水理学における非希薄な静止球の懸濁系について

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Hydrodynamics of a Suspension of Nondilute Stationary Spheres

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The linear hydrodynamics of a stationary suspension of randomly distributed, nondilute spheres in an incompressible fluid is rigorously studied with the aid of a scaling expansion method. A space-time coarse graining is carried out in a manner consistent with an expansion in sphere concentration \( c \) to obtain a linear macroscopic transport equation, which is local in time but nonlocal in space, to order \( c^2 \). Fluctuations around the macroscopic motion are also investigated and are shown to be small for dimension \( d > 2 \).

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A porous medium saturated with viscous fluid is often described by the equations

\[
\rho_0 \frac{\partial \mathbf{u}}{\partial t} = \eta \nabla^2 \mathbf{u} - (\eta_0/\kappa) \mathbf{u} - \nabla P; \quad \nabla \cdot \mathbf{u} = 0 \tag{1}
\]

in terms of the fluid velocity \( \mathbf{u} \), pressure \( P \), mass density \( \rho_0 \), shear viscosity \( \eta \) (\( \eta_0 \) for the pure fluid), and permeability \( \eta_0/\kappa \). At steady state this is Brinkman's \( \text{ad hoc} \) expression obtained by adding a Darcy-law\( \text{ damping term} (\eta_0/\kappa) \mathbf{u} \) to the Navier-Stokes equations. The Debye-Bueche discussion of dilute polymer solutions is also based on Eq. (1).\(^3\)

Adopting as a model a three-dimensional system of \( N \) stationary randomly distributed spheres of common radius \( a \) saturated with a viscous incompressible fluid in volume \( \Omega \), and noting that Eq. (1) is averaged over the sphere distribution function, we see that the viscosity and permeability are solely functions of sphere volume fraction \( \phi \). At sufficiently low sphere number density \( c = N/\Omega \) the permeability is readily found to be \( \eta_0/\kappa = 6 \pi \alpha \eta \theta \approx c \kappa = \eta_0 \kappa^2 \) where \( \kappa^{-1} \) is a hydrodynamic screening length. For nondilute spheres one must consider the correlations among the spheres. This problem has been addressed by several authors;\(^4\) here we summarize the results of a new perspective based on a scaling expansion method developed by Mori and co-workers,\(^5\) which yields new results quite different from those obtained by other methods. In a recent publication\(^6\) we studied the analogous problem of diffusion-controlled reactions among stationary random reactive spheres. The porous medium problem is a vector (anisotropic) version of the scalar reaction-diffusion problem, and can be analyzed by the same techniques.

The fluid motion is then described by the linearized Navier-Stokes equation,

\[
\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \eta_0 \nabla^2 \mathbf{v} - \mathbf{v} \times \mathbf{B} - \nabla P; \quad \nabla \cdot \mathbf{v} = 0 \tag{2}
\]

with the boundary condition \( \mathbf{v}(\mathbf{r},t) = 0 \) at a sphere surface, and the initial condition \( \mathbf{v}(\mathbf{r},t=0) = \mathbf{u}_0(\mathbf{r}) \). The fluid velocity \( \mathbf{v}(\mathbf{r},t) \) is coarse grained in space;

\[
\mathbf{v}(\mathbf{r},t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^b \frac{dz}{2\pi} \mathbf{v}(\mathbf{r},z)e^{i(\mathbf{r}-\mathbf{r}) \cdot \mathbf{z} - i\omega t}, \tag{3}
\]

where the length cutoff \( b \) is set to be much longer than the radius \( a \) but much shorter than the characteristic length \( l = \kappa^{-1} \) of the macroscopic processes concerned. The pressure \( p(\mathbf{r},t) \) is coarse grained similarly.

