Josephson effects in ferromagnetic superconductors: Noise effect due to critical spin fluctuations

Koyama T., Tachiki M.,
Physical Review. B
volume 30
number 11
page range 6463-6479
year 1984
URL http://hdl.handle.net/10097/53223
doi: 10.1103/PhysRevB.30.6463

<table>
<thead>
<tr>
<th>著者</th>
<th>作者名</th>
</tr>
</thead>
<tbody>
<tr>
<td>當代</td>
<td>Koyama T.</td>
</tr>
<tr>
<td>當代</td>
<td>Tachiki M.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>雑誌名</th>
<th>物理レビュー B</th>
</tr>
</thead>
<tbody>
<tr>
<td>巻</td>
<td>30</td>
</tr>
<tr>
<td>号</td>
<td>11</td>
</tr>
<tr>
<td>印刷</td>
<td>6463-6479</td>
</tr>
<tr>
<td>年</td>
<td>1984</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10097/53223">http://hdl.handle.net/10097/53223</a></td>
</tr>
<tr>
<td>doi</td>
<td>10.1103/PhysRevB.30.6463</td>
</tr>
</tbody>
</table>
Josephson effects in ferromagnetic superconductors: Noise effect due to critical spin fluctuations

T. Koyama and M. Tachiki
The Research Institute for Iron, Steel and Other Metals, Tohoku University, 2-1-1 Katahira, Sendai 980, Japan
(Received 3 February 1984)

A theory for the Josephson effect in the ferromagnetic superconductors is presented. Taking account of critical fluctuations of the localized-spin magnetization, we calculate the magnetic field dependence of the maximum Josephson current and $I-V$ characteristics. The Josephson current is reduced by a noise effect arising from the critical spin fluctuations at low temperatures. The anomalous behaviors observed in the dc Josephson effect in ErRh$_4$B$_4$-Lu$_6$O$_7$-In junctions are interpreted on the basis of present theory.

I. INTRODUCTION

Rare-earth ternary compounds such as ErRh$_4$B$_4$ and HoMo$_6$S$_8$ exhibit a reentrant transition from superconducting to ferromagnetic state at a lower critical temperature $T_c$. The coexistence of periodic magnetic order with superconductivity is also realized in these systems in a small temperature range just above $T_c$. The interplay between superconductivity and magnetism leads to the occurrence of unusual superconducting and magnetic properties near the reentrant transition temperature. In tunneling experiments using the junction made by the ferromagnetic superconductors, anomalous behavior has been found. The dc Josephson current observed in ErRh$_4$B$_4$ by Umbach and Goldman reveals the following outstanding features. As the temperature is lowered, the maximum Josephson current takes a maximum value at a low temperature and falls rapidly to zero at a certain temperature slightly above the reentrant temperature. The external field dependence of the maximum Josephson current also exhibits unusual behavior—Fraunhofer-type diffraction-pattern shifts and splitting of its central peak upon cooling just before the Josephson current is quenched.

In this paper we study the Josephson effect in systems such as ErRh$_4$B$_4$ and HoMo$_6$S$_8$. We should use, in general, a model in which superconducting tunneling electrons interact with rare-earth magnetic moments through both the electromagnetic and $s$-$f$ exchange interaction, in order to understand the Josephson effect in these compounds since the $s$-$f$ exchange interaction reduces the Josephson-current density $j_0$ by the pair-breaking effect, and the electromagnetic interaction controls the phase difference across the junction through the Maxwell equations. The single-particle tunneling experiments show that the temperature dependence of the superconducting energy gap is rather mild up to $T_c$. We may then approximate incorporate the effects of the $s$-$f$ exchange interaction through renormalizing the various parameters, assuming that the temperature dependence of $j_0$ arising from the pair-breaking effect is weak. In this paper, therefore, we explicitly take into account only the electromagnetic interaction between the pair tunneling current and the localized-spin magnetization, and investigate the characteristics of the Josephson junction in the ferromagnetic superconductors.

In previous papers we predicted that spontaneous magnetization bound to the surface of ferromagnetic superconductors appears in a narrow temperature range just above the transition temperature of the periodic phase. This is caused by the weakening of the diamagnetic screening effect on the interaction between rare-earth magnetic moments near the surface. Since the Josephson junction may be regarded as a kind of surface, the surface magnetization is expected to appear around the junction. The Josephson current may therefore be greatly affected by the rare-earth moments at temperatures near $T_c$.

In this paper we confine ourselves to the case for $T > T_g$. Critical fluctuations of the localized-spin magnetization near the surface arise in the temperatures just above $T_g$. Since the fluctuations of the localized-spin magnetization can cause fluctuations of the local flux density around the junction, the phase difference of the superconducting order parameter fluctuates. Thus the critical fluctuations of the localized-spin magnetization are expected to work as a noise source for the Josephson current. We therefore study the effect of the fluctuations on the Josephson current and show that the Josephson current is quenched at $T_g$ due to the fluctuations.

The structure of this paper is as follows. In Sec. II basic equations for the phase difference and localized-spin magnetization are derived from the Maxwell equations and the time-dependent Ginzburg-Landau (GL) theory. In Sec. III, taking random forces into account, we set up Langevin-type stochastic equations and derive a Fokker-Planck equation. The maximum dc Josephson current and the current-voltage characteristic are calculated using solutions of the Fokker-Planck equation. Section IV is devoted to a discussion of the experimental results of the dc Josephson effect on the basis of our present theory.

II. DERIVATION OF BASIC EQUATIONS

A. Equation for the phase difference

Let us consider a tunneling junction made by two ferromagnetic superconductors separated by a thin dielectric
layer. The electromagnetic behaviors of the tunneling junction are described by the Maxwell equations,
\[ \begin{align*} 
\nabla \times \mathbf{h} &= \frac{4\pi}{c} \mathbf{J} + \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t}, \\
\nabla \times \mathbf{E} &= \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, 
\end{align*} \]
(2.1)
(2.2)
where \( \epsilon \) and \( c \) are dielectric permeability around the junction and light velocity in free space, respectively. For the superconducting current \( \mathbf{J} \) in (2.1), we assume London's equation in a gauge-invariant form:
\[ \mathbf{J} = -\frac{c}{4\pi \lambda_L^2} \left( \frac{\Phi_0}{2e} \nabla \phi \right). \]
(2.3)
Here, \( \mathbf{A} \), \( \phi \), and \( \lambda_L \) are, respectively, the vector potential, the phase of the superconducting order parameters, and the London penetration depth. In the case of the magnetic superconductor the magnetic induction field \( \mathbf{b} \) is related to the rare-earth magnetization \( \mathbf{m} \) and the magnetic field \( \mathbf{h} \) as
\[ \mathbf{b} = \mathbf{h} + 4\pi \mathbf{m}. \]
(2.4)

In Eqs. (2.5) and (2.6) we assume that the phase \( \phi \) has a discontinuity at the junction \( (x_3=0 \text{ plane}) \). This discontinuity leads to a topological singularity in the sense that differential operations on \( \phi \) do not commute, so that the commutators in (2.5) and (2.6) do not vanish. This circumstance is understood in the following way. Consider a contour integral, \( \oint_C \nabla \phi \cdot d\mathbf{I} \), whose path is shown in Fig. 2. When the path \( C \) is divided into the paths \( C_1 \) and \( C_2 \) at the junction, the integral can be evaluated in terms of the phase jump on the junction,
\[ \oint_C \nabla \phi \cdot d\mathbf{I} = \int_{C_1} \nabla \phi \cdot d\mathbf{I} - \int_a^b \partial_1 \phi(x_1,x_3=-0,t)dx_3 + \int_{C_2} \nabla \phi \cdot d\mathbf{I} + \int_a^b \partial_1 \phi(x_1,x_3=+0,t)dx_3 \]
\[ = \frac{\Phi_0}{2e} \left( \partial_3 \phi - \frac{4\pi}{c} \partial_3 \partial_2 m(x_1,x_3) \right), \]
(2.5)

where \( c^2 = \frac{\epsilon^2 + \lambda_L^2}{\epsilon} \) and \( [\partial_1, \partial_2] = \partial_1 \partial_2 - \partial_2 \partial_1 \).
(2.7)

On the other hand, an application of the Stokes theorem for the integral gives the relation
\[ \oint_C \nabla \phi \cdot d\mathbf{I} = \int \nabla \times \nabla \phi \cdot d\mathbf{S} = \int_C [\partial_3, \partial_1] \phi dx_1 dx_3, \]
(2.10)
This result indicates the relation
\[ [\partial_3, \partial_1] \phi = \partial_1 \gamma(x_1,t) \delta(x_3). \]
(2.11)
Repeating the same arguments on the \( t-x_3 \) plane, we can also obtain
\[ \gamma(x_1,t) = \phi(x_1,x_3=0+, t) - \phi(x_1, x_3=0-, t). \]
(2.9)
JOSEPHSON EFFECTS IN FERROMAGNETIC SUPERCONDUCTORS: ... 6465

\[ \partial_{\gamma} \gamma = \partial_{x} \gamma \delta(x_1, t) \delta(x_3) . \]  

