博　士　論　文

Calabi's conjecture of the Kähler-Ricci soliton type

（ケーラー・リッチ・ソリトン型のカラビ予想）

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Contents

Abstract

In this thesis, we discuss Calabi's equation of the Kähler-Ricci soliton type on a compact Kähler manifold. This equation was introduced by Zhu as a generalization of Calabi's conjecture. We give necessary and sufficient conditions for the unique existence of a solution for this equation on a compact Kähler manifold with a holomorphic vector field which has a zero point. We also consider the case of a nowhere vanishing holomorphic vector field, and give sufficient conditions for the unique existence of a solution for this equation.

In Chapter 1, we give some preliminaries. Let (M, ω) be a compact Kähler manifold of complex dimension m with a Kähler form ω. For every holomorphic vector field X on M, the holomorphic potential $\theta_X(\omega)$ of X associated to ω is defined by

$$\mathcal{L}_X \omega = \sqrt{-1}\partial\bar{\partial}\theta_X(\omega) \tag{1.0.1}$$

with a normalization condition

$$\int_M \theta_X(\omega)\,\frac{\omega^m}{m!} = 0. \tag{1.0.2}$$

Here $\mathcal{L}_X$ denotes the Lie derivative along X. In Section 1.2, we describe some properties of holomorphic potentials on a Kähler manifold.

In order to introduce Calabi's conjecture of the Kähler-Ricci soliton type, we discuss Calabi's conjecture and Kähler-Ricci solitons in Chapter 2. In 1978, Yau [21] proved the following theorem, which was conjectured by Calabi:

Theorem 1.0.1. *Let $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form. Then there exists a unique Kähler form ω' in the Kähler class $[\omega]$ such that $\mathrm{Ric}(\omega') = \Omega$.*

Here $c_1(M)$ denotes the first Chern class of M and $\mathrm{Ric}(\omega')$ denotes the Ricci form of ω'. Yau proved this theorem by continuity method and Cao [5]

also proved this theorem by using a geometric flow. This theorem is deeply related to Kähler-Einstein metrics. For instance, as a immediate corollary of this theorem we have

Corollary 1.0.2. *If $c_1(M) = 0$, then there exists a unique Ricci-flat Kähler form ω' in Kähler class $[\omega]$.*

In Section 2.3, we discuss Kähler-Ricci solitons. A Kähler-Ricci soliton is one of the generalization of a Kähler-Einstein metric and closely related to the limiting behavior of the normalized Kähler-Ricci flow. A Kähler form ω' is called a Kähler-Ricci soliton if it satisfies

$$\mathrm{Ric}(\omega') - \omega' = \mathcal{L}_X \omega' \tag{1.0.3}$$

for some holomorphic vector field X. We remark that, in this thesis, a holomorphic vector field means a holomorphic section of holomorphic tangent bundle $T^{1,0}M$. If $X = 0$, then ω' is nothing but a Kähler-Einstein metric. Moreover, Guan [6] extended the notion of Kähler-Ricci soliton and introduced generalized quasi-Einstein metric. A Kähler metric ω' is called a generalized quasi-Einstein Kähler metric if it satisfies

$$\mathrm{Ric}(\omega') - \mathrm{HRic}(\omega') = \mathcal{L}_X \omega' \tag{1.0.4}$$

for some holomorphic vector field X, where $\mathrm{HRic}(\omega')$ is the harmonic part of $\mathrm{Ric}(\omega')$ with respect to ω'.

Motivated by the study of Kähler-Ricci solitons, Zhu [22] considered the following problem as a generalization of Calabi's conjecture:

Problem 1.0.3 (Calabi's conjecture of the Kähler-Ricci soliton type)**.** *Let $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form and X be a holomorphic vector field on M. Then, does there exist a Kähler form ω' in the Kähler class $[\omega]$ such that*

$$\mathrm{Ric}(\omega') - \Omega = \mathcal{L}_X \omega'? \tag{1.0.5}$$

We call (1.0.5) Calabi's equation of the Kähler-Ricci soliton type. It is obvious that a Kähler-Ricci soliton ω' is a solution for (1.0.5) when $\Omega = \omega'$. In his paper, Zhu [22] showed the following theorem:

Theorem 1.0.4 ([22])**.** *Let (M, ω) be a compact Kähler manifold with $c_1(M) > 0$. Let $\Omega \in 2\pi c_1(M)$ be a positive definite $(1,1)$-form on M and X be a holomorphic vector field on M. Then equation (1.0.5) has a unique solution ω' in the Kähler class $[\omega]$ if and only if both of the following conditions hold:*

(i) There exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter family $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$,

(ii) $\mathcal{L}_X\Omega$ is a real $(1,1)$-form on M.

Here $\mathrm{Aut}_0(M)$ is the identity component of the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of M.

In Chapter 3, we focus on Calabi's conjecture of the Kähler-Ricci soliton type. One of the main purposes of this thesis is to remove the assumption that M is a Fano manifold and Ω is positive definite, and give a partial answer to Problem 1.0.3. Our first main result is as follows:

Theorem 1.0.5. *Let (M,ω) be a compact Kähler manifold and $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form on M. Suppose that a holomorphic vector field X has a zero point. Then equation (1.0.5) has a unique solution ω' in the Kähler class $[\omega]$ if and only if both of the following conditions hold:*

(i) There exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter family $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$,

(ii) $\mathcal{L}_X\Omega$ is a real $(1,1)$-form on M.

