Harmonic Wavelet Analysis of Sound

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Received January 29, 2008; final version accepted June 17, 2008

We call the harmonic wavelet analysis by means of rapidly decaying harmonic wavelets the rd harmonic wavelet analysis. It is applied to data of sound, and comparison is made of the results obtained by using two kinds of rd harmonic wavelets and two ways of calculating the inner products in the numerical calculation. A practical method of applying the rd harmonic wavelet analysis is proposed.

KEYWORDS: rapidly decaying harmonic wavelet, harmonic wavelet analysis, Fourier analysis, tight frame, sound

1. Introduction

In the present paper, we study data of sound by harmonic wavelet analysis. The data of sound were taken when a person uttered a short phrase "Ah Yukida!", that means "Oh snow!", in Japanese, in a sad and in a happy mood. They are time series of length $N = 54545$ and $N = 31794$, respectively, shown in Fig. 1. We are interested in the amount of oscillation of each frequency as a function of time. It is then natural to apply the harmonic wavelet analysis, which was used to analyze the time series taken from music [5, 6].

![Fig. 1. Two data of time series $g(t)$ as a function of $t \in \mathbb{Z}_{0,N-1}$, where (a) $N = 54545$ and (b) $N = 31794$.](image)

In the harmonic wavelet expansion proposed by Newland [5, 6], the Fourier transforms of wavelets are constant in their respective supports and are zero outside them. Morita and Takeuchi [2, 4] studied the harmonic wavelet expansion of a periodic function or of a function restricted to a finite time interval. The harmonic wavelets for a periodic function are linear combinations with equal amplitudes of plane waves within a certain range of frequency. The merit of the harmonic wavelet expansion is that it is regarded as a rearrangement of the Fourier series.

The demerit of the harmonic wavelet expansion is that the wavelets are not well localized in time and decay slowly. For Shannon’s wavelet expansion [9, p. 10], which is the simplest of the harmonic wavelet expansion, Meyer showed that an orthogonal system of wavelets which decay fast can be constructed by mixing the plane waves in neighboring ranges of frequency [9, p. 166], [1, p. 137]. Morita [3] constructed an orthogonal system of rapidly decaying (rd) harmonic wavelets in the same way.

In [2–4], translationally invariant (ti) expansions which do not depend on the origin of the spatial coordinate were presented. They are expansions with the aid of an over-complete system of harmonic wavelets or of rd harmonic wavelets, each constructed from an orthogonal system.

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In the theory of wavelet, a tight frame of wavelets is as important as a complete set of orthogonal wavelets. In a tight frame, wavelets are not orthogonal with each other and yet a function can be expanded in them and the Parseval formula holds. We find a tight frame, which is closely related with the orthogonal system of rd wavelets, in the classical book of Daubechies [1, p. 73]. We show that we have a ti expansion constructed from this tight frame.

We now present a simpler ti expansion related with these. We use this expansion and the ti expansion based on the over-complete system in the following calculation.

We call the harmonic wavelet analysis by means of rd harmonic wavelets the rd harmonic wavelet analysis. The purpose of the present paper is to compare the results of numerical calculations in the rd harmonic wavelet analysis based on the above-mentioned ti expansions, and present a practical method of the calculation. This is done to give the power density associated with intervals of frequency as a function of time, for the time series shown in Fig. 1.

For two integers \(a \leq b\), \(Z_{a,b}\) is used to denote the set of all the integers \(k\) satisfying \(a \leq k \leq b\). The notation \([x]\) for a real number \(x\) represents the least integer that is not less than \(x\). In this paper except in Appendix B, we consider a time series of a finite length \(L\) of data \(\{f(t)\}\). We assume that \(L \in \mathbb{N}\) is an even integer and real numbers \(f(t)\) for \(t \in \mathbb{Z}_{0,L-1}\) are given. The function \(f(t)\) is sometimes treated as a periodic function of \(t \in \mathbb{Z}\), with period \(L\).