(2.12)

Now, general solutions to Eqs. (2.5) and (2.6) are obtained in terms of the phase difference and the rare-earth magnetization:

\[
h(x_1, x_3, t) = h_0(x_1, x_3, t) + \frac{1}{\epsilon^2} \partial_x^2 \left[ \frac{\hbar c}{2e} \partial_{x} \gamma(x_1, t) \delta(x_3) - \frac{4\pi^2}{\lambda_L^2} m(x_1, x_3, t) - \frac{4\pi}{c} \partial_{x} \gamma \delta(x_3) - \frac{4\pi}{c} \partial_{x} \partial_{x} \gamma \delta(x_3) \right] ,
\]

(2.13)

\[
E(x_1, x_3, t) = E_0(x_1, x_3, t) + \frac{1}{\epsilon^2} \partial_x^2 \left[ \frac{\hbar c}{2e} \partial_{x} \gamma(x_1, t) \delta(x_3) - \frac{4\pi}{c} \partial_{x} \gamma \delta(x_3) - \frac{4\pi}{c} \partial_{x} \partial_{x} \gamma \delta(x_3) \right] .
\]

(2.14)

Here, \( h_0 \) and \( E_0 \) are the solutions to the homogeneous equations:

\[
\begin{align*}
-\partial_x^2 \partial_x^2 + \frac{1}{\epsilon^2} \partial_x^2 + \lambda_L^{-2} h_0(x_1, x_3, t) & = 0 , \\
-\partial_x^2 \partial_x^2 + \frac{1}{\epsilon^2} \partial_x^2 + \lambda_L^{-2} E_0(x_1, x_3, t) & = 0 .
\end{align*}
\]

(2.15) (2.16)

Let us make the following assumptions: (a) The spatial variation along the \( x_1 \) direction is negligible in the scale of the London penetration depth \( \lambda_L \), and (b) the characteristic frequency of the system, \( \omega_0 \), is small, i.e., \( \lambda_L \omega_0 / \epsilon \ll 1 \). Under these assumptions we can neglect the nonlocality along the \( x_1 \) direction and the retardation effect. Then, we have the following approximate solutions from (2.13) and (2.14):

\[
h(x_1, x_3, t) = h_0(x_1, x_3, t) + \frac{\hbar c}{2e} \partial_x \gamma(x_1, t) e^{-|x_3| / \lambda_L} - \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|y_3| / \lambda_L} m(x_1, x_3, t) ,
\]

(2.17)

\[
E(x_1, x_3, t) = E_0(x_1, x_3, t) + \frac{\hbar c}{4\pi \lambda_L} \partial_x \gamma(x_1, t) e^{-|x_3| / \lambda_L} ,
\]

(2.18)

with \( \phi_0 \) being the unit flux \( \hbar c / 2e \).

Next we calculate the current passing through the junction, \( J(x_1, t) \). From Eqs. (2.1), (2.17), and (2.18), \( J(x_1, t) \) is given by

\[
J(x_1, t) = \frac{4\pi}{c} \int h(x_1, x_3, t) = \frac{4\pi}{c} I_0 x_1 ,
\]

(2.20)

\[
E_0(x_1, x_3, t) = 0 ,
\]

(2.21)

with \( I_0 \) being the bias-current density, so that relation (2.19) reduces to

\[
J(x_1, t) = \frac{4\pi}{c} I_0 x_1 + \frac{\phi_0}{4\pi \lambda_L} \left( \partial_x^2 \gamma(x_1, t) - \frac{1}{\epsilon^2} \partial_x^2 \gamma(x_1, t) \right)
\]

\[- \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|y_3| / \lambda_L} \partial_x m(x_1, y_3, t) .
\]

(2.22)

The current obtained above may be identified with the pair tunneling Josephson current and the single-quasiparticle current across the Josephson junction:

\[
J(x_1, t) = j_0 \sin[\gamma(x_1, t)] + GV(x_1, t)/L .
\]

(2.23)

Here, \( j_0 \) is the maximum pair tunneling current density, and \( G \) and \( V(x_1, t) \) are, respectively, the conductance and voltage associated with the single-particle tunneling. For simplicity, we assume that \( G \) is independent of \( V(x_1, t) \). We note that the voltage is related to the phase difference by the well-known relation

\[
\frac{\partial \gamma(x_1, t)}{\partial t} = \frac{2e}{\hbar} V(x_1, t) ,
\]

(2.24)

which is also obtained from (2.18) in the case of \( E_0 = 0 \). Now we arrive at the equation for the phase difference. Using (2.22), (2.23), and (2.24), we obtain
where $\lambda_J$ is the Josephson penetration depth,

$$
\lambda_J = \left(\frac{8\pi^2 (2\lambda_L)}{\phi_0 c} - j_0 \right)^{-1/2}
\quad \text{and} \quad g = \frac{8\pi \lambda_L}{\epsilon L} G.
$$

B. Dynamical equation for the localized-spin magnetization

In order to investigate the time-dependent phenomena, we need one more equation for describing dynamical behaviors of the localized spins. In the following we set up an equation for $m$ based on the time-dependent Ginzburg-Landau theory. For this purpose let us first study the free energy of our system in the equilibrium state.

The free energy in our system is written as

$$
F = \int d^2x \frac{\hbar j_0}{2e} \left[1 - \cos[\gamma(x_1)] \right] + \int d^3x \left[ \frac{1}{2}nm_e v^2 + \frac{b^2}{8\pi} - bm \right] + F_m,
$$

where the first term is the surface energy associated with the phase difference across the junction, the term $\frac{1}{2}nm_e v^2$ is the kinetic energy density of superconducting electrons, $n, m_e$, and $v$ being, respectively, the number density, effective mass, and velocity of the electrons, and $F_m$ is the free energy of the localized-spin system. Noting the definition of the London penetration depth,

$$
\lambda^2_L = \frac{m_e e^2}{4\pi n e^2},
$$

and using (2.3), we can rewrite the second integral in (2.27) as

$$
\int d^3x \left[ \frac{1}{8\pi} \left( \frac{\hbar \kappa}{2e} \nabla \phi \right) + \frac{b^2}{8\pi} - bm \right]
\quad \text{as}
$$

This is also expressed in terms of $\gamma$ and $m$ by using (2.11), (2.17), and (2.20):

$$
\int dx_1 dx_2 \left[ \frac{\phi_0^2}{64\pi^3 \lambda_L} \left[ \partial_1 \gamma(x_1) \right]^2 + \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1) \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} m(x_1, x_3) 
\right.
$$

$$
+ \frac{1}{2} \int dx_1 m(x_1, x_3) \left[ -4\pi + \frac{4\pi \lambda_L^2}{-\partial_3^2 + \lambda_L^2} \right] m(x_1, x_3) - \frac{\phi_0}{2\pi c} I_0 \gamma(x_1)
\left. - \frac{2\pi}{c} I_0 x_1 \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} m(x_1, x_3) \right].
$$

For the localized-spin system in paramagnetic phase we assume that the free energy $F_m$ can be expanded in powers of $m$ up to second order in the sense of the Landau theory:
\[ F_m = \int d^3 x \frac{1}{2} m(x_1, x_3) \alpha_0(T, \vec{V}) m(x_1, x_3) , \] (2.31)

where \( \alpha_0(T, \vec{V}) \) is the expansion coefficient and its uniform component is assumed to change its sign at temperature \( T_m^0 \).

