博　士　論　文

Discrete-time quantum walks on the d-dimensional integer lattices

(d次元整数格子上の
離散時間量子ウォーク)

小 松　尭

平 成 27 年

Discrete-time quantum walks on the d-dimensional integer lattices

A thesis presented

by

Takashi KOMATSU

to

The Mathematical Institute

for the degree of

Doctor of Science

Tohoku University,

Sendai, Japan

March 2016

Contents

Summary

1.1 Introduction

The thesis is concerned with discrete-time quantum walks on the d-dimensional integer lattices. The notion of quantum walks was introduced by Y. Aharonov et al. [2] as a quantum analog of the classical one-dimensional random walks. It was re-discovered in computer science by several authors, for instance, [1], [5], [28] around 2000 and recently quantum walks have been intensively studied in connection with quantum computing [4], [13], [31], [34] and quantum physics [3], [17]. It is known that the long time asymptotic behaviors of the transition probability of quantum walks on the one-dimensional lattice are quite different from that of classical random walks. The study of quantum walks on higher dimensional integer lattices has attracted much attention and explicit computation of their long time behavior were advanced by many authors intensively in [10], [14], [15], [24], [30], [35], for instance. In the thesis, we propose a new model to contribute in the study of quantum walks on higher dimensional integer lattices.

We employ the framework introduced by T. Tate in 2014 [35]. He defined the notion of a *periodic unitary transition operator* on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ consisting of square summable functions on the d-dimensional integer lattice with values in a complex vector space $\mathbb{C}^D$.

Let U be a periodic unitary transition operator on $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$. Every non-negative integer n and $\varphi \in \mathbb{C}^D$ with $|\varphi|^2_{\mathbb{C}^D} = 1$, the n-step transition probability with initial state φ and initial position $x = 0$ is defined by

$$p_n(x;\varphi) = |(U^n(\delta_0 \otimes \varphi))(x)|^2_{\mathbb{C}^D}.$$

Basic interests lie in asymptotic behaviors of the transition probability $p_n(x;\varphi)$. For example, there are following problems.

- To determine the weak-limit distribution of $p_n(x;\varphi)$ as the time n goes to the infinity.
- To investigate the localization of $p_n(x;\varphi)$.

Here, the transition probability for U with initial state $\varphi \in \mathbb{C}^D$ with $|\varphi|_{\mathbb{C}^D} = 1$ is said to be *localized* at a vertex $x \in \mathbb{Z}^d$ if

$$\limsup_{n\to\infty} p_n(x;\varphi) > 0.$$

This phenomenon is one of typical properties for discrete-time quantum walks [15], [16], [35], [36] which is not seen for usual classical random walks.

The purposes of the thesis are

- **to study of limit distributions of discrete-time quantum walks on the d-dimensional integer lattices.**
- **to construct discrete-time quantum walks with localization.**

1.2 Discrete-time quantum walks on the d-dimensional integer lattice

In this section, we explain the setting for discrete-time quantum walks on the d-dimensional integer lattice introduced by T. Tate [35]. Let U be a unitary operator on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$, where D is a positive integer. For $f, g \in \ell^2(\mathbb{Z}^d, \mathbb{C}^D)$, the inner product on $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ is defined by

$$\langle f, g\rangle = \sum_{x\in\mathbb{Z}^d} \langle f(x), g(x)\rangle_{\mathbb{C}^D}, \quad f,\ g \in \ell^2(\mathbb{Z}^d, \mathbb{C}^D), \tag{1.2.1}$$

where $|\cdot|_{\mathbb{C}^D}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^D}$ are the standard norm and the inner product on $\mathbb{C}^D$. For each $x \in \mathbb{Z}^d$ and $\varphi \in \mathbb{C}^D$, define $\delta_x \otimes \varphi \in \ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ by

$$(\delta_x \otimes \varphi)(y) = \begin{cases} \varphi & x = y, \\ 0 & x \neq y\ , \quad y \in \mathbb{Z}^d. \end{cases}$$

Definition 1.2.1 (Shift operator). *For $i = 1, 2, \cdots, d$, the shift operator τ_i on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ is defined by*

$$(\tau_i f)(x) = f(x - \boldsymbol{e}_i) \quad (f \in \ell^2(\mathbb{Z}^d, \mathbb{C}^D),\ x \in \mathbb{Z}^d),$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_d\}$ denotes the standard basis of $\mathbb{Z}^d$ over $\mathbb{Z}$.

For $\alpha = \sum_{i=1}^{d} \alpha_i \boldsymbol{e}_i \in \mathbb{Z}^d$, we define the shift operator τ^α such as

$$\tau^\alpha = \tau_1^{\alpha_1} \cdots \tau_d^{\alpha_d}.$$

Definition 1.2.2 (Periodic unitary transition operator). *A unitary operator U on $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ is said to be a periodic unitary transition operator if it satisfies the following two conditions.*

(1) *There exists a finite set $S \subset \mathbb{Z}^d$, called the set of steps, such that for any $x \in \mathbb{Z}^d$, $y \in \mathbb{Z}^d \setminus (x + S)$ and any $\varphi \in \mathbb{C}^D$, we have $(U(\delta_x \otimes \varphi))(y) = 0$.*

(2) *The unitary operator U commutes with the natural action of the abelian group $\mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$, namely for each $\alpha \in \mathbb{Z}^d$*

$$U \circ \tau^\alpha = \tau^\alpha \circ U.$$

To analyze long time behaviors of quantum walks defined by periodic unitary transition operators, we use the Fourier transform. Let T^d be the d-dimensional torus in $\mathbb{C}^d$ defined by

$$T^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d; |z_i| = 1 \ (i = 1, \ldots, d)\}.$$