In Section 2, we present two sets of rd wavelets for a system of time series of length \(L \in \mathbb{N}\). One of the sets is given in [3]. Another set related with the tight frame given in [1, p. 73] is mentioned in Appendix A. In Section 3, we give the rd harmonic wavelet expansion formulas. In Section 4, we discuss the results of numerical calculation. In the numerical calculation of Fourier analysis and harmonic wavelet analysis, we treat the Fourier series of a time series and apply the fast Fourier transform (FFT) algorithm [7]. Conclusion is given in Section 5.

In the text and in Appendix A, we study a discrete system given by \(\{f(t)\}_{t \in \mathbb{Z}_{0,L-1}}\). In Appendix B, we note that a continuous system of \(f(t)\) for \(t \in \mathbb{R}\) satisfying \(0 \leq t < L\) is related with the discrete system by the sampling theorem.

In the rest of this section, we give the Fourier series of the time series \(\{f(t)\}_{t \in \mathbb{Z}_{0,L-1}}\). It is

\[
f(t) = \sum_{p=-L/2}^{L/2-1} c_p e^{i2\pi pt/L},
\]

where the Fourier coefficients \(c_p\) are given by

\[
c_p = \frac{1}{L} \sum_{t=0}^{L-1} f(t) e^{-i2\pi pt/L},
\]

and satisfy

\[
c_{-p} = \overline{c_p}.
\]

The Parseval formula is

\[
\frac{1}{L} \sum_{t=0}^{L-1} |f(t)|^2 = \sum_{p=-L/2}^{L/2-1} |c_p|^2.
\]

(1) shows that, if the data are given in every \(\tau\) sec, the frequency \(w\) corresponding to \(p\) is \(w = p/(L\tau)\) per sec.

2. Rapidly Decaying Harmonic Wavelets

We now recall the rd harmonic wavelets proposed in [3].

For a given number \(L \in \mathbb{N}\), we choose a number \(J\) and the series \(\{n_j\}, \{m_j\}\) and \(\{l_j\}\), such that they satisfy the following condition.

**Condition 1.** (i) \(J \in \mathbb{N}\), (ii) \(n_j \in \mathbb{Z}_+\), \(m_j \in \mathbb{N}\), \(l_j \in \mathbb{R}\), \(m_j = n_{j+1} - n_j > 0\), \(l_j \geq 0\), and \(l_j + l_{j+1} \leq m_j\) for every \(j \in \mathbb{Z}_{0,J+1}\). (iii) \(n_0 = 0\), \(n_1 = 1\), \(m_0 = 1\), \(l_0 = 0\), \(l_1 = 0.1\), \(n_{j+1} = L/2 - 1\), \(n_{j+2} = L/2\), \(m_{j+1} = 1\), \(l_{j+1} = 0.1\), and \(l_{j+2} = 0\).

In the numerical calculation given later, we use \(\{n_j\}\) such that the ratio \(n_{j+1}/n_j\) is approximately equal to \(2^{1/4}\).

Now the wavelet \(\psi_j(t)\) as a function of \(t \in \mathbb{Z}_{0,L-1}\) for \(j \in \mathbb{Z}_{0,J+1}\) is given in terms of their Fourier coefficients \(\tilde{\psi}_j(p)\) for \(p \in \mathbb{Z}_{-L/2+1,L/2-1}\), by

\[
\psi_j(t) = \sum_{p=-L/2}^{L/2-1} \tilde{\psi}_j(p) e^{i2\pi pt/L}.
\]

(5) shows that, if the data are given in every \(\tau\) sec, the frequency \(w\) corresponding to \(p\) is \(w = p/(L\tau)\) per sec.
\[ \tilde{\psi}_J(p) = \begin{cases} 
\frac{1}{\sqrt{2}} e^{\pm i \frac{\pi}{4}} & p = \pm 1, \\
0 & \text{otherwise}, 
\end{cases} \]

where \( u_l(l) \) is a real-valued function of \( l \in \mathbb{Z}_{-\{l\}, \{l\}} \), satisfying the following condition.

**Condition 2.** \( u_l(l) = 0 \), \( u_l(l) \in \mathbb{R} \), and \( u_l(l)^2 + u_l(-l)^2 = 1 \) for \( j \in \mathbb{Z}_{0,l+1} \) and \( l \in \mathbb{Z}_{-\{l\}, \{l\}} \).