Making use of the above results, we finally obtain the free energy in terms of \( \gamma(x_1) \) and \( m(x_1, x_3) \):

\[
F = \int dx_1 dx_2 \left\{ \frac{\phi_0 J_0}{2 \pi c} \left[ 1 - \cos[\gamma(x_1)] \right] + \frac{\phi_0}{64 \pi^3 \lambda_L} \left[ \partial_1 \gamma(x_1) \right]^2 - \frac{\phi_0}{4 \pi \lambda_L} \partial_1 \gamma(x_1) \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} m(x_1, x_3) + \int dx_3 \frac{1}{2} m(x_1, x_3) \bar{\alpha}(T, \vec{V}) m(x_1, x_3) - \frac{\phi_0}{2 \pi c} I_0 \gamma(x_1) - \frac{\pi}{c} I_0 x_1 \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} m(x_1, x_3) \right\} ,
\]

where

\[
\bar{\alpha}(T, \vec{V}) = \alpha(T, \vec{V}) + \frac{4 \pi \lambda_L^{-2}}{\lambda_2^2 + \lambda_L^2} ,
\]

with

\[
\alpha(T, \vec{V}) = \alpha_0(T, \vec{V}) - 4 \pi .
\]

It is noted that magnetic ordering of the localized spin appears in a bulk system at the highest temperature, \( T_p \), which satisfies \( \bar{\alpha}(T_p, \vec{V}) = 0.20 \). We denote this \( T_p \) the transition temperature of the periodic phase observed in the ferromagnetic superconductors. We also introduce a ferromagnetic Curie temperature in the normal state obtained from \( \bar{\alpha}(T_m, 0) = 0 \). \( T_p \) is lower than \( T_m \) due to the diamagnetic screening effect by the superconducting current. Before proceeding to the time-dependent case, we briefly study the properties which the free energy (2.32) leads to. Variations of (2.32) by \( \gamma \) and \( m \) yield the following equations:

\[
\begin{align*}
\frac{\partial}{\partial T} \gamma(x_1) - \frac{8 \pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-x_3/\lambda_L} \partial_1 m(x_1, x_3) \\
= \frac{1}{\lambda_L^2} \sin[\gamma(x_1)] - \frac{1}{\lambda_L^2} (I_0/J_0) ,
\end{align*}
\]

(2.35)

where

\[
\bar{\alpha}(T, \partial_3) m(x_1, x_3) = \frac{\phi_0}{4 \pi \lambda_L} \bar{\partial}_1 \gamma(x_1) e^{-x_3/\lambda_L} + \frac{2 \pi}{c} I_0 x_1 .
\]

(2.36)

Introducing the inverse operator \( \alpha^{-1}(T, \partial_3) \), we can rewrite Eq. (2.36) as

\[
m(x_1, x_3) = \frac{\phi_0}{4 \pi \lambda_L} \partial_1 \gamma(x_1) \bar{\alpha}^{-1}(T, \partial_3) e^{-x_3/\lambda_L} + \frac{2 \pi}{c} I_0 x_1 \bar{\alpha}^{-1}(T, 0) .
\]

(2.37)

Here, the second term on the right-hand side in (2.37) is understood to be the localized-spin magnetization induced by the magnetic field due to the bias current. Although the characteristics of the Josephson junction are influenced by the bias current, we neglect its effects for the time being. The effects will be discussed in a later section.

Substituting (2.37) into (2.35) and neglecting the bias current, we obtain

\[
s(T) = \frac{1}{\lambda_L^2} \sin[\gamma(x_1)] ,
\]

(2.38)

with

\[
\partial_1 m(x_1, x_3) = -\beta \frac{\delta F}{\delta m(x_1, x_3)} ,
\]

(2.40)

where \( \beta \) is a positive constant, and the derivative on the right-hand side is understood to be the functional derivative with respect to the function \( m(x_1, x_3) \). In the case of the free energy (2.32), Eq. (2.40) takes the form

\[
\partial_1 m(x_1, x_3) = -\beta \bar{\alpha}(T, \partial_3) m(x_1, x_3, t) + \frac{\beta \phi_0}{4 \pi \lambda_L} \partial_1 \gamma(x_1, t) e^{-x_3/\lambda_L} + \frac{2 \pi}{c} I_0 x_1 .
\]

(2.41)

Introducing

\[
M(x_1, x_3, t) \equiv m(x_1, x_3, t) - \frac{2 \pi}{c} I_0 x_1 \bar{\alpha}^{-1}(T, 0) ,
\]

(2.42)

we can rewrite Eqs. (2.25) and (2.41) as

\[
\frac{\partial}{\partial T} \gamma(x_1, t) - \frac{1}{c^2} \frac{\partial^2}{\partial x_1^2} \gamma(x_1, t) - \frac{8 \pi^2}{\phi_0} \bar{\partial}_1 \gamma(x_1, t) + \bar{\alpha}(T, \partial_3) M(x_1, x_3, t) - \frac{1}{\lambda_L^2} \sin[\gamma(x_1, t)] - \frac{J_0}{\lambda_L^2} - \frac{2 \pi}{c} I_0 x_1 \bar{\alpha}^{-1}(T, 0) ,
\]

(2.43)

\[
\partial_1 M(x_1, x_3, t) = -\beta \bar{\alpha}(T, \partial_3) M(x_1, x_3, t) + \frac{\beta \phi_0}{4 \pi \lambda_L} \partial_1 \gamma(x_1, t) e^{-x_3/\lambda_L} .
\]

(2.44)
with

\[ J_0 = [1 - 2\pi \alpha^{-1}(T_0)] I_0 / J_0 . \]  

Equations (2.43) and (2.44) provide the basic equations for the Josephson effect in the ferromagnetic superconductors.

### III. Noise Effect Due to Critical Fluctuations of the Localized Spin

In the temperature range close to the magnetic transition temperature, critical fluctuations of the localized-spin magnetization arise. The fluctuations couple to the phase difference \( \gamma \) through the relation (2.43) and cause fluctuations of \( \gamma \). As is well known, the fluctuations of the phase difference generate noise currents and produce an important influence on the characteristic of the Josephson junction. In this section we reinterpret Eqs. (2.43) and (2.44) as stochastic ones, taking account of the fluctuations of \( \gamma \) and \( M \), and investigate their effects on the characteristics of the Josephson junction in the ferromagnetic superconductor.

Let \( \gamma(x_1), \eta(x_1) = \partial_x \gamma(x_1), \) and \( M(x_1,x_3) \) be stochastic functions. We set up Langevin-type stochastic equations, introducing random forces into Eqs. (2.43) and (2.44),

(i) \[
\frac{1}{\epsilon^2} \partial_t \eta(x_1,t) = -\frac{\epsilon^2}{\phi_0} \eta(x_1,t) + \partial_x \eta(x_1,t) + \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) \\
- \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] + \bar{\eta}_0 + \frac{1}{\lambda_L^2} R_1(x_1,t)/\epsilon^2 ,
\]

(ii) \[
\partial_t \gamma(x_1,t) = \eta(x_1,t) ,
\]

(iii) \[
\partial_t M(x_1,x_3,t) = -\beta \bar{\alpha}(T,\partial_3) M(x_1,x_3,t) \\
+ \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} \\
+ R_2(x_1,t) ,
\]

with \( \bar{\eta}_0 \) being \( J_0 / \phi_0 \lambda_L^2 \). \( R_1(x_1,t) \) and \( R_2(x_1,t) \) represent the noises, respectively, due to the resistive flow of the quasielectron current and thermal fluctuations of the localized-spin magnetization. It is assumed in the following that \( R_1(x_1,t) \) and \( R_2(x_1,t) \) are statistically independent of one another, and their distributions are given by a Gaussian—white-noise spectrum, for simplicity. We then have the following correlation functions for the noises:

\[
\langle R_1(x_1,t) R_1(x_1',t') \rangle = \frac{2}{RL} k_B T \delta(x_1 - x_1') \delta(t - t') ,
\]

\[
\langle R_2(x_1,t) R_2(x_1',t') \rangle = 2\beta k_B T \delta(x_1 - x_1') \delta(t - t') ,
\]

\[
\langle R_1(x_1,t) R_2(x_1',t') \rangle = 0 .
\]

Here, \( R \) is the resistance of the junction and \( L \) is its extent. Now let us introduce the probability density function, \( p(\gamma(\cdot),\eta(\cdot),M(\cdot),t) \). We denote, hereafter, a functional \( F \) of a function \( f(\cdot) \) by \( F[f(\cdot)] \) and its functional integral by \( \int d\gamma F[f(\cdot)] \). The probability of finding the values of the stochastic functions \( \gamma, \eta, \) and \( M \) in the domains \( \gamma(x), \gamma(x) + \delta \gamma(x) \), \( \eta(x), \eta(x) + \delta \eta(x) \), and \( M(x), M(x) + \delta M(x) \), respectively, at time \( t \) is then expressed by

\[
p(\gamma(\cdot),\eta(\cdot),M(\cdot),t) \delta \gamma(\cdot) \delta \eta(\cdot) \delta M(\cdot) .
\]

We can derive the Fokker-Planck equation for \( p(\gamma(\cdot),\eta(\cdot),M(\cdot),t) \) from Eqs. (3.1)—(3.3) with the use of (3.4)—(3.6), following the standard course of stochastic theory. The equation is given by

\[
\partial_t p[\gamma(\cdot),\eta(\cdot),M(\cdot),t] = \int dx_1 \left[ -\frac{\delta}{\delta \gamma(x_1,t)} \gamma(x_1,t) \\
+ \frac{\delta}{\delta \eta(x_1,t)} \left[ g \eta(x_1,t) \\
- \frac{\epsilon^2}{\phi_0} \partial_x \eta(x_1,t) + \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) \\
- \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] + \bar{\eta}_0 - g \frac{32\pi^2 \lambda_L}{\phi_0} k_B T \frac{\delta}{\delta \eta(x_1,t)} \right] \\
+ \int dx_3 \delta M(x_1,x_3,t) \left[ \frac{\beta \bar{\alpha}(T,\partial_3) M(x_1,x_3,t) - \phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} \\
+ \beta k_B T \frac{\delta}{\delta M(x_1,x_3,t)} \right] \right] \left[ p[\gamma(\cdot),\eta(\cdot),M(\cdot),t] \right] .
\]