We assume that the spheres are distributed as dilute hard spheres; uniformly but without overlap. The spheres are accounted for by a source term added to Eq. (2) and explicit reference to the source is eliminated by use of the boundary condition \( \mathbf{v}(\mathbf{r},t) = 0 \) at sphere surfaces. Then, after Laplace transformation, Eq. (2) can be written as a Langevin-like equation

\[
(\rho_0 \nabla^2 - \eta_0 \nabla^2) \mathbf{v}(\mathbf{r},z) = \mathbf{u}_0(\mathbf{r}) + \mathbf{v} \times \mathbf{B}(\mathbf{r},z) + \nabla P(\mathbf{r},z) - \mathbf{R}(\mathbf{r},z), \tag{4}
\]

where a memory function \( \mathbf{S} \) and a fluctuating force \( \mathbf{R} \) are expressed by a multiple-scattering expansion in terms of the sphere-free fluid propagator (the Oseen tensor) and a sphere scattering operator. \( \mathbf{R} \) satisfies \( \langle \mathbf{R} \rangle = 0 \), where the brackets denote a sphere configuration average. We note that \( \mathbf{S} \) and \( \mathbf{R} \) are related in a different way than by the conventional fluctuation-dissipation theorem since the fluctuations here are generated by the random sink distribution.\(^6\) Equation (4) is exact and is a useful starting point not only for deriving a macroscopic transport equation but also for studying the fluctuations around the causal motion.

Three kinds of expansions have been used to obtain macroscopic transport equations from microscopic equations: expansions in sphere con-
centrations $c$, in spatial gradients $\nabla$, and in slowness parameter $z$. However, they are all related and the expansion has to be carried out consistently. This is accomplished by the scaling expansion method in which asymptotic limits of the macroscopic parameters of Eq. (4) are taken. It enables us not only to extract the macroscopic processes characterized by length $l$ from the microscopic process characterized by $a$ but also to evaluate the fluctuations asymptotically. Since $l \gg b \gg a$, we introduce a scaling

$$ l \to \bar{l} = a \, l, \quad a = a $$

with $S \gg 1$, where all molecular quantities such as $\eta_0$ are kept constant. The space-time coarse graining is then given by the scaling

$$ \bar{r} \to \bar{r}, \quad t \to S^0 t $$

for distances $|\bar{r}| > b$, where $\theta$ is a positive exponent to be determined. Use of Eqs. (5) and (6)

$$ u \to \bar{u}(\bar{r}, S^0 z) = S^\alpha u(\bar{r}, z); \quad \bar{u}(\bar{r}, z) = b^{-\alpha} u(\bar{r}/b, b^0 z, \theta b), $$

$$ \delta \bar{u} = \delta u(S^0 t, S^0 z) = S^{-\delta} \delta u(\bar{r}, \bar{z}); \quad \delta(\bar{r}, z) = b^{-\delta} \delta u(\bar{r}/b, b^0 z, \theta b), $$

where $\bar{u}$ and $\delta u$ are scale invariants. Applying the scaling (7) to Eq. (4) into $S^0 t$ leads to the deterministic equation for $u(\bar{r}, z)$

$$ (\rho, \varphi - \eta_0 \nabla^2 \bar{u}(\bar{r}, z) + S^{-1} \varphi \delta(\bar{r}, S^0 z) = -S^{d+2} \delta(\bar{r}, S^0 z) \cdot \bar{u}(\bar{r}, z) $$

with $\nabla \cdot \bar{u} = 0$, and the stochastic equation for $\delta u(\bar{r}, z)$

$$ (\rho, \varphi - \eta_0 \nabla^2 \delta u(\bar{r}, z) + S^{-1} \varphi \delta(\bar{r}, S^0 z) = -S^{d+2} \delta(\bar{r}, S^0 z) \cdot \delta u(\bar{r}, z) + \bar{R}(\bar{r}, z) $$

leads to $\kappa a - S^{-1} \kappa_0, \varphi = S^{d} \varphi$, $N = S^{d-1} N$, and $\Omega \to S^d \Omega$, where $\varphi = 4\pi d c/3$ is the sphere volume fraction. The three kinds of expansions are then expressed by the scaling

$$ c \to S^{-2} c, \quad \nabla \to S^{-1} \nabla, \quad z \to S^{-2} z. $$

Thus the expansion in $S^{-1}$ enables us to carry out the space-time coarse graining in a manner consistent with the density $c$ expansion. This feature is quite different from conventional expansion in a small parameter.