In order that \( \psi_j(t) \) decay rapidly, \( u_l(l) \) are desired to be smooth functions [1]. In practice, we adopt

\[ u_l(l) = \cos\left( \frac{\pi}{2} \frac{s}{4} + \frac{\ln |l|}{4|l|} \right). \]

This is the form given in the book [1, p. 74] for a tight frame. In [3], \( u_l(l) = \cos\left( \frac{\pi}{4} + \frac{\ln |l|}{2l} \right) \) is used. It is confirmed that the expression (7) gives better results.

In particular when \( j = 0 \) and \( j = J + 1 \), \( \tilde{\psi}_J(p) \) is given by

\[ \tilde{\psi}_J(p) = \begin{cases} 
1, & p = 0, \\
\frac{1}{\sqrt{2}} e^{\pm i \frac{\pi}{4}} & p = \pm 1, \\
0 & \text{otherwise}, 
\end{cases} \]

By this choice, the orthonormal relations of \( \tilde{\psi}_J(p) \) and \( \psi_j(t) \) are given by

\[ \frac{1}{L} \sum_{l=0}^{L-1} \psi_j(t) \psi_j(t) = \sum_{|m|=L/2} \psi_j(p) \psi_J(p) = 2m \delta_{t_j}. \]

In [3], the following theorem is proved.

**Theorem 1.** Let \( J \), \( \{n_l\} \), \( \{m_l\} \) and \( \{l_l\} \) satisfy Condition 1, \( m_l \) be a divisor of \( n_l \), 2\( m_l \) be a divisor of \( L \) for every \( j \in \mathbb{Z}_{1,J} \), and \( u_l(l) \) satisfy Condition 2, and let \( \tilde{\psi}_J(p) \) be given by Choice 1. Let the set of the \( r.d \) harmonic wavelets \( \psi_j(t) = \psi_j(t - kL/(2m_l)) \), \( j \in \mathbb{Z}_{0,J+1} \) and \( k \in \mathbb{Z}_{0,2m_l-1} \), be defined by (5). Then the set of these functions \( \{\psi_j(t)\} \) of \( t \) is an orthogonal basis in the space of real-valued functions of \( t \in \mathbb{Z}_{0,L-1} \).

In Fig. 2, typical curves of \( \tilde{\psi}_J(p) \) and \( \psi_j(t) \) in Choice 1 are shown.

![Fig. 2](image_url)

**Fig. 2.** (a) \( \tilde{\psi}_J(p) \) as a function of \( p \) and (b) \( \psi_j(t) \) as a function of \( t \) for \( w_j = 215.38 \), in Choice 1. In (a), the curve taking only positive values is for \( \text{Re} \tilde{\psi}_J(p) \) and the other for \( \text{Im} \tilde{\psi}_J(p) \). In (b), \( t_{m,j} = 882 \) and \( t_{l,j} = 704 \).

The second choice of \( \tilde{\psi}_J(p) \) is obtained by using the absolute value of \( \tilde{\psi}_J(p) \) in Choice 1.

**Choice 2.** \( \tilde{\psi}_J(0) = \delta_{00}, \tilde{\psi}_J(L/2) = \delta_{J+1J}, \) and \( \tilde{\psi}_J(p) = \psi_J(-p) \) for \( p \in \mathbb{Z}_{1,J}/2-1 \), are given by

\[ \tilde{\psi}_J(p) = \begin{cases} 
\psi_j(n_j - p), & n_j - l_j \leq p \leq n_j + l_j, \\
1, & n_j + l_j \leq p \leq n_j + 1 - l_j, \\
\psi_J(p - n_j + 1), & n_j + 1 - l_j \leq p \leq n_j + 1 + l_j, \\
0, & \text{otherwise}, 
\end{cases} \]

where \( u_l(l) \) is a real-valued function of \( l \in \mathbb{Z}_{-\{l\}, \{l\}} \), satisfying Condition 2.
In place of (8), we have

\[
\tilde{\psi}_0(p) = \begin{cases} 
1, & p = 0, \\
\frac{1}{\sqrt{2}}, & p = \pm 1, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\tilde{\psi}_{j+1}(p) = \begin{cases} 
1, & p = L/2, \\
\frac{1}{\sqrt{2}}, & p = \pm (L/2 - 1), \\
0, & \text{otherwise}.
\end{cases}
\] (11)

