| 著者 | 菅野 謙孝
|---|---|
| 出版社 | 井川秋子
| 出版年月 | 2013-09-19
| URL | http://hdl.handle.net/10097/57206
| doi | 10.4036/iis.2013.135 |
Optimization for Lattices and Diophantine Approximations

Masaru ITO\(^1\) and Noriko HIRATA-KOHNO\(^2,+; \)\(^1\)

\(^1\) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Meguro, Tokyo 152-8552, Japan
\(^2\) Department of Mathematics, College of Science and Technology, Nihon University, Chiyoda, Tokyo 101-8308, Japan

Received January 16, 2013; final version accepted May 14, 2013

In this article, we optimize a growth condition such that an analytic function being almost integer-valued at integers turns out to be a polynomial. Our argument relies on a computational optimization combined with Diophantine approximations on lattices.

KEYWORDS: lattice, almost integer-valued function, optimization, Diophantine approximation

1. Introduction

Denote by \(\mathbb{N}\) the set of strictly positive rational integers. Let us investigate the nature of an entire function \(F(z)\) in one complex variable such that \(F(\mathbb{N}) \subset \mathbb{Z}\), or \(F(z)\) taking values close to integers at any \(z \in \mathbb{N}\).

Our natural questions which arise as follows.

(1) Is \(F(z)\) a polynomial?

The answer is no, because of the existence of the functions such that \(F(z) = 2^z\) or \(3^z\), etc. Then the second question arises:

(2) Does there exist a suitable condition such that \(F(z)\) becomes a polynomial?

Denote by \(\overline{\mathbb{Q}}\) the algebraic closure of \(\mathbb{Q}\) in \(\mathbb{C}\). Consider a complex function \(F(z)\) in \(z \in \mathbb{C}\) which satisfies \(F(u) \in \overline{\mathbb{Q}}\) for any \(u \in \mathbb{Q}\). We may now ask the following more general question:

(3) Is \(F(z)\) a polynomial or an algebraic function?

Indeed, there exist examples due to P. Stäckel [9] in 1895 of transcendental functions taking algebraic values at all algebraic points. Hence the answer to the third question is again, no, namely it is true that such \(F(z)\) supposed to verify \(F(u) \in \overline{\mathbb{Q}}\) for any \(u \in \mathbb{Q}\), is not necessarily algebraic function.

On the other hand, the Hermite–Lindemann theorem in transcendental number theory shows \(\exp(\alpha) \notin \overline{\mathbb{Q}}\) for any \(\alpha \in \mathbb{Q}\), \(\alpha \neq 0\), which says, there exists a transcendental function always taking transcendental values at any non-trivial algebraic point. Moreover, the Gel’fond–Schneider theorem notices \(2^{\sqrt{3}} \notin \overline{\mathbb{Q}}\), then even an entire function \(F(z) = 2^z\) with \(F(\mathbb{N}) \subset \mathbb{Z}\) does not satisfy \(F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}\).

Now let us restrict ourselves to consider an entire function \(F(z)\) such that \(F(\mathbb{N}) \subset \mathbb{Z}\). For our question (2), we refer to a fundamental result of G. Pólya [8] concerning with such functions.

Recall the definition of the order of a complex function.

**Definition 1.** Let \(F(z)\) be an entire function in \(\mathbb{C}\). Write \(|F|_r = \sup_{|z| \leq r} |F(z)|\) and define \(\alpha(F)\) the order of exponential type of \(F(z)\) by

\[
\alpha(F) = \limsup_{r \to +\infty} \frac{\log |F|_r}{r}.
\]

**Theorem A (Pólya).** Let \(F(z)\) be an entire function in \(\mathbb{C}\) with \(F(\mathbb{N}) \subset \mathbb{Z}\). Suppose \(\alpha(F) < \log 2\). Then \(F(z)\) is a polynomial.

Pólya also observed the case \(F(\mathbb{Z}) \subset \mathbb{Z}\) as follows.

