Translating Solitons of the Mean Curvature Flow in Arbitrary Codimension

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Abstract

This is a survey of the Ph.D thesis by the author. In the thesis we study the translating solitons of the mean curvature flow. Although many researchers study translating solitons in codimension one, there are few references and examples for higher codimensional case. Here, we mainly consider the mean curvature flow in arbitrary codimension. Firstly, we obtain non-existence results of Bernstein type for the translating solitons in higher codimension and the eternal solutions in codimension one. Secondly we provide many new examples of translating solitons in arbitrary codimension. We will see that these examples have the property called parallel principal normal. Finally we characterize the complete translating solitons with parallel principal normal under a certain curvature condition.

1 Introduction

This is a survey of the Ph.D thesis by the author. The main object of the thesis is *translating solitons* of the mean curvature flow in arbitrary codimension.

We consider deformations of submanifolds M^n in certain ambient spaces $\overline{M}^{n+m}$ by their mean curvature vectors. Let $F : M^n \times [0, T_0) \to \overline{M}^{n+m}$ be a one parameter family of smooth immersions with initial data F_0 and the second fundamental form B. The mean

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curvature vector is given by $\vec{H} := \mathrm{Trace}B$. Then the *mean curvature flow* is defined by

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) = \vec{H}(p,t),\ p \in M^n, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases} \tag{1.1}$$

The mean curvature flow is the most important geometric flow of submanifolds because it is the gradient flow of the volume functional and minimal submanifolds are stationary points of this flow. This implies that the mean curvature flow can be used to construct or find minimal submanifolds. Also, this flow makes a rough initial submanifold into a nice geometric shape, like standard Euclidean sphere. By observing the limiting submanifolds, if exists, we can see how the original shape, curvature, topology, etc. change under the mean curvature flow. In this way, the mean curvature flow is a very strong tool in the submanifold geometry.

However, in general, the mean curvature flow has finite time singularities where the norm of the second fundamental form $|B|$ blows up. After a singularity occurs, the smooth flow no longer continues. One way to overcome this difficulty is to use the surgery near the singularity. To perform the surgery, we need to know the structure around the singularities by rescaling.

Singularities of the mean curvature flow are categorized into two types called the type I and the type II by their blow-up rates of $|B|$. One says that M^n has a singularity of the type I if there exists a constant $c > 0$ such that

$$\max_{M_t} |B|^2 \leq \frac{c}{T_0 - t}, \quad \forall t \in [0, T_0). \tag{1.2}$$

Otherwise, one calls the singularity to be of the type II. Although the study of the type I singularities, i.e., of *self-shrinkers* is developed, the type II singularities are less known. The second fundamental form near the type I singularity is well controlled by the definition. On the other hand, the type II singularities are much more difficult to deal with.

A *translating soliton* $M^n \subset \mathbb{R}^{n+m}$ is a submanifold defined by

$$\vec{H} = T^{\perp}, \tag{1.3}$$

where $T^{\perp}$ denotes the normal component of a nonzero constant vector $T \in \mathbb{R}^{n+m}$ and $\vec{H}$ is the mean curvature vector of M^n. Translating solitons naturally arise near singularities of the type II after a rescaling. Hence it is important to study the translating solitons to characterize the type II singularities. Translating solitons are solutions to the mean curvature flow, which move by translation in the direction of T without changing their shape. Thus the translating solitons are *eternal solutions* to the mean curvature flow which exist on all the time $(-\infty, \infty)$. Since the mean curvature flow is a parabolic type

PDE, we can not solve it backward in time. This means that the translating solitons are highly special solutions to the mean curvature flow, and the geometry of translating solitons is interesting in itself.

Another important topic is the higher codimensional problem. Although the higher codimensional mean curvature flow is natural to consider, it is less known because the behavior of the second fundamental form becomes more complicated. Nevertheless, the Lagrangian mean curvature flow and the symplectic mean curvature flow in Kähler-Einstein manifolds attract much attention. An important fact is that the almost calibrated Lagrangian or the symplectic mean curvature flow in a certain class of Kähler-Einstein manifolds can not develop the type I singularities (see [4], [40]). Therefore, the study of the type II singularities, especially, of translating solitons is required.