Here \(\tilde{\psi}_j(p)\) are real and nonnegative, and \(\frac{1}{L} \sum_{k=0}^{L-1} \psi_j(t)^2 = \sum_{p=-L/2+1}^{L/2} \tilde{\psi}_j(p)^2 = 2m_j\) holds, but the orthogonal relations for \(j \neq j'\) in (9) do not hold.

In Fig. 3, typical curves of \(\tilde{\psi}_j(p)\) and \(\tilde{\psi}_j(t)\) in Choice 2 are shown.

In Appendix A, we give Choice 3, which is closely related with a tight frame presented in the book [1, p. 73].

3. Discrete Harmonic Wavelet Expansion

We define the wavelet transform \((\Psi f)(t')\) for \(t' \in \mathbb{Z}_{0,L-1}\) by

\[
(\Psi f)(t') = \frac{1}{L} \sum_{t=0}^{L-1} \psi(t) f(t' + t).
\] (12)

This is also expressed as

\[
(\Psi f)(t') = \sum_{p=-L/2+1}^{L/2} \tilde{\psi}(p) e^{2\pi ip' t/L}.
\] (13)

When the conditions in Theorem 1 are satisfied, for arbitrary \(t_1 \in \mathbb{Z}_{0,L-1}\), we obtain

\[
f(t) = \frac{1}{2m_j} \sum_{k=0}^{2m_j-1} \psi(t - t_1 - kL/(2m_j)) \{(\Psi f)(t_1 + kL/(2m_j))\},
\] (14)

and

\[
\frac{1}{L} \sum_{t=0}^{L-1} f(t)^2 = \frac{1}{2m_j} \sum_{k=0}^{2m_j-1} e_j(t_1 + kL/(2m_j)),
\] (15)

where

\[
e_j(t') = |(\Psi f)(t')|^2.
\] (16)

**Theorem 2.** Let the conditions in Theorem 1 be satisfied. Then for a real-valued function \(f(t)\) of \(t \in \mathbb{Z}_{0,L-1}\), the rd harmonic wavelet expansion (14) and the Parseval formula (15) hold.

Summing (14) and (15) with respect to \(t_1\) from 0 to \(L - 1\) and dividing the results by \(L\), we obtain

\[
f(t) = \frac{1}{L} \sum_{t=0}^{L-1} \frac{1}{L} \sum_{t'=0}^{L-1} \psi(t - t') |(\Psi f)(t')|,
\] (17)

and

\[
\frac{1}{L} \sum_{t=0}^{L-1} f(t)^2 = \frac{1}{L} \sum_{t=0}^{L-1} \frac{1}{L} \sum_{t'=0}^{L-1} e_j(t').
\] (18)
Theorem 3. Let \( J, [n_j] \) and \([l_j]\) satisfy Condition 1, let \( \hat{\psi}_j(p) \) be as in Choice 1 or Choice 2 in Section 2, and let \( \psi_j(t) \) and \( (\Psi_j f)(t') \) be given by (5) and (12). Then for a real-valued function \( f(t) \) of \( t \in \mathbb{Z}_0L^{-1} \), the \( \text{rd} \) harmonic wavelet expansion (17) and the Parseval formula (18) hold.

Substituting (5) and (13) into the righthand sides of (17) and (16), we confirm the last statement in Theorem 3, by using the equality \( \sum |\hat{\psi}_j(p)|^2 = 1 \).

(18) shows that the \( e_j(t') \) can be regarded as the energy associated to the \( j \text{th} \) wavelet at time \( t' \). When we are interested in the energy per frequency, we have to calculate \( e_j(t')/(2m_j) \).

In Appendix A, we give Choice 3 and the equations corresponding to the equations given above, in that case.