**Theorem B (Pólya).** Let \(F(z)\) be an entire function in \(\mathbb{C}\) with \(F(\mathbb{Z}) \subset \mathbb{Z}\). Suppose \(\alpha(F) < \log \left(\frac{3 + \sqrt{5}}{2}\right)\). Then \(F(z)\) is a polynomial.

\(2010\) Mathematics Subject Classification: 11C08, 11K60, 11Y70, 30D15.

*Corresponding author. E-mail: hirata@math.cst.nihon-u.ac.jp

\(^{†}\)The second author is supported by the NEXT Program, GR 087, JSPS for 2010–2013.
We see that \( F(z) \) in theorems of Pólya is necessarily a polynomial in \( \mathbb{Q}[z] \), but not in \( \mathbb{Z}[z] \) [consider for example, \( \frac{1}{2}(z + 1) \)]. The bounds of the order \( \log 2 \) and \( \log(\frac{1 + \sqrt{5}}{2}) \) are optimal because of the existence of the functions \( 2^z \) and \( (\frac{1 + \sqrt{5}}{2})^z \).

It is equivalent to say that \( F(z) \) is a polynomial with coefficients in \( \mathbb{Q} \), and that the functions \( z^k F(z)^k \) \((h, k \in \mathbb{N} \cup \{0\})\) are linearly dependent over \( \mathbb{Q} \). Then a natural generalization of works of Pólya is to seek a sufficient condition such that several functions, say, \( f_1(z), \ldots, f_t(z) \), are linearly dependent over \( \mathbb{Q} \).

For \( \xi_1, \xi_2, \cdots \in \mathbb{C} \) denote by \( r(N) := \max_{1 \leq k \leq N} |\xi_k| \). We proved in [5]:

**Theorem C ([5]).** Let \( L \) and \( N_0 \) be integers with \( 1 < N_0 < L \). There are constants \( C_1 > 0 \) and \( C_2 > 0 \) depending only on \( L, N_0 \) satisfying the following. Let \( \xi_1, \xi_2, \cdots \in \mathbb{C} \) be infinitely many pairwise distinct complex points. Let \( f_1(z), \ldots, f_t(z) \) be entire functions in \( \mathbb{C} \). Suppose \( f_j(\xi_k) \in \mathbb{Z} \) for any \( j, n \) with \( 1 \leq j \leq L \) and \( n \geq 1 \). If we have \( \max_{1 \leq j \leq L} \log |f_j(\xi_n)| \leq C_2 N \) for any \( N \geq N_0 \), then the functions \( f_1(z), \ldots, f_t(z) \) are linearly dependent over \( \mathbb{Q} \).

A consequence of Theorem C is for instance, a weaker version of Theorem A:

**Corollary D ([5]).** Let \( F(z) \) be an entire function in \( \mathbb{C} \) with \( F(\mathbb{N}) \subset \mathbb{Z} \). Suppose \( \alpha(F) \leq \frac{1}{30} \). Then \( F(z) \) is a polynomial over \( \mathbb{Q} \).

**Proof.** Consider \( \frac{z(z - 1) \cdots (z - h + 1)}{h!} \cdot F(z)^k \) as \( f_1(z), \ldots, f_t(z) \) in Theorem C \((h, k \in \mathbb{N} \cup \{0\})\). \( \square \)

## 2. Results for an Almost Integer-Valued Function

Our aim in the present article is to relax the hypothesis of theorem of Pólya from the viewpoint of Diophantine approximations.

**Definition 2.** We write \( \|z\| := \min |z - m| \), the distance between \( z \in \mathbb{C} \) and the nearest integer.

In this section, by means of computational optimization joined with Diophantine approximation method, we prove the following results. These theorems show that an almost integer-valued entire function is a polynomial over \( \mathbb{Q} \), provided that the growth of the function is sufficiently bounded.