Motivated by these backgrounds, we investigate in this thesis the translating solitons in arbitrary codimension by using geometric analysis. Our aims in the thesis are:

1. to show non-existence results,

2. to make non-trivial examples,

3. to determine the global shape of the geometric object by its curvature condition.

We give satisfactory answers to these aims through four main results.

2 Bernstein type problem of the translating solitons

Translating solitons have similar properties as minimal submanifolds in the Euclidean space. Especially, we show a Bernstein type result (non-existence result) for the translating solitons. In the case of codimension $m = 1$, Bao-Shi proved the following:

Theorem 2.1 (Bao-Shi [3], 2013). *Let $M^n \subset \mathbb{R}^{n+1}$ be an n-dimensional complete translating soliton. If the image of the Gauss map of M^n lies in a closed ball $B_\Lambda^{\mathbb{S}^n}$ of $\mathbb{S}^n$ with radius $\Lambda < \pi/2$, then M^n must be a hyperplane.*

We generalize Bao-Shi's theorem to arbitrary codimension. Let $\{e_i\} \subset TM$ be a positively oriented orthonormal frame. In arbitrary codimensional case, we need the *generalized Gauss map*

$$M^n \ni p \mapsto T_pM = e_1 \wedge \cdots \wedge e_n \in \mathbf{G}^+_{n,m}, \tag{2.1}$$

where $\mathbf{G}^+_{n,m}$ is the Grassmannian manifold consisting of positively oriented n-subspaces in $\mathbb{R}^{n+m}$. We use the so-called *w-function* on M^n to study the distribution of the image of the Gauss map. Fix a unit n-plane $A = a_1 \wedge \cdots \wedge a_n$ in $\mathbb{R}^{n+m}$ and define a function w on M^n by

$$w = w_A = \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle = \det(\langle e_i, a_j \rangle).$$

If M^n is a graph on an open domain $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, where $\mathbb{R}^n$ is spanned by $\{a_1, \ldots, a_n\}$, the w-function is always positive. Then the function $v = 1/w$ is just the coffiecient of the volume element of the graph (see for instance [8], [45]), that is,

$$v = \sqrt{\det(g_{ij})} = \sqrt{\det\Big(\delta_{ij} + \sum_\alpha \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}\Big)}.$$

Conversely, if $w_A > 0$ on M^n for some $A = a_1 \wedge \cdots \wedge a_n$, then M^n can be written as a graph at least on some open domain $\Omega \subset \mathbb{R}^n = span\{a_1, \cdots, a_n\}$.

The condition $w > 0$ also means that the image of the Gauss map is contained in one coordinate neighborhood of the Grassmannian manifold if we take the matrix coordinates on $\mathbf{G}^+_{n,m}$ as usual (see for instance [26], p.21, Example 1.51).

We use the maximum principle by a standard analysis of $|B|$ and the w-function. Then we obtain the following Bernstein type result.

Main Theorem 1 ([23])**.** *Let $M^n \subset \mathbb{R}^{n+m}, n \geq 2, m \geq 1$ be an n-dimensional complete translating soliton with flat normal bundle and R be the Euclidean distance from a fixed point on M^n to a point on M^n. If $w = w_A$ is positive for some n-plane A, and it satisfies the growth condition*

$$v = \frac{1}{w} = o(R^{\frac{1}{2}}),$$

then M^n must be an affine subspace.

The function v represents the slope of the graph. The importance of this theorem is the growth condition $v = o(R^{\frac{1}{2}})$. This is optimal because the translating paraboloid satisfies $v \sim CR^{\frac{1}{2}}$. Therefore this theorem characterizes the graphic translating soliton in arbitrary codimension by the slope condition.

Remark 2.2. Recently Xin [44] showed a Bernstein type result for the translating solitons in arbitrary codimension without flatness of the normal bundle under a stronger slope condition.