4. Numerical Calculation

Numerical calculation of \( e_j(t') \) is performed for two sets of data shown in Fig. 1. These are series of \( N \) real numbers \( g(t) \) for \( t \in \mathbb{Z}_0N^{-1} \), where \( N = 54545 \) and \( N = 31794 \), respectively, and the time interval between two successive data is \( \tau = 1/16000 \) sec.

In the numerical calculation, we apply the FFT program given in [7, 8]. We then use the value of \( L \) to be a power of 2. For the data of 54545 and 31794 real numbers, we choose \( L = 2^{16} = 65536 \), and set \( f(t) \) for \( t \in \mathbb{Z}_0L^{-1} \) as follows:

\[
f(t) = \begin{cases} 
0, & 0 \leq t \leq t_0 - 1, \\
g(t - t_0), & t_0 \leq t \leq t_0 + N - 1, \\
0, & t_0 + N \leq t \leq L - 1,
\end{cases}
\]

where \( t_0 \) is chosen such that \( 2t_0 + N \leq L \). For the two data, we adopt \( t_0 = 4096 \).

Calculation is made of \( e_j(t')/(2m_j) \) with the aid of (16) and (12) or (13). The results for the four cases shown in Table 1 are shown in Figs. 4 and 5 for the two data given in Fig. 1. The figures show the power density \( e_j(t')/(2m_j) \) as a function of abscissa \( t' \tau \) and ordinate \( w_j \) by darkness. The numbers below the abscissa show the values of \( t' \tau = (t' - t_0)\tau \) for all \( t' \in \mathbb{Z}_0 \). Here \( w_j \) is the typical frequency for the \( j \text{th} \) wavelet, which is equal to \( \sqrt{m_j w_j}/(L \tau) \). The results are shown for the frequencies satisfying \( 40 \text{Hz} < w_j < 4000 \text{Hz} \).

Calculations by using (12) for (c) and (d) were done in an approximate way. Now both functions \( \psi_j(t) \) and \( f(t) \) do not have a compact support. In order to save computer time, we choose \( t_{j,0} > 0 \) and \( t_{m,j} > 0 \) such that \( ||\psi_j(t)||/||\psi_j(t)|| < 0.01 \) for all \( t \) satisfying \( t_{j,0} < t \leq L/2 \), and \( ||\psi_j(t + t)|/||\psi_j(t)|| < 0.01 \) for all \( t \) satisfying \( t_{m,j} < t \leq L/2 \), where \( ||\psi_j(t)|| = \max_{t \in \mathbb{Z}_0L^{-1}} |\psi_j(t)| \), and define \( \psi_{j,a}(t) \) by

\[
\psi_{j,a}(t) = \begin{cases} 
\psi_j(t), & 0 \leq t \leq t_{j,0} \text{ or } L - t_{m,j} \leq t \leq L - 1, \\
0, & t_{j,0} < t < L - t_{m,j}.
\end{cases}
\]

We then use (12) with \( \psi_{j,a}(t) \) in place of \( \psi_j(t) \). Now we have \( t_{j,0} + t_{m,j} + 1 \) nonzero terms at most in (12). When we use (13), the total number of nonzero terms is \( 4m_j \) at most as seen from (6) or (10). In Fig. 6, we compare these numbers. From this figure, we do not expect much difference in computer time, but we calculate the summation in (12) for each \( t' \) when (12) is used, while a single Fourier inversion in FFT is performed when (13) is used. The computer time is then...
much more for (12) than for (13). In practice, the computation of (12) is done only for 128 values of $t_0$ in an interval of length $\tau_0t_j$ when $\tau_0t_j > 128$, and then the computer time when (12) is used is almost twice of that when (13) is used.

In Figs. 4 and 5, the vertical solid line segments at $t_0 = 0$ and at the $t_0$ which is little less than 3.5 sec or 2.0 sec, just below the abscissa, indicate the boundaries of the range where the data $g(t_0)$ takes nonzero values. We note that there exists no appreciable difference between the four graphs in each figure, except that the smearing is eminent for (a) in Fig. 4, that is when Choice 1 of $\tilde{\psi}(p)$ and (13) are adopted for the data given in Fig. 1(a). By choosing (12) or Choice 2 of $\tilde{\psi}(p)$, the result is improved. In the respect of the smearing, we may say that the best choice is (d), although no appreciable differences are observed between (b), (c) and (d).