**Theorem 1.** Let \( f(z) \) be an entire function in \( \mathbb{C} \) with \( \alpha(f) \leq 0.00437155 \). Suppose

\[
\|f(n)\| < e^{-663143n}
\]

for all sufficiently large \( n \in \mathbb{N} \). Then \( f(z) \in \mathbb{Q}[z] \).

A slight modification of the proof enables us to obtain:

**Theorem 2.** Let \( f(z) \) be an entire function in \( \mathbb{C} \) with \( \alpha(f) \leq 8.775 \times 10^{-6} \). Suppose

\[
\|f(n)\| < e^{-2n}
\]

for all sufficiently large \( n \in \mathbb{N} \). Then \( f(z) \in \mathbb{Q}[z] \).

We note 0.00437155 > \( \frac{1}{229} \) and 8.775 \times 10^{-6} > \( \frac{1}{113961} \).

Theorems 1 and 2 show, not only an integer-valued entire function but also an almost integer-valued function, namely a function taking values very close to integers, may turn out to be a polynomial. This is an improvement of a result in [5] concerning with a growth condition or an assumption for the distance from the nearest integer. Theorems 1 and 2 give explicit versions of a theorem due to Ch. Pisot dealing with an almost integer-valued function (Théorème 2 in [7]). We prove both theorems by so-called Schneider’s method (refer its applications found in [3, 10]) of Diophantine approximations, that differs from Pisot’s original proof.

## 3. Computational Lemmata

We collect here a few classical estimates. Throughout the article, we denote \( \varphi(x) = x \log x \).

**Lemma 3.** Let \( N_0 \) be a sufficiently large integer. Let \( h_0 \in \mathbb{Q} \) such that \( h_0 N_0 \in \mathbb{Z} \), \( 0 < h_0 < \frac{1}{2} \). Let \( n, h \) be integers with \( 0 \leq n \leq N_0 - 1 \), \( 1 \leq h \leq h_0 N_0 \). Then we have

\[
\log \left| \binom{n}{h} \right| \leq -N_0 [\varphi(h_0) + \varphi(1 - h)] + o(N_0).
\]

**Proof.** Since \( h_0 < \frac{1}{2} \), we have
Let $F$ be an entire function in $\mathbb{C}$ and $N \geq 2$ be an integer. Put $\Lambda = \lambda^{\frac{1}{2} - 1} (\lambda^2 - 1)^{1/2} > 1$. There exists a constant $C$ with $\frac{1}{2} < C < 4$ satisfying the following.

Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(n) = 0$ for all $n = 0, 1, \ldots, N - 1$. Then we have

$$\log |F(N)| \leq \log |F|_{1N} - N \log \Lambda + C \log N. \quad (3)$$

If $N \geq 10^5$ then we have a more precise estimate:

$$\log |F(N)| \leq \log |F|_{1N} - N \log \Lambda + \frac{1}{2} \log(N + 1) + \frac{1}{2} \log \frac{2\pi(\lambda^2 - 1)}{\lambda^2} + \theta \quad (4)$$

with

$$- \frac{73}{36} < \theta < - \frac{10}{9} + \frac{1}{2} \log \frac{\lambda^2}{\lambda^2 - 1}.$$

Proof. We put

$$Q(z) = Q_N(z) = \prod_{n=0}^{N-1} \frac{\Lambda N(z - n)}{\lambda^2 N^2 - zn}.$$  

The function $Q$ and $\frac{F}{Q}$ are analytic in the disk $|z| \leq \lambda N$.

Since $|Q(z)| = 1$, we have for $|z| = \lambda N$:

$$\left| \frac{F(N)}{Q(N)} \right| \leq \left| \frac{F}{Q} \right|_{1N} \leq |F|_{1N}. \quad (5)$$

Therefore we obtain $|F(N)| \leq Q(N)|F|_{1N}$.

On the other hand, by an asymptotic expansion of the logarithm of the Gamma function (called Stirling’s series, see [12]), we have for $x > 0$,

$$\Gamma(x) = x^{1/2} e^{-x} (2\pi)^{1/2} e^{\theta_0}$$

with certain $\theta_0$, $0 < \theta_0 < 1$.