We decompose the time evolution of $\bar{v}(\bar{r}, z)$ into a causal motion $\bar{u}(\bar{r}, z) = \bar{v}(\bar{r}, z)$ and its fluctuation $\delta\bar{u}(\bar{r}, z)$: $\bar{v} = \bar{u} + \delta\bar{u}$. This decomposition is essential since the $b$ dependence of $\delta\bar{u}$ differs from that of $\bar{u}$. Hence we define scaling exponents $\alpha$ and $\beta$ by

$$ \delta\bar{u}(\bar{r}, z); \quad \delta\bar{u}(\bar{r}, z) = \bar{u}(\bar{r}, z) \cdot \delta(\bar{r}, z). $$

The scaled memory function $S^{d+2} \bar{u}(\bar{r}, S^0 z)$ is expressed diagrammatically in Fig. 1, to order $S^{-2}$. The diagram elements, their algebraic expressions, and their order in $S^{-1}$ are given in Table I. The memory function $\bar{u}(\bar{r}, z)$ is divided into two classes, a local ($\bar{S}_L$) and a nonlocal ($\bar{S}_N$) memory function, $\bar{S}^0 = \bar{S}_L^0 + \bar{S}_N^0$; here the space dependence of $\bar{S}_L^0$ is on $\bar{r}/a$ while that of $\bar{S}_N^0$ is on $\bar{r}/l$. There are three types of interactions: (1) long-range interactions which give a local contribution to the memory function (B) and (C) of Fig. 1; (2) short-range interactions which lead to a local memory function (D); and (3) long-
range interactions which yield a nonlocal memory function (E). Note that because of a screening effect of long-range interactions, \( \langle B \rangle \), \( \langle C \rangle \), and \( \langle E \rangle \) are written in terms of a renormalized propagator. The algebraic expression of Fig. 1 is

\[
S^+ \mathcal{E}^\pm (S, r, S', r') = \Gamma \left[ 1 + S^{-1} \mu a + S^{-2} \left( \frac{5}{6} (\kappa a)^2 + \frac{1}{3} (\alpha a)^2 + \frac{3}{32} (\kappa a)^2 \ln \frac{11 \times \tau^4}{5} \right) \right] \delta (r - r')
\]

\[
- \Gamma S^2 \left( \frac{5}{2} \right) \phi \eta \varphi^{\pm} \delta (r - r') - S^{-2} c \partial_t \overline{G} (r - r', z),
\]

where \( \Gamma \) is the unit tensor, \( \varepsilon = 6 \pi \eta \mu \), \( \mu = k^2 + \alpha^2, \) \( \alpha = \rho_0 \varphi / \eta_0 \), and the renormalized propagator

\[
\overline{G} (r, t) = (2\pi)^{-3} \int d^3k e^{ik \cdot r} \eta_{\kappa} (\kappa^2 + \mu^2)^{-1} (\Gamma - \mathcal{G}^{kk} \kappa^2).
\]

The terms proportional to \( \delta (r - r') \) come from the local diagrams. The logarithmic term is due to the diagram (D), and the \( \eta_0 \) term arises from the expansion in \( \nabla^2 \) of the diagram (A). The last nonlocal term arises from the diagram (E).

In the scaling limit \( S \to \infty \), Eqs. (9a) and (10) lead to

\[
\rho_0 (\partial / \partial t) \overline{u} (r, t) = \eta_0 \nabla^2 \overline{u} - \nabla \varphi \overline{P} - c \xi \overline{u}; \quad \nabla \cdot \overline{u} = 0.
\]

The time scale of this process is \( \tau = 1 / c \). If the sphere density is not sufficiently dilute, however, the higher-order terms in \( S^{-1} \) of Eq. (10) become important on longer time scales than \( \tau \). Employing the multitime scaling, we thus obtain, on the time scale of order \( \tau (\kappa a)^{-2} \),

\[
\rho_0 (\partial / \partial t) \overline{u} (r, t) = \eta_0 \left( 1 + \frac{5}{2} \right) \nabla^2 \overline{u} - \nabla \varphi \overline{P} - c \xi \overline{u} + \nu \overline{g} (r, t); \quad \nabla \cdot \overline{u} = 0.
\]

where \( \nu (r) = \overline{g} (r, z = 0) \). Thus, to order \( S^{-2} \), the scaling expansion results in a transport equation which is local in time but nonlocal in space.

