Logic design of Josephson network. II

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By numerical calculations of the differential-difference sine-Gordon equation, we have discussed the discrete Josephson-junction transmission lines which are constructed of a series of small-area Josephson junctions connected by superconducting strips. It is shown that the discrete Josephson lines containing $D$ lines, $N$ lines, $T$ turning points, and $S$ turning points are elementarily characterized by the discreteness parameter $(2nL_{c}/\Phi_{0})^{1/2}$. On the discrete Josephson logic circuits there exists a region of forbidden propagation in the $(2nL_{c}/\Phi_{0})^{1/2}, \gamma$ (bias-current parameter) plane for single flux quanta. A single flux quantum can be stuffed in a small area of the discrete Josephson logic circuits. The discrete circuits can be conveniently and easily linked to each other in a practical fabrication of a Josephson network.

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I. INTRODUCTION

Logic devices based on the Josephson effect are extremely attractive for high-performance computers. 1-3 It is now possible to make Josephson junctions and transmission lines with reasonable mechanical and electrical stability. 4-7 In previous papers 8-11 we proposed a Josephson computing network using continuous Josephson-junction transmission lines which are described by the sine-Gordon equation having a loss term and a bias term. 12,13 Mathematical properties and physical applications of the pulselike solitary wave as the solution of the sine-Gordon equation 14-24 and the other nonlinear partial differential equations 25 have been the subject of many investigators. Moreover, research in recent years exhibits an increase of interest in the differential-difference form of the sine-Gordon equation. 26,27 The discreteness effects modify significantly the continuum dynamics given by the sine-Gordon equation which describes a continuous Josephson line. In the present paper, of which Ref. 8 is a companion paper (hereafter denoted as part I), we discuss the characteristics of discrete Josephson transmission lines and networks which are constructed of a series of small-area Josephson junctions connected by superconducting strips. In Sec. II we describe a differential-difference equation of the discrete Josephson line and its equivalent circuit, and we discuss the propagation characteristics of single quanta of magnetic flux, which can be employed as information bits, and flux-quantum-antiflux-quantum (soliton–antisoliton) collision properties by using the result of computer simulations. It can be seen from these results that the discrete Josephson lines with bias-current sources have nearly the same characteristics as the continuous lines, but there exists a region of forbidden propagation for single flux quanta depending upon the bias current and line constant.

Section III deals with turning points and line branches, 1 and it is shown that the behavior of single flux quanta arriving at a turning point are nearly the same as the case of continuous lines. In Sec. IV we compare the transmitting characteristics of flux quanta on continuous and discrete Josephson lines, and conclude briefly.

II. DISCRETE JOSEPHSON TRANSMISSION LINE

Consider a series of small-area Josephson junctions connected by superconducting strips, as shown in Fig. 1(a). We assume the junctions are small enough that they never contain an appreciable fraction of a flux quantum, and, for simplicity, the individual junction critical currents $I_{c}', s$, capacitances $C', s$, and conductances $G', s$, and the self-inductances $L'$ s of single double-junction loops which are shown in Fig. 1(b) have the same values. From the equivalent circuit shown in Fig. 1(b), the phase differences $\phi_{t+1}'$, $\phi_{t-1}'$, and $\phi_{t}'$ are restricted by the usual constraint, 28-30

$$\phi_{t}-\phi_{t-1}'=2\pi\Phi_{t+1}'/\Phi_{0}$$
$$=2\pi LI_{c}/\Phi_{0},$$
$$\phi_{t-1}'-\phi_{t}'=2\pi\Phi_{t-1}/\Phi_{0}$$
$$=2\pi LI_{c}/\Phi_{0},$$

where $\phi_{t+1}'$ and $\phi_{t-1}'$ are the net magnetic fluxes within the individual double-junction loops, $\phi_{0}$ is the flux quantum, and the equation for $I_{t}$ and $I_{t-1}$ can be written

$$I_{t}=I_{t-1}-\frac{\Phi_{0}}{2\pi} C\Phi_{t}^{2}\Phi_{t}^{2} - \frac{\Phi_{0}}{2\pi} G\Phi_{t}^{2} \Phi_{t}^{2} - I_{c} \sin \phi_{t}'.$$

From Eqs. (1)–(3), the differential-difference equa-

FIG. 1. (a) Discrete Josephson-junction transmission line and (b) equivalent circuit of a discrete Josephson line.