It is well-known that Euler’s formula implies for $z$ which is not a negative integer (chapter XII, §12, page 237 of [12]):

$$\Gamma(z + 1) = z \Gamma(z),$$

consequently we have

$$\Gamma(\lambda^2 N + 1) = \lambda^2 N (\lambda^2 N - 1) \cdots (\lambda^2 N - N + 1) \cdot \Gamma(\lambda^2 N - N + 1)$$

from which we get

$$\prod_{n=0}^{N-1} (\lambda^2 N - n) = \frac{\Gamma(\lambda^2 N + 1)}{\Gamma(\lambda^2 N - N + 1)}. \quad (6)$$

Hence we obtain

$$|Q(N)| = \frac{(\Lambda N)^N \cdot \Gamma(N + 1) \cdot \Gamma(\lambda^2 N - N + 1)}{\prod_{n=0}^{N-1} (\lambda^2 N - n)}$$

$$= \sqrt{2\pi} \cdot \Lambda^N \cdot (N + 1)^{N+\frac{1}{2}} \cdot (\lambda^2 N - N + 1)^{N+\frac{1}{2}} \cdot e^{\frac{\theta_0}{2(N + 1)^{1/2}}} \cdot e^{\frac{\theta_0}{2(\lambda^2 N - N + 1)^{1/2}}} \cdot (\lambda^2 N + 1)^{N+\frac{1}{2}} \cdot e^{\frac{\theta_0}{2(\lambda^2 N + 1)^{1/2}}}$$

with $0 < \theta_i < 1$ ($1 \leq i \leq 3$).

Thanks to a simple estimate for $x > 0$:

$$(x + 1)^x = x^x \cdot e^{\theta}$$

with certain $0 < \theta < 1$, we have
\[ \log |Q(N)| = -N \log \Lambda + \frac{1}{2} \log(N + 1) + \frac{1}{2} \log \frac{A^2 N - N + 1}{A^2 N + 1} + \frac{1}{2} \log 2\pi - 1 + a_1 + a_2 - a_3 \]
\[ + \frac{\theta_1}{12(N + 1)} + \frac{\theta_2}{12(A^2 N - N + 1)} + \frac{\theta_3}{12(A^2 N + 1)} \]
\[ = -N \log \Lambda + \frac{1}{2} \log(N + 1) + \frac{1}{2} \log \left( 2\pi \left( 1 - \frac{1}{A^2} \right) \right) + \theta \]

where \( 0 < a_i < 1 \) (\( 1 \leq i \leq 3 \)), \( 0 < \theta_i < 1 \) (\( 1 \leq i \leq 3 \)), \( -\frac{73}{36} < \theta < \frac{10}{9} + \frac{1}{2} \log \frac{A^2}{A^2 - 1} \).

This completes the proof. \( \square \)

**Lemma 5.** Let \( f(z) \) be an analytic function on a disk \( |z| \leq R \) in \( \mathbb{C} \). Let \( \zeta_0, \ldots, \zeta_l \) be pairwise distinct points in \( |z| < R \). Then we have

\[ |f(\zeta_0)| \leq E_1 + E_2 \]

where

\[ E_1 = |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{\ell} \frac{|\zeta_0 - \zeta_n|}{R - |\zeta_n|}, \]

\[ E_2 = \prod_{k=1}^{\ell} |\zeta_0 - \zeta_k| \cdot \sum_{n=1}^{\ell} \frac{|f(\zeta_0)|}{|\zeta_0 - \zeta_n|} \prod_{i=1}^{\ell} \frac{1}{|\zeta_i - \zeta_n|}. \]

**Proof.** Put

\[ A(z) = \prod_{n=1}^{\ell} \frac{R - \zeta_n}{z - \zeta_n} \]

and for \( 1 \leq n \leq \ell \) write also

\[ A_n(z) = A(z) \cdot (z - \zeta_n) = \prod_{i=1}^{\ell} (R - \zeta_n) \cdot \prod_{i=1, i \neq n}^{\ell} \frac{1}{z - \zeta_i}. \]