Let us first consider the equilibrium state without bias current. Setting \( \partial_t p[\gamma(\cdot),\eta(\cdot),M(\cdot),t] = 0 \), we have the solution
Josephson effects in ferromagnetic superconductors:

\[ p[\gamma(\cdot), \eta(\cdot), M(\cdot)] = C \exp \left[ -\frac{1}{k_B T} \frac{\phi_0^2}{32 \pi^2 \lambda_L^2} \int dx_1 \left( \frac{1}{2 e^2} \eta^2(x_1, t) ight) \right. \]

\[ + \frac{1}{2} \left[ \partial_1 \gamma(x_1, t) \right]^2 - \frac{8 \pi^2}{\phi_0} \partial_1 \gamma(x_1) \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} M(x_1, x_3) \]

\[ + \frac{I}{\lambda} \left[ 1 - \cos[\gamma(x_1)] \right] \]

\[ - \frac{I}{k_B T} \int dx_1 dx_3 \frac{1}{2} M(x_1, x_3) \tilde{a}(T, \partial_3) M(x_1, x_3) \right] , \quad (3.9) \]

where \( C \) is a normalization constant and \( l \) is a characteristic length in the \( x_3 \) direction. The role of the critical spin fluctuations for the fluctuations of \( \gamma \) becomes much clearer if we eliminate \( \eta \) and \( M \) in the solution (3.9). Performing functional integration by \( \eta \) and \( M \), we find (see Appendix B)

\[ p[\gamma(\cdot)] = \int \delta[\gamma(\cdot)] \delta[M(\cdot)] p[\gamma(\cdot), \eta(\cdot), M(\cdot)] \]

\[ = C \exp \left[ -\frac{1}{k_B T} \frac{\phi_0^2}{32 \pi^2 \lambda_L^2} \int dx_1 \left( \frac{S(T)}{2} \left[ \partial_1 \gamma(x_1) \right]^2 + \frac{I}{\lambda} \left[ 1 - \cos[\gamma(x_1)] \right] \right) \right] . \quad (3.10) \]

It is seen from (3.10) that the effect of the spin fluctuations appears through the factor \( S(T) \). This result shows that the inhomogeneous fluctuations of \( \gamma \) are enhanced near \( T_s \) because the term including \( \partial_1 \gamma \) in (3.10) decreases as \( T \rightarrow T_s \) as a result of the fluctuations of the localized-spin magnetization.

In the following we calculate the maximum dc Josephson current and the \( I-V \) characteristic in the case of a small junction, and investigate the effect of the critical spin fluctuations.

A. dc Josephson effect

We consider the case in which an external field is applied in the \( x_2 \) direction. In this case it is convenient to separate \( \gamma(x_1, t) \) and \( M(x_1, x_3, t) \) into their average values, \( \gamma_0(x_1) \) and \( M_0(x_1, x_3) \), and their fluctuations, \( \Theta(x_1, t) \) and \( m(x_1, x_3, t) \), respectively, as

\[ \gamma(x_1, t) = \gamma_0(x_1) + \Theta(x_1, t) , \quad (3.11) \]

\[ M(x_1, x_3, t) = M_0(x_1, x_3) + m(x_1, x_3, t) . \quad (3.12) \]

\( \gamma_0(x_1) \) and \( M_0(x_1, x_3) \) are the solutions of the equations

\[ \partial_1^2 \gamma_0(x_1) - \frac{8 \pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M_0(x_1, x_3) = \frac{1}{\lambda^2} \sin[\gamma_0(x_1)] , \quad (3.13) \]

\[ \tilde{a}(T, \partial_3) M_0(x_1, x_3) - \frac{\phi_0}{4 \pi \lambda_L} \partial_1 \gamma_0(x_1) e^{-|x_3|/\lambda_L} = 0 . \quad (3.14) \]

Substituting (3.11) and (3.12) into (3.9), and using (3.13) and (3.14), we obtain the probability density functional,

\[ p[\gamma(\cdot), \eta(\cdot), m(\cdot); \gamma_0, M_0] = C \exp \left[ -\frac{1}{k_B T} \frac{\phi_0^2}{32 \pi^2 \lambda_L^2} \int dx_1 \left( \frac{1}{2 e^2} \eta^2(x_1) \right) \right. \]

\[ + \frac{1}{2} \left[ \partial_1 \Theta(x_1) \right]^2 - \frac{8 \pi^2}{\phi_0} \partial_1 \Theta(x_1) \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} m(x_1, x_3) \]

\[ - \frac{1}{\lambda^2} \left[ \cos[\gamma_0(x_1) + \Theta(x_1)] - \cos[\gamma_0(x_1)] + \sin[\gamma_0(x_1)]\Theta(x_1) \right] \]

\[ - \frac{16 \pi^3}{\phi_0^2} \int dx_3 m(x_1, x_3) \tilde{a}(T, \partial_3) m(x_1, x_3) \right] . \quad (3.15) \]
The total pair tunneling current \( I \) is then obtained by averaging \( j_0 \sin \gamma \) as
\[
I = j_0 \int_{-L/2}^{L/2} dx \int \delta(\Theta(x)) \delta(\gamma(x)) \sin[\gamma(x) + \Theta(x)] p[\Theta(x); \gamma_0(x), \gamma_0, M_0]
\]
\[
= j_0 \int_{-L/2}^{L/2} dx \int \delta(\Theta(x)) \sin[\gamma_0(x) + \Theta(x)] p[\Theta(x); \gamma_0],
\]
where
\[
p[\Theta(x); \gamma_0] = C \exp\left(-\frac{1}{k_B T} \left[ \frac{\phi_0^2}{32 \pi^3 \lambda_L} \right] \int dx \left[ \frac{S(T)}{2} \left[ \delta_1 \gamma(x) \right]^2 - \frac{1}{\lambda_j^2} \sin[\gamma_0(x)] \Theta(x) \right]
\]
\[
+ \cos[\gamma_0(x) + \Theta(x)] - \cos[\gamma_0(x)] \right) \right). \tag{3.16}
\]

Since the functional integration in (3.16) can not be performed without a certain approximation, we make the following approximate calculation. Let us write the probability density functional (3.17) as
\[
p[\Theta(x); \gamma_0] = C p_0[\Theta(x); \gamma_0] \exp\left[ -\frac{1}{k_B T} \left[ \frac{\phi_0^2}{32 \pi^3 \lambda_L} \right] \int dx \left[ \frac{S(T)}{2} \left[ \delta_1 \gamma(x) \right]^2 + \frac{1}{2\lambda_j^2} \cos[\gamma_0(x)] \Theta^2(x) \right] \right], \tag{3.17}
\]
where
\[
p_0[\Theta(x); \gamma_0] = \exp\left[ -\frac{1}{2} \int dx dy \Theta(x) K(x,y) \Theta(y) \right], \tag{3.18}
\]
with
\[
K(x,y) = \frac{\phi_0^2}{32 \pi^3 \lambda_L} \frac{1}{k_B T} \left[ -S(T) \delta_1^2 + \frac{1}{\lambda_j^2} \cos[\gamma_0(x)] \right] \delta(x - y). \tag{3.19}
\]

Expanding the exponential factor in (3.18) into a Taylor series, and keeping only the terms up to first order, we have
\[
p[\Theta(x); \gamma_0] = C p_0[\Theta(x); \gamma_0] \left[ 1 + \frac{\phi_0^2}{32 \pi^3 \lambda_L k_B T} \frac{1}{\lambda_j^2} \int dx \left[ \cos[\gamma_0(x) + \Theta(x)] + \sin[\gamma_0(x)] \right] \Theta(x)
\]
\[
+ \frac{1}{2} \cos[\gamma_0(x)] \Theta^2(x) - \cos[\gamma_0(x)] \right) \right). \tag{3.20}
\]

The constant \( C \) is determined by the normalization condition,
\[
\int \delta(\theta) p[\Theta(x); \gamma_0] = 1. \tag{3.21}
\]
Substituting (3.21) into (3.22), we obtain
\[
C = \exp\left[ -\frac{1}{2} \int dx \ln K^{-1}(x,x) \right] \left[ 1 + \frac{\phi_0^2}{32 \pi^3 \lambda_L k_B T} \frac{1}{\lambda_j^2} \int dx \left[ \cos[\gamma_0(x)] \right] e^{-(1/2) K^{-1}(x,x)}
\]
\[
+ \frac{1}{2} \cos[\gamma_0(x)] K^{-1}(x,x) - \cos[\gamma_0(x)] \right) \right]^{-1}, \tag{3.22}
\]
where the inverse function \( K^{-1}(x,y) \) is the solution of the equation
\[
\left\{ \frac{\phi_0^2}{32 \pi^3 \lambda_L k_B T} \right\} \left[ -S(T) \delta_1^2 + \frac{1}{\lambda_j^2} \cos[\gamma_0(x)] \right] K^{-1}(x,y) = \delta(x - y). \tag{3.23}
\]
The functional integrations appearing in this section are summarized in Appendix B.