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tion for $\phi_{i+1}$, $\phi_i$, and $\phi_{i-1}$ can be written

$$\frac{\Phi_0}{2\pi L c} (\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{G\Phi_0}{2\pi l_c} \frac{\partial^2 \phi_i}{\partial t^2} - \frac{G\Phi_0}{2\pi l_c} \frac{\partial \phi_i}{\partial t} = \sin \phi_i. \tag{4}$$

When time is measured in units of $\tau_f = (C\Phi_0/2\pi L_c)^{1/2}$ and bias current $I_B(\delta L)$ is applied to every junction, Eq. (4) can be rewritten

$$\frac{\Phi_0}{2\pi L c} (\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\partial^2 \phi_i}{\partial t^2} - \frac{\Gamma}{\partial \phi_i} = \sin \phi_i - \gamma_i, \tag{5}$$

where $\Gamma = G(\Phi_0/2\pi L_c)C^{1/2}$ and $\gamma_i = I_B(\delta L)$. If we replace the second-order partial derivative of $\phi$ with respect to space in the equation describing the continuous Josephson line without the effect of losses associated with the flow of normal electrons parallel to the junction by the differences of $\phi_i$, the equation has the same form as Eq. (5); but there are differences in the definitions of the units of time, the units of distance, $\Gamma$, and the bias current between the continuous lines and the discrete lines. This partial differential-difference equation (5) can also describe the mechanical transmission line reported in Ref. 31. The discreteness parameter $(2\pi L c/\Phi_0)^{1/2}$ in Eq. (5) corresponds to $(mgl/K)^{1/2}$ in the mechanical system where $m$ is the mass of the bob of the single pendulum, $g$ is the acceleration due to gravity, $l$ is the length of the single pendulum, and $K$ is the torque constant of the single section of spring between two pendulums.

Standing single flux quantum solutions are shown in Fig. 2 which are obtained from Eq. (5) ($\gamma_i = 0$) by using numerical calculation. It can be seen from Fig. 2 that the number of the loops where a single flux quantum is spread decreases with an increase in the value of $(2\pi L c/\Phi_0)^{1/2}(=1/\Lambda_f)$. When $1/\Lambda_f$ is larger than ~3, a single flux quantum is almost stuffed in one double-junction loop.

It may be expected that single flux quanta cannot propagate down on the discrete Josephson line if the value of bias currents is smaller than some critical one. Figure 3 shows the results of numerical calculation on this critical value of bias currents. For simplicity, we made the individual junction bias currents $\gamma_i$'s equal. In Fig. 3, at the region denoted by $\Pi$, single flux quanta cannot begin to propagate down on a discrete Josephson transmission line. When $1/\Lambda_f = 0.5$, at the regions denoted by $I-1$, $I-2$, $I-3$, and $I-4$ single flux quanta can propagate down steadily; at the region denoted by $I-5$ single flux quanta propagation causes a prompt increase of phase difference, namely, many pairs of flux quanta with opposite signs are created. By decreasing the value of $\Gamma$ to 0.3, 0.1, and 0.01, the region where single flux quanta can propagate down steadily decreases to $I-1$, $I-2$, $I-3$, and $I-5$. It can be seen from Fig. 3 that single flux quanta can propagate down with enough bias current, whenever $1/\Lambda_f$ is large. If single flux quanta propagate down initially on the discrete transmission line, the region of allowed propagation I is spread out in the direction of region $\Pi$, corresponding to the kinetic energy of the single flux quanta.
FIG. 4. Behaviors of two flux quanta with opposite signs after a head-on collision on a discrete Josephson line with bias current sources; the region under the solid line is the same region of allowed propagation for single flux quanta as shown in Fig. 3. The broken lines denoted by $I_1$, $I_2$, and $I_3$ distinguish discrete Josephson lines between $D$ (the region on the left-hand side of $I_1$, $I_2$, and $I_3$) and $N$ lines (the region on the right-hand side of $I_1$, $I_2$, and $I_3$) when $\Gamma = 0.1$, 0.3, and 0.5, respectively.