The residue formula gives

\[ A(\zeta_0) f(\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma} A(\zeta) f(\zeta) \frac{d\zeta}{\zeta - \zeta_0} + \sum_{n=1}^{\ell} A_n(\zeta_0) f(\zeta_0) \cdot (\zeta_0 - \zeta_n)^{-1} \]

where we denote \( \Gamma = \{ \zeta \in \mathbb{C}; |\zeta| = R \} \). For \( \zeta \in \Gamma \), we get

\[ |A(\zeta)| \leq \prod_{n=1}^{\ell} \frac{|R - \zeta_n|}{|R - |\zeta_n||} \]

then the required statement appears. \( \square \)

**Lemma 6.** Let \( f(z) \) be an analytic function on a disk \( |z| \leq R \) in \( \mathbb{C} \). Let \( \zeta_0, \ldots, \zeta_l \) be pairwise distinct points in \( |z| < R \). Then

\[ |f(\zeta_0)| \leq G_1 + G_2 \]

where

\[ G_1 = |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{\ell} \frac{|R - \zeta_n|}{|R^2 - \zeta_0 \zeta_n|} \]

and

\[ G_2 = \sum_{n=1}^{\ell} \left( |f(\zeta_n)| \cdot \prod_{i=1}^{\ell} \frac{|R^2 - \zeta_n \zeta_i|}{|R^2 - \zeta_0 \zeta_i|} \prod_{k=1, k \neq n}^{\ell} \frac{|\zeta_0 - \zeta_k|}{|\zeta_n - \zeta_k|} \right) \]

**Proof.** Write Blaschke product by
Proof of Theorem 1

Let $N_0$ be a sufficiently large integer. Let $h_0 \in \mathbb{Q}$ such that $h_0N_0 \in \mathbb{Z}, 0 < h_0 < \frac{1}{2}$. Let $k_0 \in \mathbb{Z}$ with $k_0 \geq 1$. Let $g > 0$ such that

$$
\|f(n)\| < e^{-gn}.
$$

Finally, let $\lambda > 1$ be a parameter $\in \mathbb{R}$.

[First step]: construction of an auxiliary function.

Write

$$
f(z) = \left( \frac{z}{h} \right)^k f(z) \]

for $1 \leq h \leq h_0N_0, 0 \leq k \leq k_0 - 1, 1 \leq j = h(k + 1) \leq L = h_0k_0N_0 \in \mathbb{Z}$.

Suppose that the entire function $f(z)$ satisfies for sufficiently large $r > 0$:

$$
\log |f|_r < \alpha r.
$$

Let $b_n \in \mathbb{Z}$ such that $\|f(n)\| = |f(n) - b_n| \leq e^{-gn}$ for all sufficiently large $n$ and put

$$
a_{jn} = \left( \frac{n}{h} \right) b_n^k
$$

for $1 \leq h \leq h_0N_0, 0 \leq k \leq k_0 - 1, 1 \leq j = h(k + 1) \leq L = h_0k_0N_0$.

We construct an auxiliary function

$$
F(z) = \sum_{1 \leq j \leq L} p_j f(z)
$$

with $p_j \in \mathbb{Z}$ not all zero such that

$$
\sum_{1 \leq j \leq L} p_j a_{jn} = 0, \quad u(N_0 - 1) \leq n \leq N_0 - 1
$$

with a real parameter $u$ satisfying $0 < u < 1$.