Let us calculate the total current (3.16). The current is written in this approximation as

$$I = I_0 + I_1 ,$$

with

$$I_0 = j_0 C \int_{-L/2}^{L/2} dx \int \delta \Theta_A \sin[\gamma_0(x) + \Theta(x)] p_0(\Theta(x); \gamma_0) \tag{3.26}$$

and

$$I_1 = j_0 C \frac{\phi_0^2}{32\pi^2 \lambda_L k_B T} \frac{1}{\lambda_j^2} \int_{-L/2}^{L/2} dx \int dy \left\{ \cos[\gamma_0(x) + \Theta(x)] + \sin[\gamma_0(x)] \Theta(x) + \frac{1}{2} \cos[\gamma_0(x)] \Theta^2(x) \right\}$$

$$- \cos[\gamma_0(x)] p_0(\Theta(x); \gamma_0) .$$

The functional integrations in (3.26) and (3.27) can be performed using (B9)–(B15) in Appendix B. The results are given by

$$I_0 = j_0 C \int_{-L/2}^{L/2} dx \sin[\gamma_0(x)] \exp\left[-\frac{1}{2} K^{-1}(x, x)\right] ,$$

$$I_1 = j_0 C \left[ \frac{\phi_0^2}{32\pi^2 \lambda_L k_B T} \frac{1}{\lambda_j^2} \int_{-L/2}^{L/2} dx \exp\left[-\frac{1}{2} K^{-1}(x, x)\right] \right]$$

$$\times \int_{-L/2}^{L/2} dx_1 \left[ \cos[\gamma_0(x)] \cos[\gamma_0(x_1)] \left( \exp\left[-K^{-1}(x, x_1)\right] - K^{-1}(x, x_1) \right) \right]$$

$$+ \left( \frac{1}{2} K^{-1}(x, x_1) - \frac{1}{2} K^{-1}(x, x_1) \right)^2 - 1)$$

$$+ \sin[\gamma_0(x_1)] \cos[\gamma_0(x_1)] K^{-1}(x, x_1) \right] , \tag{3.29}$$

where

$$C = \left[ 1 + \frac{\phi_0^2}{32\pi^2 \lambda_L k_B T} \frac{1}{\lambda_j^2} \int dx \cos[\gamma_0(x)] \left( \exp\left[-\frac{1}{2} K^{-1}(x, x)\right] + \frac{1}{2} K^{-1}(x, x) - 1\right) \right]^{-1} . \tag{3.30}$$

We note that the Josephson current is reduced by the factor \( \exp\left[-\frac{1}{2} K^{-1}(x, x)\right] \). In the case of no external field we have

$$K^{-1}(x, x) = \text{const} = \nu t / [2S(T) \cos \gamma_0]^{1/2} ,$$

setting \( \gamma_0(x) \) as a constant \( \gamma_0 \) in Eq. (3.24), where

$$\nu = 32\pi^2 \lambda_L \lambda_j k_B T_m / \phi_0^2$$

and \( t = T/T_m \). This result shows that the reducing factor vanishes at \( T_s \) since \( K^{-1} \to \infty \) for \( T \to T_s \) as a consequence of the critical spin fluctuations. Therefore, we conclude that the dc Josephson effect in the ferromagnetic superconductor is quenched at \( T = T_s \), the temperature at which surface magnetization appears.

The numerical results for the maximum Josephson current are shown in the following. In the calculations we confine ourselves to the cases of the small junction, that is, \( L \ll \lambda_j \). Then, \( \gamma_0(x) \) is given by

$$\gamma_0(x) = 2\pi \Phi x / L \phi_0 + \gamma_0 , \tag{3.31}$$

where \( \Phi \) is the total flux within the junction. The function \( S(T) \) was obtained using a previous theory\(^5\) of the surface of the ferromagnetic superconductors. When we choose the values of the parameters, \( T_m \approx 2 K, \lambda_j \approx 0.1 \ cm, \) and \( \lambda \approx 10 \lambda_j \) as a typical case, we find \( \nu \approx 0.1 \). Then we set the values of \( \nu \) around 0.1 in the following calculations. The calculated result of \( S(T) \) is shown by a dotted-dashed line in Fig. 3(a). In the calculation we used the values of the parameters \( c = 4\pi C/T_m = 2 \) and \( d = D/\lambda_L^2 T_m = 0.01 \), where \( C \) is the Curie constant and \( D \) is the magnetic stiffness constant. The above choice of the parameters leads to the transition temperatures, \( T_s/T_m = 0.761 \) and \( T_p/T_m = 0.745 \).

We present the temperature dependence of the maximum current for the parameter values \( \nu = 0.05, 0.1, \) and 0.2 in the case of no external field in Fig. 3(a). As seen from the figure, the maximum current shows a sharp decrease just above \( T_s \). Such a behavior of the temperature variation has been observed above the reentrant transition temperature \( T_{c2} \) by Rowell et al.,\(^3\) and by Umbach and Goldman\(^7\) in ErRh4B4. In Fig. 3(b) the curves \( I_c \) versus...
T. KOYAMA AND M. TACHIKI

\[ \Phi/\phi_0 \] are plotted for various temperatures. As seen in the figure, the maximum Josephson current in the small-flux region is greatly reduced as the temperature is lowered. Note that when we plot \( I_c \) as a function of an applied field, the period of the Fraunhofer pattern shrinks as the temperature decreases since the localized-spin magnetization induced by the external field contributes to the total flux \( \Phi \).

**B. I-V characteristics**

The effect of the critical spin fluctuations is also expected in the ac Josephson effect. In this section we study, as an example, its effect on the current-voltage (I-V) characteristics in the case of a current bias. It is very difficult to obtain the solution of the Fokker-Planck equation (3.8) for the state of finite voltage. Therefore, we begin with the following Langevin-type equation corresponding to the case of a negligible capacitance of the Josephson junction:

\[
\frac{g}{e^2} \partial_t \gamma(x_1,t) = \frac{\delta}{\delta \gamma(x_1,t)} \left[ \gamma(x_1,t) - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) + \tilde{J}_0 - \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] - \frac{1}{\lambda_L} R(x_1,t)/\rho_0 \right] + \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} + R_2(x_1,x_3,t).
\]

Equation (3.32) was obtained from Eq. (3.1) by neglecting the term \( \partial \eta/e^2 \), since this term is proportional to the capacitance. The Fokker-Planck equation can then be derived from Eqs. (3.32) and (3.33) as

\[
\frac{\partial}{\partial t} p[\gamma(t),M(t)] = \int dx_1 \left[ \frac{\delta}{\delta \gamma(x_1,t)} \left[ \gamma(x_1,t) - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) + \tilde{J}_0 - \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] - \frac{1}{\lambda_L} R(x_1,t)/\rho_0 \right] + \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} \right] + \frac{\beta \sigma(T,\partial_3) M(x_1,x_3,t)}{\delta M(x_1,x_3,t)} - \frac{\beta \phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} + \frac{\beta k_B T}{\delta M(x_1,x_3,t)} \right] p[\gamma(t),M(t),t] \right].
\]

Equation (3.34) was obtained from Eq. (3.31) by neglecting the term \( \partial \eta/e^2 \), since this term is proportional to the capacitance. The Fokker-Planck equation can then be derived from Eqs. (3.32) and (3.33) as

\[
\frac{\partial}{\partial t} p[\gamma(t),M(t)] = \int dx_1 \left[ \frac{\delta}{\delta \gamma(x_1,t)} \left[ \gamma(x_1,t) - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) + \tilde{J}_0 - \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] - \frac{1}{\lambda_L} R(x_1,t)/\rho_0 \right] + \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} \right] + \frac{\beta \sigma(T,\partial_3) M(x_1,x_3,t)}{\delta M(x_1,x_3,t)} - \frac{\beta \phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} + \frac{\beta k_B T}{\delta M(x_1,x_3,t)} \right] p[\gamma(t),M(t),t] \right].
\]

Let us solve Eq. (3.34) for a steady state, i.e.,

\[
\frac{\partial}{\partial t} p[\gamma(t),M(t)] = 0.
\]