In our logic design of Josephson networks\textsuperscript{8-11} we have used the $D$ or the $N$ lines, on which two flux quanta with opposite signs are annihilated after head-on collision, or pass through each other after head-on collision, respectively. The result shown in Fig. 4 represents the behaviors of two flux quanta with opposite signs after a head-on collision on a discrete Josephson line with bias current sources. In Fig. 4 the region under the solid line is the same as the region of allowed propagation for single flux quanta shown in Fig. 3. When $\Gamma = 0.1$, on the left-hand side of the line denoted by $I_1$ in Fig. 4, Josephson lines have characteristics of the $D$ line; on the right-hand side of line $I_1$, Josephson lines are the $N$ lines. When $\Gamma = 0.3$, Josephson lines are the $D$ lines on the left-hand side of line $I_2$, and the $N$ lines on the right-hand side of line $I_2$. The line denoted by $I_3$ in Fig. 4 distinguishes between $D$ and $N$ lines when $\Gamma = 0.5$.

III. TURNING POINTS

A. TTP (trigger turning point) (Ref. 8)

TTP can be constructed of discrete Josephson lines such as shown in Fig. 5(a). An equivalent circuit of TTP is shown in Fig. 5(b). The equations for phase differences at the TTP which are restricted by the usual constraint are derived as the following:

\begin{align*}
\frac{d\phi_0}{dt} &= \frac{2\pi L_1 I_1}{2\pi L_1 I_1} \frac{\partial^2 \phi_0}{\partial t^2} - \frac{G_1^2 \sin \phi_0}{2\pi L_1 I_1} \frac{\partial \phi_0}{\partial t} = 0,
\end{align*}

where the $\phi$'s are the individual junction phase differences, in which the subscripts of the left-hand side (1, 2, and 3) are used to designate parameters belonging to discrete Josephson lines 1, 2, and 3, respectively, and the subscripts on the right-hand side denote the junction number for each line, the $L$'s are the self-inductances of individual double-junction loops, in which the subscripts 1, 2, and 3 denote the line numbers, and the $I$'s are the flowing currents such as shown in Fig. 5(b). The differential difference equation for $\phi_{N-1}$, $\phi_N$, $\phi_{N+1}$, $\phi_{N+2}$, $\phi_{N+3}$, and $\phi_{N+4}$ can be written

\begin{align*}
\frac{d^2 \phi_{N-1}}{dt^2} &= \frac{2\pi L_1 I_1}{2\pi L_1 I_1} \frac{\partial^2 \phi_{N-1}}{\partial t^2} - \frac{G_1^2 \sin \phi_{N-1}}{2\pi L_1 I_1} \frac{\partial \phi_{N-1}}{\partial t} = 0.
\end{align*}

FIG. 5. (a) TTP constructed of discrete Josephson lines and (b) equivalent circuit of TTP.

FIG. 6. Behaviors of single flux quanta propagating toward a TTP as a function of the bias current and $(2\pi I_f/\phi_0)^{1/2}$, • and ○ denote the behaviors of a flux quantum inserted schematically in this figure.

\begin{align*}
\phi_0 &= 2\pi L_1 I_1 \frac{\partial^2 \phi_0}{\partial t^2} - \frac{G_1^2 \sin \phi_0}{2\pi L_1 I_1} \frac{\partial \phi_0}{\partial t} = 0,
\end{align*}

FIG. 7. Behaviors of two flux quanta with opposite signs after a head-on collision at the $T$ turning point (a) and at $S$ turning points (b) and (c).
FIG. 8. (a) STP constructed of discrete Josephson lines and (b) equivalent circuit of STP.

\[
\frac{\phi_0}{2\pi L_1 C_2} \left( L_2 \frac{\partial^2 \phi_1}{\partial t^2} - L_1 \frac{\partial \phi_1}{\partial t} \right) - \frac{L_2}{2\pi I_2 C_2} - \frac{L_1}{2\pi I_1 C_2} = \sin^2 \phi_1,
\]

and

\[
\frac{\phi_0}{2\pi L_3 C_3} \left( L_3 \frac{\partial^2 \phi_1}{\partial t^2} - L_2 \frac{\partial \phi_1}{\partial t} \right) - \frac{L_3}{2\pi I_3 C_3} = \sin^2 \phi_1,
\]