In Table 2, we show the standard deviations $\sigma_{t,j}$ and $\sigma_{p,j}$ of $\psi(t)$ and $\tilde{\psi}(p)$, respectively, and related quantities, for typical values of $j$. Here $\sigma_{t,j}$ and $\sigma_{p,j}$ are defined by

$$
\sigma_{t,j}^2 = \langle \tau^2 \rangle_t - \langle \tau \rangle_t^2, \quad \langle \tau^2 \rangle_t = \frac{1}{2m_j L} \left\{ \sum_{t=1}^{L/2} \tau^2 \psi_j(t)^2 + \sum_{t=L/2+1}^{L-1} \tau^2 \tilde{\psi}_j(t)^2 \right\},
$$

(21)

$$
\sigma_{p,j}^2 = \langle |p|^2 \rangle_p - \langle |p| \rangle_p^2, \quad \langle |p|^2 \rangle_p = \frac{1}{2m_j} \sum_{p=-L/2+1}^{L/2} |p|^2 |\psi_j(p)|^2.
$$

(22)

Fig. 5. The power density $e_j(t')/(2m_j)$ for the data of Fig. 1(b). See Fig. 4 for explanations.

Table 1. Four cases of choosing Choices 1 or 2 of $\tilde{\psi}(p)$, and Eq. (12) or Eq. (13) for the inner product.

<table>
<thead>
<tr>
<th>Choice 1 or 2 of $\tilde{\psi}(p)$</th>
<th>(12) or (13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Choice 1 (13)</td>
<td></td>
</tr>
<tr>
<td>(b) Choice 2 (13)</td>
<td></td>
</tr>
<tr>
<td>(c) Choice 1 (12)</td>
<td></td>
</tr>
<tr>
<td>(d) Choice 2 (12)</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6. The numbers of nonzero terms in the inner products calculated by using (12) and (13). The curve from left bottom to right up is for (13), and the curve from left up to right bottom is for (12). The latter consists of two curves, of which the upper and lower ones are for Choices 2 and 1, respectively, of $\tilde{\psi}(p)$. The abscissa is the typical frequency of the $j$th wavelet $w_j$. 

In Figs. 4 and 5, the vertical solid line segments at $t' = 0$ and at the $t' = \tau_0 t_j$ which is little less than 3.5 sec or 2.0 sec, just below the abscissa, indicate the boundaries of the range where the data $g(t')$ takes nonzero values. We note that there exists no appreciable difference between the four graphs in each figure, except that the smearing is eminent for (a) in Fig. 4, that is when Choice 1 of $\tilde{\psi}(p)$ and (13) are adopted for the data given in Fig. 1(a). By choosing (12) or Choice 2 of $\tilde{\psi}(p)$, the result is improved. In the respect of the smearing, we may say that the best choice is (d), although no appreciable differences are observed between (b), (c) and (d).
Appendix A: Daubechies’ Choice

5. Conclusion

We proposed four ways of applying 1D harmonic wavelet analysis. They are different in the way of choosing the wavelets and in calculating the inner product. The conclusion we get is that all the four choices (a)–(d) are as useful when the smearing-out, mentioned at the end of Section 4, does not cause a serious problem. The programming is simpler for (a) or (b).

We obtained Figs. 4 and 5 for the data shown in Figs. 1(a) and 1(b), respectively. We note much differences between them, e.g. if we compare the lowest curves which connect the maximum power density at each time. We obtained Figs. 4 and 5 for the data shown in Figs. 1(a) and 1(b), respectively. We note much differences between them, e.g. if we compare the lowest curves which connect the maximum power density at each time.