We impose a periodic boundary condition,

\[
p[\gamma(t) + 2\pi,M(t)] = p[\gamma(t),M(t)]
\]

Let us introduce an unknown functional \( \omega[\gamma(t),M(t)] \) defined by

\[
p[\gamma(t),M(t)] = p_0[\gamma(t),M(t)] \omega[\gamma(t),M(t)],
\]

with

\[
p_0[\gamma(t),M(t)] = \exp \left[ -\frac{1}{k_B T} \int dx_1 \left[ \frac{\phi_0}{16\pi^3 \lambda_L} \frac{1}{2} |\partial_1 \gamma(x_1)|^2 - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) - \tilde{J}_0 \gamma(x_1) \right] + \frac{1}{\lambda_L^2} \left[ 1 - \cos[\gamma(x_1)] \right] + \int dx_3 \frac{1}{2} \partial_1 M(x_1,x_3) \tilde{a}(T,\partial_3) M(x_1,x_3) \right] \right].
\]

Substituting (3.37) into (3.34), we have

\[
-\frac{\delta}{\delta \gamma(x_1)} \left[ \frac{\delta}{\delta \gamma(x_1)} \left[ \gamma(x_1) - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx_3 e^{-|x_3|/\lambda_L} \partial_1 M(x_1,x_3,t) + \tilde{J}_0 - \frac{1}{\lambda_L^2} \sin[\gamma(x_1,t)] - \frac{1}{\lambda_L} R(x_1,t)/\rho_0 \right] + \frac{\phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} \right] + \frac{\beta \sigma(T,\partial_3) M(x_1,x_3,t)}{\delta M(x_1,x_3,t)} - \frac{\beta \phi_0}{4\pi \lambda_L} \partial_1 \gamma(x_1,t) e^{-|x_3|/\lambda_L} + \frac{\beta k_B T}{\delta M(x_1,x_3,t)} \right] p_0[\gamma(t),M(t)] \frac{\delta \omega[\gamma(t),M(t)]}{\delta \gamma(x_1)} = 0.
\]
In this case to calculate the I-V characteristics. Then, writing \( p[\gamma(\cdot)] \) as

\[
p[\gamma(\cdot)] = \int \delta M(\cdot) p[\gamma(\cdot), M(\cdot)]
\]

we obtain the equation for \( w[\gamma(\cdot)] \):

\[
\int dx \frac{\delta}{\delta \gamma(x)} \left[ p_0[\gamma(\cdot), M(\cdot)] \frac{\delta w[\gamma(\cdot)]}{\delta \gamma(x)} \right] = 0 .
\]

(3.40)

As will be seen later, we need only

\[
p[\gamma(\cdot)] = \int \delta M(\cdot) p[\gamma(\cdot), M(\cdot)]
\]

in this case to calculate the I-V characteristics. Then, writing \( p[\gamma(\cdot)] \) as

\[
p[\gamma(\cdot)] = \int \delta M(\cdot) p_0[\gamma(\cdot), M(\cdot)] w[\gamma(\cdot)]
\]

\[
\equiv p_0[\gamma(\cdot)] w[\gamma(\cdot)] ,
\]

(3.41)

where

\[
p_0[\gamma(\cdot)] = \exp \left[-\frac{1}{k_B T} \left( \frac{\phi_0^2}{16\pi^3 \lambda_L} \right) \right.
\]

\[
\times \int dx \left\{ \frac{S(T)}{2} \left[ \partial_x \gamma(x) \right]^2 - J_0 \gamma(x)
\]

\[
+ \frac{1}{\lambda^2} \left[ 1 - \cos[\gamma(x)] \right] \right\} ,
\]

(3.42)

Note that \( p_0[\gamma(\cdot), M(\cdot)] \) corresponds to the solution in the equilibrium state (\( V = 0 \)). We further assume that the localized-spin system is close to thermal equilibrium. Under this assumption we can set \( w[\gamma(\cdot), M(\cdot)] \) independent of \( M(x_1, x_3) \), so that Eq. (3.39) reduces to

\[
w[\gamma(\cdot)] = \frac{-e^\sigma}{e^\sigma - 1} \int_0^1 \frac{dt}{t} \exp \left[ \frac{t^2}{k_B T} \left( \frac{\phi_0^2}{16\pi^3 \lambda_L} \right) \right.
\]

\[
\times \int dx \left[ \gamma(x) \gamma(x) + 2\pi \frac{\delta Z[\gamma(\cdot) + 2\pi]}{\delta \gamma(x)} - \gamma(x) \frac{\delta Z[\gamma(\cdot)]}{\delta \gamma(x)} \right] .
\]

(3.47)

The functional derivative of \( w[\gamma(\cdot)] \) can be obtained by noting the relation

\[
X[\gamma(\cdot)] - X[0] = \int_0^1 \frac{dt}{t} \int dx \gamma(x) \frac{\delta X[\gamma(\cdot)]}{\delta \gamma(x)}
\]

(3.48)

Using (3.48) and requiring

\[
Z[\gamma(\cdot) + 2\pi] = e^{-\sigma} Z[\gamma(\cdot)] ,
\]

(3.49)

we find, from (3.47),
The boundary condition (3.49) insures that \( w(\gamma) \) given by (3.47) satisfies the relation (3.44). Substituting (3.50) into (3.43), we obtain the following equation:

\[
\int dx \frac{\delta}{\delta \gamma(x)} \exp \left[ \frac{1}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \gamma(x) - \frac{1}{\lambda_j^2} \left( 1 - \cos[\gamma(x)] \right) \right) \frac{\delta Z[\gamma(\cdot)]}{\delta \gamma(x)} \right] = 0.
\]

(3.51)

The above equation is equivalent to the differential equation

\[
\lim_{n \to \infty} \sum_{l=1}^{n} \frac{\partial}{\partial \gamma_l} \prod_{j=1}^{n} \exp \left[ \frac{\Delta x}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \gamma_l - \frac{1}{\lambda_j^2} \left( 1 - \cos[\gamma_l] \right) \right) \frac{\delta Z[\gamma_1, \gamma_2, \ldots, \gamma_n]}{\delta \gamma_l} \right] = 0,
\]

(3.52)

where \( \Delta x = L / n \) and \( \gamma_l = \gamma(x_l) \). A solution to Eq. (3.52) is easily obtained in the case in which \( Z[\gamma_1, \gamma_2, \ldots, \gamma_n] \) can be factorized as

\[
Z[\gamma_1, \gamma_2, \ldots, \gamma_n] = \prod_{l=1}^{n} Z[\gamma_l].
\]

(3.53)

In this case, Eq. (3.52) is satisfied when

\[
\exp \left[ \frac{\Delta x}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \gamma_l - \frac{1}{\lambda_j^2} \left( 1 - \cos[\gamma_l] \right) \right) \right] \frac{\partial}{\partial \gamma_l} Z[\gamma_1, \gamma_2, \ldots, \gamma_n] = c_l,
\]

(3.54)

with \( c_l \) a constant. We then obtain the following solution:

\[
Z[\gamma(\cdot)] = \lim_{n \to \infty} \prod_{l=1}^{n} c_l \int \gamma_l d\Theta_l \exp \left[ - \frac{\Delta x}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \Theta_l - \frac{1}{\lambda_j^2} \left( 1 - \cos[\Theta_l] \right) \right) \right]
\]

(3.55)

\[
= \lim_{n \to \infty} \prod_{l=1}^{n} c_l \int \gamma_l^{\gamma(\cdot)} \delta \Theta(\cdot) \exp \left[ - \frac{1}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \Theta(x) - \frac{1}{\lambda_j^2} \left( 1 - \cos[\Theta(x)] \right) \right) \right].
\]

(3.56)

The integral domains in (3.55) and (3.56) should be taken such that the boundary condition (3.49) is fulfilled, so that we have

\[
Z[\gamma(\cdot)] = \lim_{n \to \infty} \prod_{l=1}^{n} c_l \frac{e^{-\sigma_l}}{e^{-\sigma_l} - 1} \int \gamma_l^{\gamma(\cdot) + 2\pi} \delta \Theta(\cdot) \exp \left[ - \frac{1}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \Theta(x) - \frac{1}{\lambda_j^2} \left( 1 - \cos[\Theta(x)] \right) \right) \right],
\]

(3.57)

where

\[
\sigma_l = \frac{\Delta x}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) 2\pi \tilde{J}_0 = \sigma / n,
\]

(3.58)

and the \( c_l \)'s are determined by the normalization condition

\[
\int \delta \gamma(\cdot) p[\gamma(\cdot)] = 1.
\]

(3.59)

Now we summarize the result obtained above. The probability density functional is given as follows:

\[
p[\gamma(\cdot)] = \int \delta M(\cdot)p[\gamma(\cdot), M(\cdot)]
\]

\[
= e^{\sigma} - e^{-\sigma} \exp \left[ - \frac{1}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \gamma(x) \right)^2 \right] \int dx \left( \frac{S(T)}{2} \left[ \partial_x \gamma(x) \right]^2 - \frac{1}{\lambda_j^2} \left( 1 - \cos[\gamma(x)] \right) \right]
\]

\[
\times \int_0^1 dt \exp \left[ \frac{t^2}{k_B T} \left( \frac{\phi_{ij}}{16 \pi^2 \lambda_L} \right) \int dx \left( \tilde{J}_0 \gamma(x) \right)^2 \right] \int dx \left( [\gamma(x) + 2\pi] \frac{\delta Z[\gamma(\cdot) + 2\pi]}{\delta \gamma(x)} - \gamma(x) \frac{\delta Z[\gamma(\cdot)]}{\delta \gamma(x)} \right),
\]

(3.60)
where \( Z[\gamma(\cdot)] \) is given by (3.57).