As a corollary of Theorem 1.0.5, we have

Corollary 1.0.6. *Let (M,ω) be a compact Kähler manifold. Let $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form on M and X be a holomorphic vector field on M. Suppose $H^1(M;\mathbb{R}) = 0$. Then equation (1.0.5) has a unique solution ω' in the Kähler class $[\omega]$ if and only if both of the following conditions hold:*

(i) There exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter family $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$,

(ii) $\mathcal{L}_X\Omega$ is a real $(1,1)$-form on M.

In particular, if M is a Fano manifold, i.e., $c_1(M) > 0$, then M satisfies the condition $H^1(M;\mathbb{R}) = 0$. Zhu used the continuity method in the proof of his theorem, but on the other hand we show Theorem 1.0.5 by using a geometric flow. Now we provide an outline of the proof of Theorem 1.0.5, in particular, the existence of the solution.

Let X be a holomorphic vector field on M. We assume that there exists a maximal compact subgroup $K \subset \mathrm{Aut}_0(M)$ such that $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}} \subset K$. By changing ω if necessary, we may assume that ω is a K-invariant Kähler form. Let $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form such that $\mathcal{L}_X\Omega$ is a real $(1,1)$-form. Since $\Omega \in 2\pi c_1(M)$, there exists a real-valued function f on M such that

$$\begin{cases} \mathrm{Ric}(\omega) - \Omega = \sqrt{-1}\partial\bar{\partial} f, \\ \int_M e^f \frac{\omega^m}{m!} = \int_M \frac{\omega^m}{m!}. \end{cases} \tag{1.0.6}$$

We consider the following parabolic complex Monge-Ampère equation:

$$\begin{cases} \dot{\varphi}_t = \log \frac{\omega_t^m}{\omega^m} - f + \theta_X(\omega) + X(\varphi_t), \\ \varphi_0 = 0, \end{cases} \tag{1.0.7}$$

where $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$. In Section 3.2, we prove the short-time existence. In Section 3.3, we show some a priori estimates in order to prove the long-time existence. Section 3.3 consists of the following three parts:

(1) Volume ratio estimate (Subsection 3.3.1)

(2) C^2 estimate (Subsection 3.3.2)

(3) C^3 estimate (Subsection 3.3.3)

We use the assumption that X has a zero point in Subsection 3.3.1. Specifically, we use the following lemma:

Lemma 1.0.7 (see [9], [22])**.** *Let (M, ω) be a compact Kähler manifold and X be a holomorphic vector field on M which has a zero point. Let $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form. Suppose that $\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega_\varphi$ are real $(1,1)$-forms. Then $\|\theta_X(\omega)\|_{C^0} = \|\theta_X(\omega_\varphi)\|_{C^0}$.*

In Section 3.4, we prove the convergence of the flow by standard arguments of parabolic partial differential equations, and hence, complete the proof of the existence of the solution ω' in Theorem 1.0.5.

We also consider the case of a nowhere vanishing holomorphic vector field X in Section 3.5. Under the condition that X has no zero point, we show the following theorem:

Theorem 1.0.8. *Let (M, ω) be a compact Kähler manifold and $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form on M. Let X be a holomorphic vector field which has no zero point. Assume that both $\{\exp(t \operatorname{Re} X)\}_{t\in\mathbb{R}}$ and $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$ are periodic. Moreover, suppose that $\mathcal{L}_X\Omega$ is a real $(1,1)$-form on M. Then equation (1.0.5) has a unique solution ω' in the Kähler class $[\omega]$.*

From the proof of Theorem 1.0.5, we can see that Theorem 1.0.8 follows from the next lemma:

Lemma 1.0.9. *Let (M, ω) be a compact Kähler manifold and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form. Let X be a nowhere vanishing holomorphic vector field. Suppose that $\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega_\varphi$ are real $(1,1)$-forms. Assume both $\{\exp(t\,\mathrm{Re}\,X)\}_{t\in\mathbb{R}}$ and $\{\exp(t\,\mathrm{Im}\,X)\}_{t\in\mathbb{R}}$ are periodic. Then there exists a constant C independent of φ such that*

$$|X(\varphi)| \leq C. \tag{1.0.8}$$

At first, we consider the case M is a 1-complex torus. In this case, we can remove the assumption that $\{\exp(t\,\mathrm{Re}\,X)\}_{t\in\mathbb{R}}$ and $\{\exp(t\,\mathrm{Im}\,X)\}_{t\in\mathbb{R}}$ are periodic, and we obtain the following lemma:

Lemma 1.0.10. *Let M be a 1-complex torus and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form. Let X be a holomorphic vector field. Suppose that $\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega_\varphi$ are real $(1,1)$-forms. Then*

$$|X(\varphi)| \leq C \tag{1.0.9}$$

for some constant C depending only on M, X and ω.

If $m \geq 2$, then we can see that M is foliated by 1-complex tori from the assumption that both $\{\exp(t\,\mathrm{Re}\,X)\}_{t\in\mathbb{R}}$ and $\{\exp(t\,\mathrm{Im}\,X)\}_{t\in\mathbb{R}}$ are periodic. Therefore, Lemma 1.0.9 follows from Lemma 1.0.10.

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