Finally, we discuss the fluctuations. Similarly to Eq. (11), on the time scale of order \( \tau \), use of Eqs. (9b) and (10) leads to the Langevin equation

\[
\rho_0 (\partial / \partial t) \delta \overline{u} (r, t) = \eta_0 \nabla^2 \delta \overline{u} - \nabla \delta \varphi \overline{P} - c \xi \delta \overline{u} + \overline{R} (r, t); \quad \nabla \cdot \delta \overline{u} = 0.
\]

The correlation function of the fluctuating force \( \overline{R} (r, t) \) also takes the asymptotic form, to the lowest order in \( S^{-1} \),

\[
\langle \overline{R} (r, t) \rangle = \delta (r - r') c \varepsilon^2 \delta \overline{u} (r, t),
\]

where we have chosen the initial \( \overline{u} (r, t) \) as a constant \( \overline{u}_0 \). Since \( \overline{u}_0 \to S^{-2} + \overline{u}_0 \) and \( \overline{R} (r, t) \to S^{-3} + \overline{u}_0 \times \overline{R} (r, t) \), use of Eq. (14) leads to \( \beta = \alpha = (d - 2) / 2 \). Thus when \( d = 3 \), \( \beta > \alpha \). Therefore, the fluctuations are small compared to the causal motion and obey a Gaussian process. The physical basis for the Gaussian process is as follows. The spatial coarse-graining cell size \( b \) was taken to be large so that the number of spheres for a cell \( cb^d \) is large even in the low-density limit \( c \to 0 \). In fact, Eq. (7) leads to \( cb^d \to S^{-2} + cb^d \). Therefore, this is valid as far as \( d > 2 \). Since Eq. (13) is linear, the fluctuating force \( \overline{R} (r, t) \) is also Gaussian. As is easily seen from Eq. (14), however, \( \overline{R} (r, t) \) is not white noise. This property prevents the steady-state variance of \( \delta \overline{u} (r, t) \) from having a long-range correlation, which is expected from the Langevin equation, Eq. (13), when \( \overline{R} (r, t) \) is white noise. In fact, use of Eqs. (13) and (14) leads to the short-range correlation

\[
\lim_{t \to \infty} \langle \delta \overline{u} (r, t) \rangle \langle \delta \overline{u} (r', t) \rangle = \frac{1}{2} (\kappa a) u_0^2 \exp (- \kappa |r - r'|).
\]

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\[\text{References}\]

Role of Diffractive Coupling and Self-Focusing or Defocusing in the Dynamical Switching of a Bistable Optical Cavity

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Optical bistability transients have been computed including one transverse Cartesian coordinate. With a Gaussian transverse input profile but weak diffractive coupling, the switchup intensity is radially dependent, and the output requires hundreds of round trips to reach steady state. Strong diffractive coupling results in whole-beam switchup and much faster convergence to steady state.

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Transverse effects, which arise from the transverse Laplacian term in the wave equation, are known to be of unavoidable importance in self-focusing and superfluorescence. In some phenomena such as self-induced transparency, a uniform-plane-wave experiment can be performed in one limit, but transverse effects can dominate the coherent pulse propagation in the other limit. Similarly, a uniform-plane-wave optical bistability experiment can be realized by use of a hybrid device in which the electrical feedback affects all parts of the light beam in the same way. However, intrinsic (purely optical) bistability experiments involve nonlinearities which depend upon the local light intensity; transverse effects are inescapable. This article describes interesting and significant differences in the hysteresis loops and switching times arising from transverse effects. The significance of diffraction can be greatly accentuated by the spatial intensity discontinuity caused by the bistability in a large-Fresnel-number case. It may be of interest to note that interesting extensions of this work have already been made to purely absorptive bistability, to competition between nearby beams, and to Ikeda instabilities.

Here optical bistability switching transients are calculated numerically including transverse effects and studied as a function of Fresnel number for the first time. For large Fresnel numbers switchup occurs rapidly for radii out to the plane-wave switchup intensity $I_1$; see Figs. 1(a) to 1(e) in agreement with Ref. 6. The diffraction

FIG. 1. Optical bistability switchup in response to a step-function input as a function of Fresnel number. (a) Input profile $I_1/I_s$. (b)-(f) For defocusing ($\Delta = -5$); (g) for focusing ($\Delta = +5$). (b)-(d) $F = 2200$; round trips 40-50; (e) $F = 220$, round trips 98-100; (f) $F = 2.2$; round trips 40-50. For comparison the plane-wave $I_1$ is ~200$I_s$ and $I_1$ is 60$I_s$, so that the beam-center intensity here is ~2.7$I_s$. Not shown: $I_1/I_s$ for $F = 22$ and $\Delta = -5$ falls off gradually, vanishing by $x/w_0 \approx 1.2$; for $F = 0.22$ diffraction overcomes self-focusing and whole-beam switching occurs.

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