Interdisciplinary Information Sciences Vol. 22, No. 1 (2016) 143–146

ISSN 1340-9050 print/1347-6157 online
DOI 10.4036/iis.2016.S.01

SHORT COMMUNICATION

Quantum Probability Aspects to Lexicographic and Strong Products of Graphs

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The adjacency matrix of the lexicographic product of graphs is decomposed into a sum of monotone independent random variables in a certain product state. The adjacency matrix of the strong product of graphs admits an expression in terms of commutative independent random variables in a product state. Their spectral distributions are obtained by using the monotone, classical and Mellin convolutions of probability distributions.

KEYWORDS: adjacency matrix, convolution of probability distributions, lexicographic product, spectral distribution, strong product

1. Products of Graphs

A graph $G = (V, E)$ is a pair, where V is a non-empty set of vertices and E a set of edges, i.e., a subset of unordered pairs of distinct vertices. If $\{x, y\} \in E$, we say that x and y are adjacent and write $x \sim y$. We deal with both finite and infinite graphs, but always assume that a graph is locally finite, i.e., $\deg(x) < \infty$ for all vertices $x \in V$. The adjacency matrix of G, denoted by $A = A[G]$, is a matrix with index set $V \times V$ defined by

$$(A)_{xy} = \begin{cases} 1, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ there is a large variety of forming their product to obtain a larger graph, see e.g., [4] and references cited therein. From the quantum probability viewpoint we have so far studied the Cartesian, star, comb and free products of graphs [1, 6, 9, 10]. In this paper, being based on a similar spirit, we will discuss the lexicographic and strong products of graphs, and derive their spectral distributions using certain concepts of independence in quantum probability.

Definition 1.1. The *lexicographic product* of G_1 and G_2, denoted by $G_1 \rhd_L G_2$, is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 \sim x_2$; or (ii) $x_1 = x_2$ and $y_1 \sim y_2$.

Lemma 1.2. *Let G_1 and G_2 be graphs with adjacency matrices A_1 and A_2, respectively. Then the adjacency matrix of the lexicographic product $G_1 \rhd_L G_2$ satisfies*

$$A[G_1 \rhd_L G_2] = A_1 \otimes J_2 + I_1 \otimes A_2, \tag{1.1}$$

where J_2 is the matrix with index set $V_2 \times V_2$ whose entries are all one, and I_1 is the identity matrix with index set $V_1 \times V_1$. In particular, the graph operation $\rhd_L$ is associative: $(G_1 \rhd_L G_2) \rhd_L G_3 \cong G_1 \rhd_L (G_2 \rhd_L G_3)$, but it is not commutative.

The proof is straightforward by definition and is omitted. The *Cartesian product* of G_1 and G_2, denoted by $G_1 \times_C G_2$, is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 = x_2$ and $y_1 \sim y_2$; or (ii) $x_1 \sim x_2$ and $y_1 = y_2$. The adjacency matrix of $G_1 \times_C G_2$ is given by

$$A[G_1 \times_C G_2] = A_1 \otimes I_2 + I_1 \otimes A_2. \tag{1.2}$$

The Cartesian product is associative and commutative. By definition, $G_1 \times_C G_2$ is a subgraph of $G_1 \rhd_L G_2$, which is viewed also from the adjacency matrices (1.1) and (1.2).

Definition 1.3. The *strong product* of G_1 and G_2, denoted by $G_1 \times_S G_2$, is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 = x_2$ or $x_1 \sim x_2$; and (ii) $y_1 = y_2$ or $y_1 \sim y_2$.

Received February 19, 2016; Accepted February 28, 2016; J-STAGE Advance published March 18, 2016
2016 Mathematics Subject Classification: Primary 05C76, Secondary 05C30, 05C50, 81S25.
This work is supported by JSPS Grant-in-Aid for Exploratory Research No. 26610018 and Scientific Research (B) No. 26310203.
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Lemma 1.4. *Let G_1 and G_2 be graphs with adjacency matrices A_1 and A_2, respectively. Then the adjacency matrix of the strong product $G_1 \times_S G_2$ satisfies*

$$A[G_1 \times_S G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2. \tag{1.3}$$

The proof is obvious. In the recent paper [7] we studied the spectral distribution of the *Kronecker product* $G_1 \times_K G_2$, which is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever $x_1 \sim x_2$ and $y_1 \sim y_2$. Then, the adjacency matrix is given by

$$A[G_1 \times_K G_2] = A_1 \otimes A_2. \tag{1.4}$$

It is noted that the Kronecker product is a subgraph of the distance-2 graph of the Cartesian product $G_1 \times_C G_2$.