As a direct corollary, we obtain a Bernstein type result for entire graphic minimal submanifolds ($T \equiv 0$) with flat normal bundle under the growth condition of the slope. The classical Bernstein theorem says that there are no non-trivial entire minimal graphs in $\mathbb{R}^3$. This is also true for entire minimal graphs (hypersurfaces) up to in $\mathbb{R}^8$ and false in higher dimensions. In the case of higher codimension, few results are known about Bernstein type problem. Our result gives a partial answer to this problem.

Corollary 2.3. *An entire graph of a minimal submanifold or a translating soliton $M^n = \{(x, u(x)) | x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+m}$ given by m functions $u^\alpha(x^1, \cdots, x^n)$ with flat normal bundle satisfying*

$$v = \sqrt{\det\left(\delta_{ij} + \sum_\alpha \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}\right)} = o(R^{\frac{1}{2}}),$$

where $R = (|x|^2 + |u(x)|^2)^{\frac{1}{2}}$, must be an affine subspace.

Remark 2.4. For entire graphs of minimal submanifolds with flat normal bundle (not translating solitons), a better result is known by Smoczyk, Wang and Xin ([36], Theorem 1 and Corollary 1). Their growth condition is $v = o(R)$.

3 Bernstein type problem of eternal solutions to the mean curvature flow

Next, we show a Bernstein type theorem of *eternal solutions* to the mean curvature flow. In general, after rescaling near a singularity of the type II, we obtain eternal solutions (not necessarily translating solitons). Hence it is natural to consider the eternal solutions for the characterization of the type II singularities. Although the translating solitons are time-independent, the eternal solutions are time-dependent. This makes the study of the eternal solutions difficult.

We mainly use the method of the harmonic map heat flow on noncompact complete manifolds to show the curvature estimate of eternal solutions (see [37] and [39]). Although the idea of the proof is similar to Main theorem 1, we essentially need a time-dependent analysis in this case.

Main Theorem 2 ([24]). *Let $F : M^n \times (-\infty, \infty) \to \mathbb{R}^{n+1}$ be a complete eternal solution to the mean curvature flow. If there exist a positive constant C_1 and a nonnegative constant C_2 such that $w(p,t) \geq C_1$ and $|\vec{H}(p,t)| \leq C_2$ for any point in $M^n \times (-\infty, \infty)$, then $M_t = F(M^n, t)$ must be a hyperplane for any $t \in (-\infty, \infty)$.*

This is also a generalization of Bao-Shi's theorem to space-time since translating solitons satisfiy $|\vec{H}| \leq |T|$.

4 Examples of the translating solitons in arbitrary codimension

In this section, we consider examples of translating solitons in arbitrary codimension. Few examples are known even in the hypersurface case. One dimensional translating soliton in

$\mathbb{R}^2$ is known to be only the *grim reaper* $y = -\log \cos x$ which lies between two vertical lines $x = \pm\frac{\pi}{2}$ (see Fig.1). A trivial generalization of the grim reaper is the product immersion of the grim reaper and $\mathbb{R}^{n-1}$ which is called the *grim reaper cylinder*. The grim reaper cylinder can be written as a complete graph on a strip region in $\mathbb{R}^n$. In hypersurface case, we know rotationally symmetric translating solitons called the *translating paraboloid* and the *translating catenoid*. The former is a convex entire graph and grows quadratically at infinity. The latter is complete, non-convex and non-graphical translating soliton which is made by the *wing-like curves* (see Fig.2 and [6]).

Inspired by the construction of rotationally symmetric translating solitons, we construct a lot of new non-trivial examples of translating solitons in arbitrary codimension by using minimal submanifolds in the sphere.