Kaneko, one of the authors, has been studying the problem of detecting characteristic features of sound data spoken in place of (12)–(18), we have

\[ \sigma_{\psi_j} = \sigma_{\psi_j}/Lr \] are the standard deviations in sec and 1/sec, respectively. The product of these, \( 2\pi \sigma_{\psi_j} \sigma_{\psi_j} \), is calculated to be 0.5814 ~ 0.5830 and 0.5752 ~ 0.5763 for Choices 1 and 2, respectively, of \( \psi_j(p) \). These values are slightly greater than 0.5, which is the optimal value due to the uncertainty principle in quantum mechanics. \( \sigma_{\psi_j} \) given in the 4th column is seen to be little smaller than \( \Delta w_j = w_{j+1} - w_j \) given in the 5th column. This is expected from the construction of the present wavelet. The values of \( \sigma_{\psi_j} \) are given in the 6th and 8th columns. These values are comparable with 128 or greater for \( w_j \leq 200 \). In the graphs in Figs. 4 and 5, darkness are given for every 128 values of \( t' \). Hence we clearly observe smearing out in \( t' \) direction for \( w_j \leq 200 \). In Figs. 2 and 3, \( \psi_j(p) \) and the wavelet \( \psi_j(t) \) for \( w_j = 215.38 \) are shown for Choices 1 and 2, respectively, of \( \psi_j(p) \).

Excluding the smearing-out, we observe no appreciable differences between the four graphs in Fig. 4 as well as in Fig. 5.

Table 2. The standard deviations \( \sigma_{\psi_j} \) and \( \sigma_{\psi_j} \) of \( \psi_j(t) \) and \( \psi_j(p) \), respectively.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( w_j )</th>
<th>( \sigma_{\psi_j} )</th>
<th>( \sigma_{\psi_j} )</th>
<th>( \Delta w_j )</th>
<th>( \sigma_{\psi_j} )</th>
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5. Conclusion

We proposed four ways of applying 1D harmonic wavelet analysis. They are different in the way of choosing the wavelets and in calculating the inner product. The conclusion we get is that all the four choices (a)–(d) are as useful when the smearing-out, mentioned at the end of Section 4, does not cause a serious problem. The programming is simpler for (a) or (b).

We obtained Figs. 4 and 5 for the data shown in Figs. 1(a) and 1(b), respectively. We note much differences between them, e.g. if we compare the lowest curves which connect the maximum power density at each time.

Kaneko, one of the authors, has been studying the problem of detecting characteristic features of sound data spoken at different moods, and the present authors started the present investigation, seeking the possibility of applying the harmonic wavelet analysis to this problem. Further calculation is now in progress and the result will be reported in near future.

Appendix A: Daubechies’ Choice

Following the expansion in a tight frame given in [1, p. 73], we now adopt the following wavelets.

\( \psi(t) = 1, \psi_j(t) = (-1)^j, \) and \( \psi_j(t) \) for \( j \in \mathbb{Z}_{0,j+1} \) are given by

\[ \psi_j(t) = \sum_{p=-L/2+1}^{L/2} \psi_j(p) e^{2\pi i pt/L}, \] (A-1)

\[ \psi_j(p) = \begin{cases} \psi_j(p), & p > 0, \\ 0, & p \leq 0, \end{cases} \] (A-2)

Here \( \psi_j(p) \) on the right hand sides are those given by (10) in Choice 2.

The normalization conditions are \( \frac{1}{L} \sum_{t=0}^{L-1} |\psi_j(t)|^2 = \sum_{p=-L/2+1}^{L/2} |\psi_j(p)|^2 = m_j = \frac{1}{2}(\delta_{j,0} + \delta_{j,1}). \)

In place of (12)–(18), we have

\[ (\psi_j f)(t') = \frac{1}{L} \sum_{t=0}^{L-1} \psi_j(t) f(t' + t) = \sum_{p=-L/2+1}^{L/2} \psi_j(p) e^{2\pi i pt/L}. \] (A-3)

\[ f(t) = c_0 + \sum_{j=1}^{J+1} \sum_{k \in \mathbb{Z}} \frac{1}{2m_j} \sum_{s=0}^{2m_j-1} \psi_j(s+k, t - (t_1 - kL/(2m_j))) (\psi_j(s+k, t_1 + kL/(2m_j))) \]