There are quite a few concepts of "graph product" and the terminologies have not been unified in literatures. Our definitions are mostly in accordance with those in the handbook [4]. The Kronecker product is called conjunction in [2], the cardinal product in [3], the direct product in [4], and the strong product in [8].

2. Adjacency Matrices As Algebraic Random Variables

Let G be a (locally finite) graph and A the adjacency matrix. The *adjacency algebra* of G is the $*$-algebra generated by A and $I = A^0$ (identity matrix), and is denoted by $\mathcal{A}(G)$. Equipped with a state, $\mathcal{A}(G)$ becomes an algebraic probability space and the adjacency matrix A is regarded as a real algebraic random variable, where a state means a linear function $\varphi : \mathcal{A}(G) \to \mathbb{C}$ satisfying $\varphi(a^*a) \ge 0$ and $\varphi(I) = 1$. Then it is well known (see e.g., [6, 10]) that there exists a probability distribution μ, called the *spectral distribution* of A in the state φ, such that

$$\varphi(A^m) = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m \ge 0. \tag{2.1}$$

The left-hand side is called the *mth moment* of A in the state φ, while the right-hand side is the usual mth moment of a probability distribution μ. Note that the spectral distribution μ is not necessarily determined uniquely by (2.1) due to the famous indeterminate moment problem, but it is unique if $\sup\{\deg(x); x \in V\} < \infty$.

Let $C(V)$ be the space of all $\mathbb{C}$-valued functions on V. A matrix T with index set $V \times V$ acts on $C(V)$ by means of usual matrix multiplication:

$$Tf(x) = \sum_{y \in V} (T)_{xy} f(y),$$

whenever the right-hand side converges absolutely. It is convenient to define the "inner product" of $f, g \in C(V)$ by

$$\langle f, g \rangle = \sum_{x \in V} \overline{f(x)} g(x),$$

whenever the right-hand side converges absolutely. With each $x \in V$ we define $e_x \in C(V)$ by $e_x(y) = \delta_{xy}$. Then we have $\langle e_x, e_y \rangle = \delta_{xy}$ and $(T)_{xy} = \langle e_x, Te_y \rangle$.

A state φ on $\mathcal{A}(G)$ is called a *vector state* if it is of the form $\varphi(a) = \langle \xi, a\xi \rangle$, where $\xi \in C(V)$ with $\langle \xi, \xi \rangle = 1$. In particular, the *vacuum state* at a vertex $o \in V$ is defined by

$$\langle a \rangle_o = \langle e_o, ae_o \rangle = (a)_{oo}, \quad a \in \mathcal{A}(G).$$

If a graph G is finite, the *normalized trace* is defined by

$$\varphi_{\mathrm{tr}}(a) = \frac{1}{|V|} \mathrm{Tr}(a) = \frac{1}{|V|} \sum_{x \in V} (a)_{xx} = \frac{1}{|V|} \sum_{x \in V} \langle e_x, ae_x \rangle, \quad a \in \mathcal{A}(G).$$

We are also interested in the vector state with state vector given by

$$\psi = \frac{1}{\sqrt{|V|}} \sum_{x \in V} e_x.$$

In fact, slightly abusing symbols, we see easily that

$$\psi(a) = \langle \psi, a\psi \rangle = \frac{1}{|V|} \sum_{x,y \in V} \langle e_x, ae_y \rangle = \frac{1}{|V|} \sum_{x,y \in V} (a)_{xy}, \quad a \in \mathcal{A}(G). \tag{2.2}$$