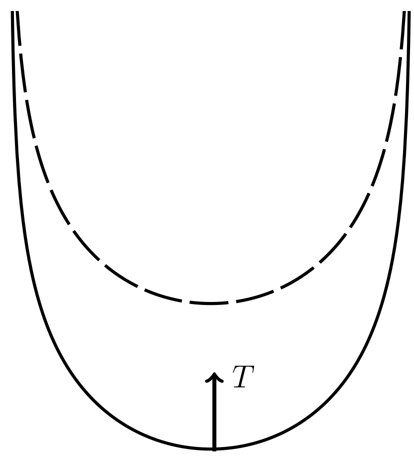

Figure 1: Two grim reapers

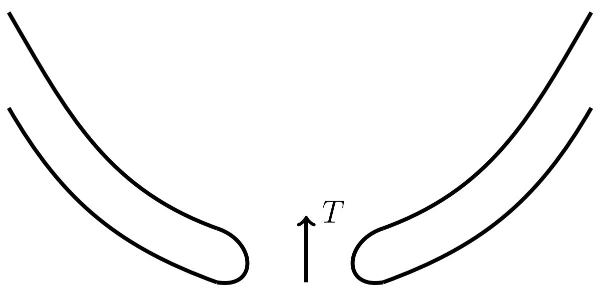

Figure 2: Wing-like curves

In the following, we always assume $n \geq 2$ and $m \geq 1$.

Main Theorem 3 ([25])**.** *Let N^{n-1} be any complete minimal submanifold of the unit sphere $S^{n+m-2}(1) \subset \mathbb{R}^{n+m-1}$, and $r(s) : \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying the ODE*

$$\ddot{r}(s) = \big(1 + \dot{r}(s)^2\big)\left(1 - \frac{(n-1)\dot{r}(s)}{s}\right). \tag{4.1}$$

Let $M^n := \mathbb{R}_+ \times N^{n-1}$ and define an immersion $F : M^n \to \mathbb{R}^{n+m}$ by

$$F(s, q) := \big(sq, r(s)\big) \in \mathbb{R}^{n+m}, \tag{4.2}$$

where $q \in N^{n-1} \subset S^{n+m-2}(1) \subset \mathbb{R}^{n+m-1}$ and $s \in \mathbb{R}_+$. Then a submanifold $F(M^n) \subset \mathbb{R}^{n+m}$ is a translating soliton with the direction $T = (0, 0, \cdots, 1) \in \mathbb{R}^{n+m}$.

By using this theorem, we construct many non-trivial codimension two complete translating solitons with flat normal bundle, which are generated from a minimal hypersurface in the unit sphere $S^{n+m-2}(1) \subset \mathbb{R}^{n+m-1}$ and the wing-like curve.

Example 4.1. Let $r(s)$ be a solution of (4.1) with $n = 3$. Then the following immersion is a translating soliton in the direction of $T = (0, 0, 0, 0, 1) \in \mathbb{R}^5$ with flat normal bundle.

$$F(s, x, y) := \left(\frac{s}{\sqrt{2}}\cos x, \frac{s}{\sqrt{2}}\sin x, \frac{s}{\sqrt{2}}\cos y, \frac{s}{\sqrt{2}}\sin y, r(s)\right) \in \mathbb{R}^5,$$

where (x, y) is a local coordinate of the Clifford torus $S^1\big(\frac{1}{\sqrt{2}}\big) \times S^1\big(\frac{1}{\sqrt{2}}\big) \subset S^3(1) \subset \mathbb{R}^4$. Furthermore if we take the wing-like curve as $r(s)$, then the immersion is a codimension two complete translating soliton with flat normal bundle.

Let $\nu := \vec{H}/|\vec{H}|$, $(|\vec{H}| \neq 0)$ be the *principal normal* of an immersion. We say that an immersion has *parallel principal normal* (PPN) if

$$\nabla^{\perp}\nu \equiv 0. \tag{4.3}$$

We find that examples in Main Theorem 3 have the property PPN.

5 Splitting theorem of the translating soliton

Finally, we characterize the complete translating soliton with PPN. A product immersion of the grime reaper and a complete minimal submanifold in the Euclidean space is a simple example having the property PPN. To characterize a translating soliton with PPN, we use the quantity $P := \langle B, \vec{H}\rangle$, that is, the second fundamental form in the direction of $\vec{H}$.

