Computability and Relative Randomness

著者	彭　寧寧
学位授与機関	Tohoku University
学位授与番号	11301甲第15405号
URL	http://hdl.handle.net/10097/57117

Computability and Relative Randomness

A dissertation presented

by

NingNing Peng

to

Mathematical Institute
for the degree of
Doctor of Science

Tohoku University
Sendai, Japan
September 2013

Contents

Abstract

In this thesis, we investigate three kinds of aspects of infinite random sequences.

First, we stduy the computability of relative versions of randomness originated by P. Martin-Löf [1] in 1966. One of the goals of algorithmic randomness theory is to measure the strength of randomness notions in the context of computability. To do this, we use the relative randomness. The relative randomness was first studied by Gaifman and Snir in 1971, and further studied by Schnorr [8], Nies, Stephan and Terwijn [3], Miller and Yu [2], Yu [10] and others. In Part I, we introduce some new relative randomness notions Γ-randomness and semi Γ-randomness. Then we show hierachical results comparing with already well known random notion such as weak n-randomness. As the Γ-randomess notion could sometimes produce alternative proofs of existing results, those results would give us a new approarch to calibrate the class of randomness notions. Next, we define randomness via another randomness. The main result is that Schnorr randomness relative to the halting problem is equivalent to Martin-Löf randomness relative to all low 1-generic reals. This is an affirmative answer to a question by Yu [9].

In Part II, we study the query complexity of algortihms with randomness. We apply Saks and Wigderson's method to compute the query complexity of many Boolean functions with randomized algorithms. Here we also treat specific game trees that have never been studied before, that is so-called unbalanced game trees as Boolean functions. The results give some positive evidence of Saks and Wigderson's conjecture.

In Part III, we investigate reverse mathematics on probability theory. Reverse mathematics is a well known research program in the foundations of mathematics, started by Harvey Friedman and Stephen Simpson in the 1970's. The main goal of reverse mathematics is to classify mathematical theorems according to which set existence axioms we need to prove them. The chief subsystem in this thesis is $\mathsf{WWKL_0}$ cosisting $\mathsf{RCA_0}$ plus a set existence axiom WWKL. Roughly speaking, WWKL asserts that there is a relative Martin-Löf random sequence. Our results shows some statements of probability theory, such as the Borel-Cantelli lemma, are equivalent to $\mathsf{WWKL_0}$, although many other properties can be proved even in $\mathsf{RCA_0}$.

Part I: Algorithmic randomness

The first part is concerned with the notion of randomness as originated by P. Martin-Löf [1] in 1966. One of the goals of algorithmic randomness theory is to measure the strength of randomness notions. The main concern of Part I is relative randomness, which is a way to measure the strength of randomness notions.

The first two chapter is devoted to define the terminology of algorithmic randomness, kolmogorov complexity and martingales. We also introduce some basic notions and descriptions of algorithmic randomness that are necessary for understanding the argument presented in sequent chapters.

Chapter 3 contains our results on K-random extension lemma. A binary string σ is K-random if $K(\sigma) \geq |\sigma|$, where K denotes prefix-free Kolmogorov complexity. Our main contribution shows that any finite string has a K-random extension i.e., $(\forall\sigma)(\exists\tau)$ such that $K(\sigma\tau) \geq |\sigma\tau|$.

Chapter 4 contains our results on relative randomness. We introduce some new relative randomness notions: Γ-randomness and semi Γ-randomness. At first, we see some important notions of randomness.

Definition 1 (Martin-Löf, 1966 [1])**.** A Martin-Löf test is a uniformly c.e. sequence $(G_m)_{m\in\omega}$ of open sets such that $\mu(G_m) \leq 2^{-m}$ for all $m \in \omega$. A set Z is Martin-Löf random if Z passes each Martin-Löf test i.e., $Z \notin \bigcap_m G_m$.

Definition 2 (Schnorr, 1971)**.** A Schnorr test is a Martin-Löf test $(G_m)_{m\in\mathbb{N}}$ such that $\mu(G_m)$ is computable uniformly in m. Z is Schnorr random if Z passes each Schnorr test.

Let Γ be a set of functions on the natural numbers. A set Z is Γ-random if Z is Martin-Löf random relative to f for all $f \in \Gamma$. The Γ-randomess notion can produce alternative proofs of existing results. For instance, some properties of $\emptyset'$-Schnorr randomness are proved more easily by the characterization due to $\mathbb{L}$-randomness than the usual methods. We also define the weak versions of Γ-randomness, called by semi Γ-randomness. Taking Γ as a Δ^0_n set, we have the following results.

Theorem 3. (1) Weak n-randomness is strictly stronger than semi Δ^0_n-randomness, for $n > 2$.

(2) Weak 2-randomness is equivalent to semi Δ^0_2-randomness.

Weak 2-randomness is a natural notion, first introduced by Kurtz in 1981.

In Chapter 5, we investigate the definability of randomness via another randomness. To compare two randomness notions with each other, we ask whether a given randomness notion can be defined via another randomness notion. Inspired by Yu's pioneering study [10], we formalize our question using the concept of relativization of randomness. We give some solutions to our formalized questions. The following is the main theorem of this chapter.

Theorem 4. For any $\emptyset'$-Schnorr test $\{\mathcal{U}_e\}_{e\in\omega}$, there exist a low 1-generic real Z and a Z-Martin-Löf test $\{\mathcal{V}_e\}_{e\in\omega}$ with $\bigcap_{e\in\omega}\mathcal{U}_e \subset \bigcap_{e\in\omega}\mathcal{V}_e$.