\[ + c_{L/2}(-1)^J, \] (A-4)
\[ \frac{1}{L} \sum_{t=0}^{L-1} f(t)^2 = c_0^2 + \sum_{j=0}^{J+1} \frac{1}{2m_j} \sum_{t=0}^{2m_j-1} e_j(t_1 + kL/(2m_j)) + c_L^2. \]  
(A-5)

\[ e_j(t') = \sum_{v \in \pm} |\psi_j(v, t')|^2, \]  
(A-6)

\[ f(t) = c_0 + \sum_{j=0}^{J+1} \sum_{t=0}^{L-1} \int_{-L/2}^{L/2} \psi_j(t-t') |\psi_j(t')|^2 dt' + c_L^2(-1)^j. \]  
(A-7)

\[ \frac{1}{L} \sum_{t=0}^{L-1} f(t)^2 = c_0^2 + \sum_{j=0}^{J+1} \sum_{t=0}^{L-1} e_j(t') + c_L^2. \]  
(A-8)

In this case, we substitute (A-1) and (A-3) in (A-4)–(A-7) and confirm (A-4), (A-5), (A-7) and (A-8), where the fact that \( \psi_{j+k}(p) \) have a compact support of length less than \( 2m_j \), and the equality \( \delta_{p,0} + \delta_{p,L/2} + \sum_{j=0}^{J+1} \sum_{v \in \pm} |\psi_j(v, p)|^2 = 1 \) are used.

**Appendix B: \( f(t) \) for \( t \in \mathbb{R} \) Satisfying \( 0 \leq t < L \)**

In the text and in Appendix A, we considered \( f(t) \) for discrete values of \( t \in \mathbb{Z}_{0,L-1} \). Here we consider \( f(t) \) for continuous values of \( t \) in \([0, L)\), which is the set of \( t \in \mathbb{R} \) satisfying \( 0 \leq t < L \). The Fourier series of this function \( f(t) \) is given by

\[ f(t) = \sum_{p=-\infty}^{\infty} c_p e^{2\pi i pt/L}, \quad c_p = \frac{1}{L} \int_0^L f(t) e^{-2\pi i pt/L} dt. \]  
(B-1)

In this place, we assume that \( c_p = 0 \) for \( p > L/2 \) and for \( p \leq -L/2 \). Now (1) is valid even for \( t \in [0, L) \), and the following sampling theorem applies:

\[ f(t) = \sum_{n=0}^{L-1} f(n) \frac{\sin[\pi(t-n)]}{\pi(t-n)/L}. \]  
(B-2)

In addition to (2)–(4), we have

\[ \frac{1}{L} \int_0^L f(t)^2 dt = \frac{1}{L} \sum_{n=0}^{L-1} f(n)^2. \]  
(B-3)

We now define \( \psi_j(t) \) for \( t \in [0, L) \) by (5), and then

\[ \frac{1}{L} \int_0^L \psi_j(t) \psi_j(t) dt = \frac{1}{L} \sum_{n=0}^{L-1} \psi_j(t) \psi_j(t). \]  
(B-4)

We define the wavelet transform (\( \Psi_j f(t') \)) for \( t' \in [0, L) \) by

\[ (\Psi_j f)(t') = \frac{1}{L} \int_0^L \psi_j(t) f(t' + t) dt, \]  
(B-5)

and then (12) and (13) are also valid for \( t' \in [0, L) \).

Now all the equations in the text as well as in this appendix hold valid, and also (14) and (17) are valid for \( t \in [0, L) \). (14) and (15) are valid for \( t_1 \in [0, L) \). In addition to (17) and (18), the following equalities are valid:

\[ f(t) = \sum_{j=0}^{J+1} \frac{1}{L} \int_0^L \psi_j(t-t') |(\Psi_j f)(t')| dt', \]  
(B-6)

\[ \frac{1}{L} \int_0^L f(t)^2 dt = \sum_{j=0}^{J+1} \frac{1}{L} \int_0^L e_j(t') dt'. \]  
(B-7)

**Acknowledgments.** The authors are grateful to Prof. Ken-ichi Sato of College of Engineering, Nihon University, for valuable discussions on wavelets and tight frames.

**REFERENCES**