It is noteworthy that the moments $\varphi(A^m)$ are related to counting walks in the graph G. Let $W_m(x, y; G)$ denote the number of m-step walks from a vertex x to another y in a graph G. As is easily verified by definition, we have

$$W_m(x, y; G) = (A^m)_{xy} = \langle e_x, A^m e_y \rangle, \quad m \ge 0.$$

Therefore, $\langle A^m \rangle_o$ coincides with the number of m-step walks from a fixed vertex $o \in V$ to itself. Moreover, $\varphi_{\mathrm{tr}}(A^m)$ is the average number of m-step walks from a vertex to itself (the average is taken over all vertices). Let $W_m(x, *; G)$

denote the number of m-step walks starting from x, and $\bar{W}_m$ the average of $W_m(x,*;G)$ over all vertices $x \in V$. Then we have

$$\bar{W}_m = \frac{1}{|V|}\sum_{x\in V} W_m(x,*;G) = \frac{1}{|V|}\sum_{x,y\in V} W_m(x,y;G) = \frac{1}{|V|}\sum_{x,y\in V}\langle e_x, A^m e_y\rangle = \langle \psi, A^m\psi\rangle = \psi(A^m).$$

3. Lexicographic Products

Theorem 3.1. *Let $G = G_1 \rhd_L G_2$ be the lexicographic product of two graphs G_1 and G_2, where the latter is assumed to be finite. Then the adjacency matrix $A = A[G_1 \rhd_L G_2]$ is expressed as in* (1.1) *and the right-hand side is a sum of monotone independent random variables in the product state $\varphi \otimes \psi$, where φ is an arbitrary state on $\mathcal{A}(G_1)$ and ψ is the vector state on $\mathcal{A}(G_2)$ defined as in* (2.2)*.*

Proof. Set $T_1 = A_1 \otimes J_2$ and $T_2 = I_1 \otimes A_2$. It is sufficient to show the following factorization property:

$$\varphi\otimes\psi(\cdots T_1^{\alpha}T_2^{\beta}T_1^{\gamma}\cdots) = \varphi\otimes\psi(T_2^{\beta})\cdot\varphi\otimes\psi(\cdots T_1^{\alpha+\gamma}\cdots), \quad \alpha,\beta,\gamma \geq 1.$$

The verification is straightforward from definition. In fact, since J_2 is a constant multiple of a rank-one projection such that $J_2\psi = |V_2|\psi$, the argument is similar to the case of comb products [1, 6]. □

Corollary 3.2. *Notations and assumptions being as in Theorem 3.1, let μ_1 and μ_2 be the spectral distributions of A_1 in φ and that of A_2 in ψ, respectively. Let μ be the spectral distribution of $A = A[G_1 \rhd_L G_2]$ in $\varphi\otimes\psi$. Then $\mu = (D\mu_1) \rhd \mu_2$, where $D\mu_1$ is the dilation defined by $D\mu_1(dx) = \mu_1(|V_2|^{-1}dx)$ and $\rhd$ is the monotone convolution.*

Proof. Since $J_2^m = |V_2|^{m-1}J_2$ we have

$$\varphi\otimes\psi((A_1\otimes J_2)^m) = \varphi(A_1^m)\psi(J_2^m) = \varphi(A_1^m)|V_2|^{m-1}\psi(J_2) = \varphi(A_1^m)|V_2|^{m-1}\cdot|V_2| = |V_2|^m\varphi(A_1^m), \quad m \geq 0.$$

Hence the spectral distribution of $A_1 \otimes J_2$ in the product state $\varphi\otimes\psi$ is $D\mu_1$. On the other hand, the spectral distribution of $I_1 \otimes A_2$ in the product state $\varphi\otimes\psi$ is μ_2. Since $A = A_1 \otimes J_2 + I_1 \otimes A_2$ is a sum of monotone independent random variables, the spectral distribution of A is given by the monotone convolution of $D\mu_1$ and μ_2. □

For explicit calculation of the monotone convolution $\mu = \mu_1 \rhd \mu_2$ we may employ Muraki's formula (see e.g., [5]). For a probability distribution μ on $\mathbb{R}$ the *Stieltjes transform* and the *reciprocal Stieltjes transform* are defined by

$$G_\mu(z) = \int_{\mathbb{R}}\frac{\mu(dx)}{z-x}, \quad H_\mu(z) = \frac{1}{G_\mu(z)},$$

respectively. Then, for $\mu = \mu_1 \rhd \mu_2$ we have

$$H_\mu(z) = H_{\mu_1}(H_{\mu_2}(z)), \quad \operatorname{Im} z > 0.$$

As a simple consequence, for the point masses we have $\delta_a \rhd \delta_b = \delta_{a+b}$ for $a, b \in \mathbb{R}$. We should remind that the monotone convolution is not commutative, namely, $\mu_1 \rhd \mu_2$ does not coincide with $\mu_2 \rhd \mu_1$ in general.