Part II: Randomized Complexity

In this part, we study the query complexity of algortihms with randomness. We handle the deterministic and the randomized decision trees for Boolean functions. The deterministic query complexity of Boolean function f is denoted by $D(f)$, and the randomized one is denoted by $R(f)$. A major open problem is how small $R(f)$ can be with respect to $D(f)$. The following is one of the most important conjectures in this field. We will focus on this conjecture.

Conjecture 5 (Saks and Wigderson, [12])**.** For any Boolean function f, $R(f) = \Omega[D(f)^{0.753}]$.

In Chapter 8, we mainly investigate the distributional query complexity of certain unbalanced trees. The eigen-distribution was defined by Liu and Tanaka [13]. This particular distribution is defined as the worst distribution on assignments to variables of T_2^k regarding a best algorithm. We obtained eigen-distribution on an unbalanced game tree $(T_2^-)^k$.

In section 8.1, 8.2, 8.3, we investigate the distributional query complexity of some unbalanced game trees: $(T_m^-)^k$ and $(T_m')^k$. We give a simple approximation on the unbalanced game trees in the CD case, by which we can determine the complexity easily. Then we give some positive evidences of Saks and Wigdersons conjecture finaly.

Theorem 6. 1. For any $(T_2^-)^k$, the distributional complexity $R((T_2^-)^k) = \Theta((\frac{3+\sqrt{3}}{2})^k) = \Theta(n^{0.784})$.

2. For any tree $(T_2')^k$, $R((T_2')^k) = \Theta(n^{0.955})$.

3. For any $(T_m^-)^k$, the distributional complexity $R((T_m^-)^k) = \Theta(n \log m^{m+1})$.

Here, $f(n) = \Theta(g(n))$ is Bachmann-Landau's Big theta notation.

Part III: Reverse Mathematics

In Part III, we study the proof-theoretic strength of statements on randomness when we formalise ordinal mathematics in the second order arithmetic.

Reverse mathematics ([15] for the basic reference) is a well known research program in the foundations of mathematics. It was started by Harvey Friedman and Stephen Simpson in the 1970's and developed in many publications. A basic goal of reverse mathematics is to determine which axiom are systems needed to prove particular theorems of mathematics.

The subsystem RCA_0 is commonly used as a base system for reverse mathematics. The axiom system WWKL (weak weak König's Lemma) was first introduced by Simpson and Yu [10] and shown to be strictly intermediate between RCA_0 and WKL_0 as well as equivalent to some statements on Lebesgue and Borel measure. Roughly speaking, WWKL asserts that there is a relative Martin-Löf random sequence. It is equivalent to some statements on Lebesgue and Borel measure. WWKL was further studied by Giusto and Simpson (2000); and by Brown, Giusto and Simpson (2002).

In Chapter 10, we investigate reverse mathematics on probability theory. One of the main purpose is to establish equivalence between law of large number theorems and set existence axioms of second order arithmetic. We summarize some of our results in this chapter.

Theorem 7. (i) Borel-Cantelli lemma is equivalent to WWKL over RCA_0.

(ii) The weak law of large numbers is provable in RCA_0.

(iii) The de Moivre-Laplace theorem can be proved in RCA_0.

Bibliography

[1] Per. Martin-Löf: The definition of random sequences. *Information and Control, vol. 9, no. 6, pp. 602-619, 1966.*

[2] Joseph S. Miller and Liang Yu: On initial segment complexity and degrees of randomness. Transactions of the American Mathematical Society, Vol. 360, No. 6, pp. 3193-3210, 2008.

[3] André Nies, Frank Stephan and Sebastiaan A. Terwijn: Randomness, Relativization and Turing Degrees. The Journal of Symbolic Logic, vol. 70, No.2, pp. 515-535, 2005.

[4] NingNing Peng, Kojiro Higuchi, Takeshi Yamazaki and Kazuyuki Tanaka Relative Randomness for Martin-Löf random Sets. *Lecture Notes in Computer Science, vol. 7318, pp. 581-588, 2012.*

[5] NingNing Peng, Takeshi Yamazaki, Kazuyuki Tanaka and ChenGuang Liu, A Lemma on K-random Extension, A processing of the 3rd IEEE International Conference on Computer and Electrical Engineering, vol. 3, 2010.

[6] NingNing Peng, Takeshi Yamazaki, Kazuyuki Tanaka and ChenGuang Liu: A Lemma on K-random Extension. A processing of the 3rd IEEE International Conference on Computer and Electrical Engineering, Vol. 3, 2010.

[7] NingNing Peng: A Report on Studies of Relative Randomness. RIMS Kôkyûroku, Vol. 1832, pp. 154-157, 2013.

[8] Claus Peter Schnorr: A unified approach to the definition of a random sequence. Mathematical Systems Theory, Vol. 5, pp. 246-258, 1971.

[9] Liang Yu: Personal Communication, 2012.

[10] Liang Yu: Characterizing strong randomness via Martin-Löf randomness. *Annals of Pure and Applied Logic, vol. 163, no. 3, pp. 214-224, 2012.*

[11] André Nies: Computability and Randomness. *Oxford University Press, 2009.*

[12] Saks M., Wigderson A.: Probabilistic Boolean Decision Trees and the Complexity of Evaluating Game Trees. In Proceedings of 27th Annual IEEE Symposium on Foundations of Computer science(FOCS). 29-38, 1986.

[13] Liu C.G., and Tanaka K.: Eigen-Distribution on Random Assignments for Game Trees. Information Processing Letters, 104(2): 73-77, 2007.

[14] Xiaokang Yu: *Measure theory and weak Königs lemma* Archive Math. Logic, 30, 171-180, 1990

[15] Stephen G. Simpson: *Subsystems of Second Order Arithmetic,* Cambridge University Press, 1999.