Next, we obtain the relation between current and voltage. In the case of a current bias the mean voltage \( V \) is obtained from the relation

\[
V = \frac{2e}{\hbar} \int_{-L/2}^{L/2} dx \int \delta \gamma(\cdot) \delta M(\cdot) \gamma(x) p[\gamma(\cdot), M(\cdot)] / L
\]

\[= \frac{2e}{\hbar} \frac{\sigma^2}{gL} \int_{-L/2}^{L/2} dx \int \delta \gamma(\cdot) \delta M(\cdot) \left[ \delta_x^2 \gamma(x) - \frac{8\pi^2}{\phi_0} \int_{-\infty}^{\infty} dx e^{-|x|} \delta_x M(x) \right. \]

\[\left. + \overline{J}_0 - \frac{1}{\lambda_j^2} \sin[\gamma(x)] + R_1(x) \right] p[\gamma(\cdot), M(\cdot)]. \]

Here, use was made of Eq. (3.32). Noting the relations

\[\int \delta \gamma(\cdot) \delta M(\cdot) R(\cdot) p[\gamma(\cdot), M(\cdot)]=0 \]

and

\[\int dx \int \delta \gamma(\cdot) \frac{\delta}{\delta \gamma(x)} p[\gamma(\cdot), M(\cdot)]=0 , \]

we can rewrite (3.62) as

\[V = \frac{2e}{\hbar} \frac{\sigma^2}{gL} \int_{-L/2}^{L/2} dx \int \delta \gamma(\cdot) \delta M(\cdot) \left[ \delta_x^2 \gamma(x) - \frac{8\pi^2}{\phi_0} \delta_x M(x) + \overline{J}_0 - \frac{1}{\lambda_j^2} \sin[\gamma(x)] - \frac{16\pi^2}{\phi_0^2} g k_B T \frac{\delta}{\delta \gamma(x)} \right] p[\gamma(\cdot), M(\cdot)]. \]

\[= \frac{2e}{\hbar} \frac{\sigma^2}{gL} \int_{-L/2}^{L/2} dx \int \delta \gamma(\cdot) p_0[\gamma(\cdot)] \frac{\delta w[\gamma(\cdot)]}{\delta \gamma(x)} \]

\[= \frac{2e}{\hbar} \frac{\sigma^2}{gL} \sum_i 2\pi c_i \prod_j^{(i)} c_j \left[ 1 - e^{-\sigma_j} \right] \int_0^{2\pi} d\gamma_j \exp \left[ \frac{\Delta x}{k_B T} \frac{\phi_j}{16\pi^2\lambda_j} \right] \int dx' \left[ \overline{J}_0 \gamma(x') - \frac{1}{\lambda_j^2} \sin[\gamma(x') - \phi_j] \right] \]

\[\times \int \gamma_j^{\gamma j + 2\pi} d\gamma_j \exp \left[ \frac{\Delta x}{k_B T} \frac{\phi_j}{16\pi^2\lambda_j} \right] \left[ \overline{J}_0 \Theta_j - \frac{1}{\lambda_j^2} \sin[\gamma(x') - \phi_j] \right] , \]

where

\[\prod_j^{(i)} c_j \equiv \prod_j c_j / c_i , \]

(3.66)

and use was made of relations (3.41), (3.50), and (3.56).

The explicit expression is hardly obtained from (3.65) at general temperatures. Therefore, in order to see the effect of the critical spin fluctuation, we confine ourselves to the special case of \( T = T_c \). The probability density functional at \( T_c \) is easily obtained from (3.60) since \( S(T_c)=0 \) as

\[\begin{align}
p[\gamma(\cdot)] &= \lim_{n \to \infty} \prod_{i=1}^n c_i \left[ 1 - e^{-\sigma_i} \right] \exp \left[ \frac{\Delta x}{k_B T} \left[ \overline{J}_0 \gamma_i - \frac{1}{\lambda_j^2} (1 - \cos \gamma_i) \right] \right] \int_{\gamma_i}^{\gamma_i+2\pi} d\Theta_i \exp \left[ - \frac{\Delta x}{k_B T} \left[ \overline{J}_0 \Theta_i - \frac{1}{\lambda_j^2} (1 - \cos \Theta_i) \right] \right] . \end{align}\]

(3.67)

The normalization condition (3.59) for (3.67) gives

\[c_i = \frac{1 - e^{-\sigma_i}}{\sigma_i} Q^{-1} , \]

(3.68)

where

\[Q = \int_0^{2\pi} d\gamma \exp \left[ \frac{\Delta x}{k_B T} \left[ \overline{J}_0 - \frac{1}{\lambda_j^2} (1 - \cos \gamma) \right] \right] \int_\gamma^{\gamma+2\pi} d\Theta \exp \left[ - \frac{\Delta x}{k_B T} \left[ \overline{J}_0 \Theta - \frac{1}{\lambda_j^2} (1 - \cos \Theta) \right] \right] . \]

(3.69)
Substituting (3.68) into (3.65), we have
\[
V = \frac{2e}{\hbar} \frac{e^2}{gL} \lim_{n \to \infty} \sum_i 2\pi \frac{1 - e^{\alpha_i}}{e^{\alpha_i}} Q^{-1}.
\] (3.70)

For \( n \to \infty \) or \( \Delta x \to 0 \), we find
\[
Q = 4\pi^2 \quad \text{and} \quad \frac{1 - e^{\alpha_i}}{e^{\alpha_i}} = \frac{\Delta x}{k_B T} \left[ \frac{\phi_0 l}{16\pi^2 \lambda L} \right] 2\pi J_0.
\] (3.71)

These results lead to the following Ohmic-type \( I-V \) characteristic at \( T_s \),
\[
V = \frac{2e}{\hbar} \frac{e^2}{g} \frac{1}{k_B T_s} \left[ \frac{\phi_0 l}{16\pi^2 \lambda L} \right] J_0.
\] (3.72)

Therefore, we conclude that the noise effect due to the critical spin fluctuations may be observed also in the \( I-V \) characteristic as the temperature approaches \( T_s \).

### IV. DISCUSSION

We discuss here the experimental results of the dc Josephson effect observed in the ferromagnetic superconductor ErRh4B4, on the basis of our theory. Let us first consider the temperature dependence of the Josephson current. The experimental result shows that the maximum Josephson current takes a maximum at \( T = 1.4 \) K and falls to zero at about 1.1 K, which is slightly higher than \( T_{c0} \), the reentrant transition temperature. The reduction of the current in the temperature region from 1.4 to 1.1 K can be understood as a result of the critical spin fluctuations which leads to the surface magnetic state. We may therefore identify the temperature at which the Josephson current vanishes with \( T_s \), the transition temperature of the surface magnetic state. It should be noted that \( T_s \) is always higher than \( T_p \), the transition temperature of the periodic phase no matter what values the material parameters take. This fact requires that there is no spontaneous magnetization in the superconductor in the temperature range where the Josephson current is observed.

In the experiments for the field dependence of the maximum Josephson current \( I_c \), an unusual Fraunhofer-type pattern has been observed at low temperatures by Umbach and Goldman.\(^{11}\) The observed curves of \( I_c \) versus \( H \) show a shift of the Fraunhofer pattern in which \( I_c \) takes a maximum for a finite applied field, and a splitting of the central peak at a temperature just before the Josephson current is quenched. To understand the experimental results we should note that the ErRh4B4 electrodes are made of the polycrystalline samples. ErRh4B4 has a large magnetic anisotropy with the easy-magnetization \( a \) axes, as observed in the magnetization measurements on single crystals.\(^{23}\) We consider the case of a finite current bias and no applied field in the junction. When the uniform Josephson current flows through the junction, it creates a magnetic field whose intensity linearly depends on \( x_1 \) [Fig. 4(a)]. The magnetic field induces, in turn, an irregular localized-spin magnetization because of the random orientation of the easy directions of grains of the polycrystalline electrode. The spatial dependence of the magnetic induction \( b_0(x) = h_0(x) + 4\pi m_0(x) \) in this case is schematically shown in Fig. 4(b). As seen in the figure, the total magnetic flux \( \int_{-L/2}^{L/2} dx b_0(x) \) does not vanish even when no external field is applied. This fact implies that the shift of the Fraunhofer pattern observed by Umbach and Goldman just above \( T_s \) is caused by the self-field effect due to the Josephson current, but not by the appearance of the spontaneous magnetization below \( T_s \).