where \( I_{C_j} \) 's are the individual junction critical currents, in which the subscripts 1, 2, and 3 denote the line numbers 1, 2, and 3, respectively. Figure 6 shows the result obtained by numerical analyses for the behavior of single flux quanta propagating toward the TTP as a function of the bias current and \( \Gamma L_j \), when \( L_1 = L_2 = L_3 = L_1,2,3, I_{C_1} = I_{C_2} = I_{C_3}, G_{1} = G_{2} = G_{3}, \gamma_{1} (= I_{B_1}/I_{C_1}) \), \( \gamma_{2} (= I_{B_2}/I_{C_2}) \), and \( \gamma_{3} (= I_{B_3}/I_{C_3}) \) are the junction bias currents on lines 1, 2, and 3, respectively, and \( \Gamma = G_{1}/2\pi I_{C_1} C_1 \). In Fig. 6 the dashed lines show the same region boundary of allowed stable propagation as is shown in Fig. 3. If bias currents are relatively small, a single flux quantum propagating toward the TTP cannot pass through the turning point and stops at the turning point. However, if the biases are relatively large, a propagating single flux quantum can pass through the TTP and initiate a single flux quantum into both connected lines. In the case of the above line constant conditions, it can be seen from the result that a single flux quantum cannot pass through the TTP without some amount of kinetic energy. This kinetic energy seems to be converted into the potential energy of a single flux quantum. It is also possible to give this potential energy to the halting flux quantum, if enough bias current is applied. When a single flux quantum stops at the turning point after propagating on one line, and then a single flux quantum with opposite sign propagates toward the turning point on the other line, after a head-on collision a single flux quantum is initiated into the third line as shown in Fig. 7(a).

B. STP (selective turning point) (Ref. 8)

STP can be also constructed of discrete Josephson lines such as shown in Fig. 8(a). An equivalent cir-

cuit of STP is shown in Fig. 8(b). The equations for phase differences at the STP which are restricted by the usual constraint are derived as the following:

\[
\phi_{N-1} = 2\pi L_1 I_{C_1}/\phi_0,
\]
\[
\phi_{N-2} - \phi_1 = 2\pi (L_1,2,3 I_{C_1} + L_{1,2,3,4} I_{C_4})/\phi_0,
\]
\[
\phi_1 - \phi_2 = 2\pi L_2 I_{C_2}/\phi_0,
\]
\[
\phi_1 - \phi_2 = 2\pi L_2 I_{C_2}/\phi_0,
\]
\[
\phi_1 - \phi_2 = 2\pi L_3 I_{C_3}/\phi_0.
\]

If one assumes the current distribution shown in Fig. 8(b), the differential-difference equation for \( \phi_{N-1} \), \( \phi_{N-2} \), \( \phi_1 \), \( \phi_2 \), and \( \phi_3 \) can be written

\[
\frac{\phi_0}{2\pi L_1 C_1} \left( \frac{L_1}{L_1,2} \phi_{N-1} + \frac{L_1/2}{L_1,2} \phi_1 - \frac{L_1/3}{L_1,2} \phi_2 \right) = \sin^2 \phi_1,
\]

and

\[
\frac{\phi_0}{2\pi L_2 C_2} \left( \frac{L_2}{L_1,2} \phi_{N-1} + \frac{L_2}{L_1,2} \phi_1 - \frac{L_2}{L_1,2} \phi_2 \right) = \sin^2 \phi_1,
\]

and

\[
\frac{\phi_0}{2\pi L_3 C_3} \left( \frac{L_3}{L_1,2} \phi_{N-1} + \frac{L_3}{L_1,2} \phi_1 - \frac{L_3}{L_1,2} \phi_2 \right) = \sin^2 \phi_1,
\]

where \( \phi_0 \) is the input phase at the junction, \( \phi_{N-1} \) is the phase difference at the junction, \( \phi_1 \) is the phase difference at the junction, \( \phi_2 \) is the phase difference at the junction, and \( \phi_3 \) is the phase difference at the junction.

In Fig. 9, behaviors of single flux quanta propagating toward a STP as a function of each line bias current \( \gamma_1, \gamma_2, \gamma_3 \) and \( \Gamma L_j \) are shown. Figure 9(a) gives the behaviors of flux quanta inserted schematically in this figure; (a) \( \gamma_2 = \gamma_3 \); (b) \( \gamma_2 = \gamma_3 + 0.05 \); (c) \( \gamma_2 = \gamma_1 \) and \( \gamma_3 = 0 \).
When $I_{1} = L_{2} = I_{3} = L_{2} + L_{3}$, the relation of $\Psi_{1}^{\prime} = \Psi_{1}^{\prime}$, and $\phi_{0}$, derived from Eqs. (18)-(20) is reduced to the boundary condition for the STP of a continuous Josephson line. Figure 9 shows the result obtained by numerical analyses of Eqs. (18)-(20) for the behavior of single flux quanta propagating toward the STP as a function of the bias current of each line and $1/\Lambda_{2}$, when $L_{1} = L_{2} = L_{3} = L_{1,2} + L_{2,3}$, $C_{1} = C_{2} = C_{3}$, $I_{C1} = I_{C2} = I_{C3}$, $G_{1} = G_{2} = G_{3}$, and $\Gamma = 0.3$. In the numerical calculations we let the bias currents for individual junctions have the following relation:

$$\sin^{2} \gamma_{1} = \sin^{2} \gamma_{2} + \sin^{2} \gamma_{3},$$

where $\gamma_{1}$, $\gamma_{2}$, and $\gamma_{3}$ are the individual junction bias currents of Josephson lines 1, 2, and 3, respectively. When $\gamma_{2} = \gamma_{3}$, it can be seen from Fig. 9(a) that if $\gamma_{1}$ is relatively small, a single flux quantum propagating toward the STP on line 1 cannot pass through the turning points and stops at the turning point; however, if $\gamma_{1}$ is relatively large, it initiates a flux quantum into lines 2 and 3, and also initiates a flux quantum with an opposite sign on line 1. This phenomenon is denoted by squares in Fig. 9, and it has been used for computing the logic design of our previous papers as $S_{1}, T_{1}, P$. Increasing further, the arriving flux quantum causes a prompt increase of phase difference at the turning point which is denoted by filled triangles in Fig. 9. If $\gamma_{2} = \gamma_{3}$, there is a region denoted by open circles, where a single flux quantum propagating toward the STP on line 1 can pass through the STP, and it is initiated on either line 2 or 3. This region falls between the region denoted by filled circles and squares, as shown in Fig. 9(b). The phenomenon denoted by open circles has been used for the logic design of our previous papers as $S_{2}, T_{2}, P$. Increasing the difference between $\gamma_{2}$ and $\gamma_{3}$, the $S_{2}, T_{2}, P$ region increases. When $\gamma_{2} = 0$, the STP does not have the characteristics of $S_{2}, T_{2}, P$ as shown in Fig. 9(c). When a single flux quantum stops at the turning point after propagating on line 1, and then a single flux quantum with opposite sign propagates toward the turning point on line 2 or line 3, after the head-on collision the two flux quanta are annihilated, which is shown schematically in Fig. 7(b); but if $\gamma_{1}$ is relatively large, after the head-on collision between the stopped flux quantum and the propagating flux quantum they pass through each other and begin to propagate into lines 1 and 2 (or line 3), respectively, which is shown in Fig. 7(c). The phenomenon shown in Fig. 7(c) is observed in a relatively narrow range of $\gamma_{1}$. Using the above-mentioned characteristics, complete logic capability can be achieved with networks of discrete Josephson lines alone, as in the case of continuous lines. Not only all the computing logic circuits using continuous lines reported in our previous papers, but also new logic networks can be constructed of the discrete Josephson transmission lines.

### IV. CONCLUSIONS

By numerical calculations we have discussed the discrete Josephson-junction transmission lines which are constructed of a series of small-area Josephson junctions connected by superconducting strips. We have also studied the behaviors of single flux quanta propagating on the discrete Josephson lines with TTP and STP. Comparing with the continuous line, the features of the discrete line are the following: (1) there exists a region of forbidden propagation in the $1/\Lambda_{2}$-$\gamma$ plane, (2) it may be easy to adjust the value of the line constants $L_{1}, G_{1}, G_{2}$, and $I_{2}$ at a desired position, (3) TTP's, STP's, and lines are able to be linked easily, and (4) it is possible to have a single flux quantum stuffed in a small area by adjusting the line constant. In the case of a continuous line, single flux quants can propagate down smoothly; therefore there does not exist such energy dissipation coming from the small oscillation generated by single flux quanta transmitting on a discrete line. In order to get the flux quantum speed almost up to $\Lambda_{2}/\gamma_{2}$ (section/see) on a discrete line, it is necessary to increase not only $\gamma$ but also $\Gamma$ to damp down the small oscillation on the line. In the continuous line with increasing $\gamma$ only, one can easily obtain the flux quantum transmitting velocity which is near the limiting value $1/(LC)^{1/2}$. It is expected that the useful and practical Josephson logic networks can be made by the combination of discrete and continuous lines.

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