt]}^2(f(n)) = \sum_{j=1}^{[nt]} (M_{j/n} - M_{(j-1)/n})^2 \rightarrow \langle M \rangle_t \quad \text{in probability} \]
as \( n \rightarrow \infty \).

Let now \( I^{(m)}(n, t) = (I^{(m)}(n, t), \mathcal{F}_{n,t})_{0 \leq t \leq 1} \) \((0 \leq t \leq 1)\), which is the iterated martingale transform of degree \( m \geq 1 \) with respect to martingale \( f(n) \) for every \( n \geq 1 \).

**Theorem 2.** Let \( M = (M_t, \mathcal{F}_t)_{t \geq 0} \) be a continuous martingale. Then for every \( t \in [0, 1] \)
\[ I^{(m)}(n) \rightarrow H_m(M_t, \langle M \rangle_t) \quad \text{in probability} \quad (m = 1, 2, 3, \ldots, n) \quad (10) \]
as \( n \rightarrow \infty \).

**Proof.** We shall inductively prove the theorem, without loss of generality, we may assume \( t = 1 \).

First, the theorem is true if \( m = 1 \) and \( m = 2 \). Indeed,
\[ I^{(1)}_n(n) = M_1 = H_1(M_1, \langle M \rangle _1) \]
and by (4)
\[ I^{(2)}_n(n) = I^{(1)}_n M_1 - \sum_{j=0}^{n} (M_{j/n} - M_{(j-1)/n})^2 \rightarrow M_1^2 - \langle M \rangle_1 = H_2(M_1, \langle M \rangle_1) \quad \text{in probability} \]
as \( n \rightarrow \infty \).

Next, we suppose that (10) is established for \( m - 1 \) in place of \( m \). Applying this to (4) and noting that
\[ \sum_{k=1}^{n} |a_k|^2 = \sum_{k=1}^{n} |M_{k/n} - M_{(k-1)/n}|^2 \leq \max_{0 \leq k \leq n} |M_{k/n} - M_{(k-1)/n}|^2 \quad \sum_{k=1}^{n} (M_{k/n} - M_{(k-1)/n})^2 \rightarrow 0 \quad (n \rightarrow \infty) \]
for all \( j > 2 \), we get
\[ \Delta_{m,n} \to 0 \text{ in probability } (n \to \infty) \]

and so
\[ I_n^{(m)}(n) \to H_{m-1}(M_1, \langle M \rangle_1)M_1 - (m - 1)H_{m-2}(M_1, \langle M \rangle_1)\langle M \rangle_1 \text{ in probability} \]
as \( n \to \infty \). Thus, the recursion relation of the Hermite polynomials
\[ H_m(x, y) = H_{m-1}(x, y)x - (m - 1)H_{m-2}(x, y)y \]
implies that
\[ I_n^{(m)}(n) \to H_m(M_1, \langle M \rangle_1) \text{ in probability} \]
as \( n \to \infty \). This completes the proof.

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