The inhomogeneity originating from the anisotropic grain structure of the ErRh4B4 electrode can also cause a phase modulation. This is a possibility to explain the splitting of the central peak. However, as discussed by Umbach and Goldman, the grain size might be too small to give a reasonable wavelength of the phase modulation. Another possibility for the explanation may come from the self-field effect. As shown in Sec. III the effective Josephson penetration depth in the ferromagnetic superconductor is shortened near \( T_s \). This effect may cause the following change in the \( L \)-versus-\( H \) curve. In the temperature range much higher than \( T_s \), we have the usual Fraunhofer-type pattern in the small-junction case \( (L < \lambda_J) \), but as the temperature is lowered close to \( T_s \), the curve changes to the large-junction type even when the relation \( L < \lambda_J \) is satisfied, that is, the "zero to one vortex mode" in the \( L \)-versus-\( H \) curve overlaps the "one to two vortex mode" just above \( T_s \).\(^{24}\) Furthermore, the maximum Josephson current in the small-flux region is greatly reduced by the critical spin fluctuations compared to that in the large-flux region, as shown in Sec. III. These two changes might be observed as a splitting of the

![FIG. 4. (a) Magnetic field created by uniform current through the junction. (b) Magnetic induction field in polycrystalline junction with magnetic anisotropy. Dotted lines denote the grain boundaries.](image-url)
central peak in the \( I_v \)-versus-\( H \) curve. The detailed calculation of the \( I_v \)-versus-\( H \) curves are planned to be published elsewhere.

**ACKNOWLEDGMENTS**

We would like to thank Professor H. Matsumoto, Dr. S. Maekawa, Dr. S. Takahashi, and Mr. I. Kondo for many valuable discussions.

**APPENDIX A: SURFACE MAGNETIZATION**

The magnetic field near the surface of a superconductor can be calculated on the basis of the same formalism as in Sec. II since the phase jump is observed at the surface and the field distribution is controlled by the singularity associated with the phase jump. The difference between the junction and the surface lies in boundary conditions for the current at the surface. The normal component of the current vector should vanish on the surface that separates the vacuum and the superconductor. On the other hand, it is given by (2.23) in the case of the Josephson junction.

In the case of the surface of the ferromagnetic superconductor, we have the following magnetic field from (2.17), dropping the homogeneous term:

\[
\begin{align*}
  h(x_1,x_3) &= \left\{ \frac{\phi_0}{2\pi} \left[ \frac{1}{2\lambda_L} \partial_1 \gamma(x_1) e^{-|x_3|/\lambda_L} ight. \\
  & \quad - \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|x_3-y_3|/\lambda_L} m(x_1,y_3) \bigg] \right\}.
\end{align*}
\]

(A1)

For the localized-spin magnetization, we can use Eq. (2.37) without the bias term:

\[
\begin{align*}
  m(x_1,x_3) &= \frac{\phi_0}{4\pi\lambda_L} \partial_1 \gamma(x_1)e^{-|x_3|/\lambda_L}.
\end{align*}
\]

(A2)

The vertical component of the current to the surface is then obtained as follows:

\[
\begin{align*}
  m(x_1,x_3) &= \frac{\phi_0}{4\pi\lambda_L} \partial_1 \gamma(x_1).
\end{align*}
\]

(A2)

The condition discussed above for the current on the surface leads to

\[
\begin{align*}
  1 - \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|x_3-y_3|/\lambda_L} \gamma(x_1) \equiv S(T) \partial_1^2 \gamma(x_1) = 0.
\end{align*}
\]

(A4)

We can take a solution

\[
\gamma(x_1) = Ax_1,
\]

with \( A \) a constant. Substituting (A5) into (A1) and (A2) we have

\[
\begin{align*}
  h(x_3) &= \frac{\phi_0}{4\pi\lambda_L} A \left\{ e^{-|x_3|/\lambda_L} \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|x_3-y_3|/\lambda_L} \gamma(x_1) \right\}.
\end{align*}
\]

(A6)

\( A \) is determined by observing that \( h(0) \) equals the value of the external field \( H_{ex} \).

\[
A = \frac{4\pi\lambda_L}{\phi_0} \frac{H_{ex}}{S(T)},
\]

(A7)

so that

\[
\begin{align*}
  h(x_3) &= \left\{ e^{-|x_3|/\lambda_L} \frac{2\pi}{\lambda_L} \int_{-\infty}^{\infty} dy_3 e^{-|x_3-y_3|/\lambda_L} \gamma(x_1) \right\} H_{ex}/S(T).
\end{align*}
\]

(A8)

Note that \( h(x_3) \) becomes singular at \( T_s \) since \( S(T_s) = 0 \). This means that an infinitesimal external field can create a finite magnetic field inside the ferromagnetic superconductors at \( T_s \). Thus the temperature \( T_s \) can be identified with the transition temperature of the surface magnetic state.

In the case of the Josephson junction the situation is slightly different from that in the surface case due to the existence of tunneling current. However, the tunneling current is usually quite small. Therefore we can also expect the surface magnetization to appear around the junction at \( T_s \) in the superconducting state.

**APPENDIX B: FUNCTIONAL INTEGRALS**

Calculated results of the functional integrals appearing in Sec. III are summarized here. The basic formula we need for obtaining them is

\[
F[k(\cdot)] \equiv \int \delta \left[ \frac{u(\cdot)}{\sqrt{2\pi}} \right] \exp[i(ku) - \frac{1}{2}(uBu)].
\]

(B1)

where the abbreviations
\[(ku) \equiv \int dx \, k(x)u(x), \quad (Bu) \equiv \int dx \, du(x)B(x,y)u(y), \]

through the relation
\[
\int dy \, B(x,y)B^{-1}(y,z) = \delta(x - z).
\]

Using formula (B1) we can evaluate the following types of functional integral:

\[
\int \delta \left[ \frac{\Theta(t)}{\sqrt{2\pi}} \right] \Theta(x) \sin^n[\Theta(y)] \cos^n[\Theta(z)] \exp \left[ -\frac{1}{2}(\Theta B \Theta) \right]
\]

are used, and the inverse functions \(B^{-1}\) are defined

\[
B'(x,y)B(y,z) = \int dy B'(x,y)B(y,z).
\]

Now consider the functional integration of \(p_0[\Theta(t); \gamma_0]\). Using (B1), we have

\[
\int \delta \Theta(t) p_0[\Theta(t); \gamma_0] = C \int \delta \left[ \frac{\Theta(t)}{\sqrt{2\pi}} \right] \exp \left[ -\frac{1}{2} \int dx \, \Theta(x)K(x-y)\Theta(y) \right]
\]

From the normalization condition (3.22), \(C\) is chosen as

\[
C = \exp \left(-\frac{1}{2} \text{Tr} \ln K^{-1} \right).
\]

Using relations (B6) and (B9), we obtain the functional integrals appearing in Sec. III:

1. \(\int \delta \Theta(t) \Theta^2(x) p_0[\Theta(t); \gamma_0] = K^{-1}(x,x)
\]

2. \(\int \delta \Theta(t) \cos[\Theta(x)] p_0[\Theta(t); \gamma_0] = e^{-K^{-1}(x,x)/2}
\]

3. \(\int \delta \Theta(t) \Theta(x) \sin[\Theta(x)] p_0[\Theta(t); \gamma_0] = K^{-1}(x,x_1)e^{-K^{-1}(x,x_1)/2}
\]

4. \(\int \delta \Theta(t) \Theta^2(x_1) \cos[\Theta(x)] p_0[\Theta(t); \gamma_0] = \left[K^{-1}(x_1,x_1) - K^{-1}(x_1,x_1) \right] e^{-K^{-1}(x,x_1)/2}
\]

5. \(\int \delta \Theta(t) \cos[\Theta(x)] \cos[\gamma_0(x_1) + \Theta(x_1)] p_0[\Theta(t); \gamma_0]
\]

6. \(\int \delta \Theta(t) \sin[\Theta(x)] \cos[\gamma_0(x_1) + \Theta(x_1)] p_0[\Theta(t); \gamma_0]
\]


10This result was first observed by Rowell et al. (Ref. 5).
JOSEPHSON EFFECTS IN FERROMAGNETIC SUPERCONDUCTORS:  