Example 3.3. Let $G = K_n$ be the complete graph on n vertices and A the adjacency matrix. The spectral distribution of A in the state ψ defined as in (2.2) is the point mass δ_{n-1}, since we have $\psi(A^m) = \bar{W}_m = (n-1)^m$ for $m \geq 0$. Now let $G_1 = K_m$ and $G_2 = K_n$. It follows from Corollary 3.2 that the spectral distribution of $A = A[K_m \rhd_L K_n]$ in the product state $\psi_1 \otimes \psi_2$ is given by the monotone convolution:

$$(D\delta_{m-1}) \rhd \delta_{n-1} = \delta_{n(m-1)} \rhd \delta_{n-1} = \delta_{n(m-1)+(n-1)} = \delta_{mn-1}. \tag{3.1}$$

On the other hand, we see easily that $K_m \rhd_L K_n \cong K_{mn}$ (hence we meet an exceptional case where $K_m \rhd_L K_n \cong K_n \rhd_L K_m$). Moreover, the product state $\psi_1 \otimes \psi_2$ coincides with the state ψ similarly defined for K_{mn}. Hence the spectral distribution of $A = A[K_{mn}]$ in ψ is the point mass δ_{mn-1}, which, of course, coincides with (3.1).

4. Strong Products

For two probability distributions μ_1 and μ_2 on $\mathbb{R}$, the (classical) convolution $\mu_1 * \mu_2$ is defined by

$$\int_{\mathbb{R}} f(z)\mu_1 * \mu_2(dz) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x+y)\mu_1(dx)\mu_2(dy), \quad f \in C_c(\mathbb{R}),$$

where $C_c(\mathbb{R})$ stands for the space of continuous functions on $\mathbb{R}$ with compact supports. It is apparent that $\delta_a * \delta_b = \delta_{a+b}$ for $a, b \in \mathbb{R}$. Similarly, the Mellin convolution $\mu_1 *_M \mu_2$ is defined by

$$\int_{\mathbb{R}} f(z)\mu_1 *_M \mu_2(dz) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(xy)\mu_1(dx)\mu_2(dy), \quad f \in C_c(\mathbb{R}).$$

The above definition is a natural extension of the standard one for the Mellin convolution of probability distributions supported by the half line $[0, \infty)$. We see immediately that $\delta_a *_M \delta_b = \delta_{ab}$ for $a, b \in \mathbb{R}$.

Theorem 4.1. *Let $G = G_1 \times_S G_2$ be the strong product of two graphs G_1 and G_2. Let μ_i be the spectral distribution of the adjacency matrix A_i of G_i in φ_i for $i = 1, 2$. Then, the spectral distribution μ of $A = A[G_1 \times_S G_2]$ in $\varphi_1 \otimes \varphi_2$ is given by $\mu = S^{-1}(S\mu_1 *_M S\mu_2)$, where S is the shift defined by $S\mu(dx) = \mu(dx - 1)$.*

Proof. Since $A + I_1 \otimes I_2 = (A_1 + I_1) \otimes (A_2 + I_2)$, we see that $S\mu$ is the Mellin convolution of $S\mu_1$ and $S\mu_2$. □

Example 4.2. We keep the same notations and assumptions as in Example 3.3. We apply Theorem 4.1 to obtain

$$S^{-1}(S\delta_{m-1} *_M S\delta_{n-1}) = S^{-1}(\delta_m *_M \delta_n) = S^{-1}\delta_{mn} = \delta_{mn-1}. \tag{4.1}$$

On the other hand, since $K_m \times_S K_n \cong K_{mn}$, which is easily verified by definition, the spectral distribution of $A = A[K_{mn}]$ in ψ is the point mass δ_{mn-1}, which coincides with (4.1).

Acknowledgements

The notion of lexicographic product was brought to the author by Professor Edy Tri Baskoro through his work [11] and that of strong product by Professor Akihiro Munemasa. The author thanks them for interesting conversation and for their instructing references.

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Note added in proof: The author is grateful to Professor Takahiro Hasebe who pointed out an error in Section 4.