EXAMPLE 5.1. Let $N^{n-1} \subset \mathbb{R}^{n+m-2}$ be a complete minimal submanifold, $\gamma(s) = \{(s, r(s)) | r(s) = -\log\cos s, -\pi/2 < s < \pi/2\} \subset \mathbb{R}^2$ be the grim reaper. Then the following product immersion

$$M^n = N^{n-1} \times \gamma \subset \mathbb{R}^{n+m-2} \times \mathbb{R}^2 = \mathbb{R}^{n+m}$$

is a complete translating soliton in the direction of $T = (0, \cdots, 0, 1) \in \mathbb{R}^{n+m}$ with parallel principal normal. Moreover this immersion satisfies the condition $|P|^2 \equiv |\vec{H}|^4$ on M^n.

REMARK 5.2. In hypersurface case, $|P|^2 \equiv |\vec{H}|^4$ implies $|B|^2 \equiv |\vec{H}|^2$, that is, flatness of the scalar curvature of M^n.

Main Theorem 4 concerns the converse of this fact. We obtain the following splitting theorem.

Main Theorem 4 ([25])**.** *A complete translating soliton $F : M^n \to \mathbb{R}^{n+m}$ with parallel principal normal such that $|P^2|/|\vec{H}|^4$ attains its maximum on M^n can only be the product immersion*

$$M^n = \gamma \times N^{n-1} \subset \mathbb{R}^2 \times \mathbb{R}^{n+m-2}$$

of the grim reaper $\gamma \subset \mathbb{R}^2$ and a complete minimal submanifold $N^{n-1} \subset \mathbb{R}^{n+m-2}$.

SKETCH OF THE PROOF. Let $u := \langle F, T \rangle$ be the height function in the direction of T. The key point of the proof is to show the following relation by using the property PPN:

$$\begin{aligned} \Delta\Big(\frac{|P|^2}{|\vec{H}|^4}\Big) = & \frac{2}{|\vec{H}|^4}\Big|\nabla|\vec{H}| \otimes \frac{P}{|\vec{H}|} - |\vec{H}|\nabla\Big(\frac{P}{|\vec{H}|}\Big)\Big|^2 \\ & - \Big\langle \nabla u, \nabla\Big(\frac{|P|^2}{|\vec{H}|^4}\Big)\Big\rangle - \frac{2}{|\vec{H}|}\Big\langle \nabla|\vec{H}|, \nabla\Big(\frac{|P|^2}{|\vec{H}|^4}\Big)\Big\rangle. \end{aligned} \tag{5.1}$$

Under our assumption, the strong maximum principle for (5.1) implies that $|P|^2/|\vec{H}|^4$ is constant. Applying the technique by Huisken in [16] and Smoczyk in [34], we then have $|P|^2 = |\vec{H}|^4$. Now the distributions $\mathcal{D}(x)$ and $\mathcal{D}^\perp(x)$ of TM defined by

$$\mathcal{D}(x) := \{X \in T_xM | PX = 0\},$$
$$\mathcal{D}^\perp(x) := \{X \in T_xM | PX = |\vec{H}|^2 X\},$$

so that $T_xM = \mathcal{D}^\perp(x) \oplus \mathcal{D}(x)$, turn out to be parallel and integrable. Hence we can apply the de Rham decomposition theorem, and M^n is isometric to a product manifold:

$$M^n \cong \gamma \times N^{n-1}.$$

Moreover we use the lemma by Moore ([30], Section 2, Lemma, p.163) to show that the immersion F is actually a product immersion:

$$F = F_1 \times F_2,$$

where $F_1 : \gamma \to \mathbb{R}^2$ is the grim reaper and $F_2 : N^{n-1} \to \mathbb{R}^{n+m-2}$ is a minimal immersion. $\square$

REMARK 5.3. We use the technique by Huisken [16] for self-shrinkers in codimension one. Smoczyk [34] developed Huisken's technique to higher codimensional self-shrinkers with PPN. Moreover Martin, Savas-Halilaj and Smoczyk [29] used a similar technique for translating solitons in codimension one. Fortunately their technique is applicable to translating solitons in higher codimension with PPN.

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