By definition of $a_{jn}$ and our assumption on the growth of $f(z)$, if $h_0 < \min(u, 1/2)$, then we have

$$
\log \max_{1 \leq j \leq L} |a_{jn}| \leq c_1N_0 + o(N_0)
$$

where

$$
c_1 = \alpha(k_0 - 1) - \varphi(h_0) - \varphi(1 - h_0).
$$

Suppose $h_0k_0 > 1 - u$. Then by Siegel’s lemma (confer for example [11]) there exist integers $p_j, 1 \leq j \leq L$ which are not all zero such that

$$
\max_{1 \leq j \leq L} \log |p_j| \leq \frac{1 - u}{h_0k_0 + u - 1} \times c_1N_0 + o(N_0).
$$

For $N \geq N_0$ and $\lambda > 1$, it follows

$$
\log |F|_{\lambda N} \leq \left( \frac{1 - u}{h_0k_0 + u - 1} \times c_1 + c_2 \right) N + o(N) + o(N_0)
$$
with
\[ c_2 = \varphi(\lambda + h_0) - \varphi(\lambda) - \varphi(h_0) + \alpha\lambda(k_0 - 1). \]

**[Second step]: extrapolation.**

We consider for each \( N \geq N_0 \) the following two properties:
\[
A(N) : \quad \sum_{1 \leq j \leq L} p_j a_m = 0, \quad u(N - 1) \leq n \leq N - 1, \\
B(N) : \quad \log |F|_N < -\gamma N,
\]
where \( \gamma > 0 \) is independent of \( N_0 \) and \( N \).

We shall show \( A(N) \Rightarrow B(N) \) and \( B(N) \Rightarrow A(N + 1) \).

**Proof of** \( A(N) \Rightarrow B(N) \). For \( u(N - 1) \leq n \leq N - 1 \) and \( 0 \leq k \leq k_0 - 1 \), we have:
\[
\max_{1 \leq j \leq L} \log |f_j(n) - a_m| \leq \max_{1 \leq j \leq L} \log \left( \frac{n}{h_j} |f(n)^k - b_n^k| \right) \\
\leq c_1 N - \alpha N - guN + o(N).
\]
By hypothesis \( A(N) \), it follows
\[
\log \sum_{u(N - 1) \leq n \leq N - 1} |F(n)| \leq \frac{h_0 k_0}{h_0 k_0 + u - 1} c_1 \times N - (\alpha + gu)N + o(N) + o(N_0)
\]
By Lemma 6, we have
\[
|F|_N \leq T_1 + T_2
\]
with
\[
T_1 = |F|_N \frac{\lambda}{\lambda - 1} \prod_{n} \frac{A(N + n)}{\lambda^2 (N + 1)^2 + N n}
\]
and
\[
T_2 = \left( \sum_{n} |F(n)| \right) \frac{1}{(\Gamma(\frac{1}{2} N - 1))^2} \prod_{n} \frac{(\lambda^2 (N - 1) + N(N + n))}{\lambda^2 N^2 + N n}
\]
because \((z - n)(\lambda^2 N^2 - zn)^{-1}\) takes the maximal absolute value on the disk \(|z| = N\) at the point \( z = -N \).

Therefore we obtain
\[
\log T_1 \leq \log |F|_N + N(1 - u) \log \lambda + \log \frac{\Gamma(2N) \cdot \Gamma((\lambda^2 + u)N)}{\Gamma((1 + u)N) \cdot \Gamma((\lambda^2 + 1)N) + o(N) + o(N_0)} \\
\leq \left( \frac{1 - u}{h_0 k_0 + u - 1} \times c_1 + c_2 + c_3 \right) N + o(N) + o(N_0)
\]
where
\[
c_3 = (1 - u) \log \lambda + \varphi(2) + \varphi(\lambda^2 + u) - \varphi(1 + u) - \varphi(\lambda^2 + 1)
\]
and also
\[
\log T_2 \leq \log \left( \sum_{n} |F(n)| \right) + \log \frac{\Gamma((\frac{1}{2} N - u)N) \cdot \Gamma((\lambda^2 + u)N) \cdot \Gamma(2N)}{\Gamma((\frac{1}{2} N - 1)N) \cdot \Gamma((\lambda^2 + 1)N) \cdot \Gamma((1 + u)N) \cdot \Gamma((\lambda^2 + 2)N)\Gamma(\frac{1}{2} N)} \\
+ N(1 - u) \log u + o(N) + o(N_0) \\
\leq \left( \frac{h_0 k_0}{h_0 k_0 + u - 1} \times (\alpha + gu) + c_4 \right) N + o(N_0) + o(N)
\]
where
\[
c_4 = \varphi(\frac{\lambda^2}{u} - u) + \varphi(\lambda^2 + u) + \varphi(2) - \varphi(\frac{\lambda^2}{2} - 1) - \varphi(\lambda^2 + 1) - \varphi(1 + u) - 2\varphi \left( \frac{1 - u}{2} \right) + (1 - u) \log u.
\]
Hence the property \( B(N) \) is true for \( N_0 \) sufficiently large, whenever the following inequalities are satisfied:
\[
1 - u \times c_1 + c_2 + c_3 < 0 \quad (6)
\]
and
We apply Liouville’s theorem. Obviously:

\[
\frac{h_0 k_0}{h_0 k_0 + u - 1} \times c_1 - (\alpha + gu) + c_4 < 0. \tag{7}
\]

Proof of \(B(N) \implies A(N + 1)\). We are going to show that the property \(B(N)\) implies \(A(N + 1)\).

We apply Liouville’s theorem. Obviously:

\[
\left| \sum_j p_j a_j \right| \leq \left| \sum_j p_j (a_j - f_j(N)) \right| + |F(N)|
\]

where by assumption \(B(N)\) we have,

\[
\log |F(N)| \leq \log |F|_N < -\gamma N.
\]

On the other hand, Lemma 3 shows

\[
\log \left| \sum_j p_j (a_j - f_j(N)) \right| \leq \left( \frac{1 - u}{h_0 k_0 + u - 1} \times c_1 - \varphi(h_0) - \varphi(1 - h_0) + (k_0 - 2)\alpha - g \right) N + o(N) + o(N_0).
\]

Then the property \(A(N)\) is satisfied when \(N_0\) is sufficiently large and whenever the following condition holds:

\[
\frac{1 - u}{h_0 k_0 + u - 1} \times c_1 - \varphi(h_0) - \varphi(1 - h_0) + (k_0 - 2)\alpha - g < 0. \tag{8}
\]

Carrying out technical computational optimizations, we choose parameters such that the three conditions (6), (7), and (8) are simultaneously satisfied.

We set:

\[
h_0 = 0.108496, \; k_0 = 16, \; u = 0.108496, \; \lambda = 11.9194, \; g = 663143,
\]

then the inequalities (6), (7), and (8) are verified with

\[
\alpha = 0.00437155 \left( \geq \frac{1}{229} \right).
\]

[Third step]: conclusion.

By the construction of the auxiliary function \(F(z)\), the property \(A(N_0)\) is true. Since \(A(N) \implies B(N)\) and \(B(N) \implies A(N + 1)\), both properties \(A(N)\) and \(B(N)\) hold for any \(N \geq N_0\). If the function \(F(z)\) is not identically zero, then by Liouville’s theorem, we obtain for \(N \to \infty\):

\[
|F|_N \to \infty
\]

which contradicts \(B(N)\).

As a consequence, the function \(F(z)\) is identically zero, namely the functions \(f_j(z)\) are linearly dependent over \(\mathbb{Q}\). Since it is equivalent to say that \(f(z)\) is a polynomial with coefficients in \(\mathbb{Q}\) and that the functions \(f_j(z) = \left( \frac{1}{h} \right)^k f(z)^k (h, k \in \mathbb{N} \cup \{0\})\) are linearly dependent over \(\mathbb{Q}\), hence the proof of Theorem 1 is achieved.

5. Proof of Theorem 2

Take

\[
h_0 = 0.000991044, \; k_0 = 640, \; u = 0.78256, \; \lambda = 1.5253, \; g = 3,
\]

then the conditions (6), (7), and (8) are valid with

\[
\alpha = 8.775 \times 10^{-6}
\]

which establishes Theorem 2.

REFERENCES