Let $L^2(T^d, \mathbb{C}^D)$ be the Hilbert space consisting of all square integrable functions on T^d with values in $\mathbb{C}^D$ and ν_d be the normalized Lebesgue measure on T^d; namely

$$L^2(T^d, \mathbb{C}^D) = \Big\{ f : T^d \longrightarrow \mathbb{C}^D; \|f\|^2 = \int_{T^d} |f(z)|^2_{\mathbb{C}^D} \, d\nu_d < \infty \Big\}.$$

Definition 1.2.3. *The Fourier transform of $f \in L^2(T^d, \mathbb{C}^D)$ is defined by the integral*

$$(\mathcal{F} f)(x) = \int_{T^d} z^x f(z) d\nu_d(z),$$

where a lattice point $x = (x_1, \cdots, x_d) \in \mathbb{Z}^d$ and we write $z^x = z_1^{x_1} \cdots z_d^{x_d}$ for a point $z = (z_1, \cdots, z_d)$ in the complex tours $(\mathbb{C} \setminus \{0\})^d$.

Then the inverse of the Fourier transform $\mathcal{F}^*$ is given by

$$(\mathcal{F}^* g)(z) = \sum_{x \in \mathbb{Z}^d} g(x) z^{-x} \quad (g \in \ell^2(\mathbb{Z}^d, \mathbb{C}^D), \ z \in T^d).$$

Definition 1.2.4. *For a periodic unitary transition operator U, we define a unitary operator $\mathcal{U}$ on $L^2(T^d, \mathbb{C}^D)$ by the formula*

$$\mathcal{U} = \mathcal{F}^* U \mathcal{F}.$$

Remark 1.2.1. *For any $f \in L^2(T^d, \mathbb{C}^D)$ and $z \in T^d$, we have*

$$(\mathcal{U} f)(z) = \hat{U}(z) f(z),$$

where a $D \times D$ unitary matrix $\hat{U}(z)$ is given by

$$\hat{U}(z)\varphi = \mathcal{U}(1 \otimes \varphi)(z).$$

It is important for us to analyze the eigenvalues of the matrix-valued function $\hat{U}(z)$ on T^d. This method was proposed in G.Grimmett-S.Janson-P.F.Scudo [11]. It has a key role on showing Theorem A, Theorem D and Theorem E.

1.3 Discrete-time quantum walks on the square lattice

To explain Theorem A, let $A = (a_{i,j})_{i,j=1,2,3,4}$ be a four-by-four unitary matrix. The matrix A is decomposed as

$$A = P_1 + P_2 + P_3 + P_4$$

to define a quantum walk on the square lattice. Here P_i is defined by

$$P_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (i = 1, 2, 3, 4).$$

Definition 1.3.1 (Quantum walk). *A quantum walk on the square lattice with four-state is described by a unitary operator $U_A : \ell^2(\mathbb{Z}^2, \mathbb{C}^4) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ defined by*

$$U_A = P_1 \tau_1 + P_2 \tau_1^{-1} + P_3 \tau_2 + P_4 \tau_2^{-1}. \tag{1.3.1}$$

The equation (1.3.1) means that a particle moves at each step either one unit to the right with matrix P_1 or one unit to the left with matrix P_2 or one unit to the up with matrix P_3 or one unit to the down with matrix P_4.

Let (X_n, Y_n) be a random vector taking values in $\mathbb{Z}^2$ with the distribution μ given by the transition probability $\mu(A) = \sum_{(x,y)\in A} p_n((x,y);\varphi)$, $A \subset \mathbb{Z}^2$. The joint moments of two-random variables X_n and Y_n is defined by

$$\mathbb{E}[X_n^\alpha Y_n^\beta] = \sum_{(x,y)\in\mathbb{Z}^2} x^\alpha y^\beta p_n((x,y);\varphi).$$

We have the following limit theorem for the discrete-time quantum walk associated with a unitary matrix A_1

$$A_1 = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \tag{1.3.2}$$

Theorem A (**[19]**). *For any initial state* $\varphi = {}^t\big(\varphi_1, \varphi_2, \varphi_3, \varphi_4\big) \in \mathbb{C}^4$ *with* $\displaystyle\sum_{i=1}^{4} |\varphi_i|^2 = 1$, *the following holds.*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^\alpha \left(\frac{Y_n}{n}\right)^\beta\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^\alpha\ y^\beta\ \frac{4\chi_\Omega(x,y)}{\pi^2(1-4x^2)(1-4y^2)}\ m(x,y)dxdy,$$

where

$$\Omega = \left\{(x,y); x^2 + y^2 < \left(\frac{1}{2}\right)^2\right\}$$

and the weight function $m(x,y)$ *of the density function is given by*

$$m(x,y) = 1 - 2\bigg((|\varphi_2|^2 - |\varphi_1|^2)x + 2\mathfrak{Re}(\varphi_2\overline{\varphi_1})y\bigg) - 2\bigg((|\varphi_4|^2 - |\varphi_3|^2)y + 2\mathfrak{Re}(\varphi_3\overline{\varphi_4})x\bigg).$$

The transition probability for this quantum walk is not localized at any point on the square lattice. A particle diffuses in four directions, right, left, up and down.

1.4 Alternate quantum walks

1.4.1 Relationships between our quantum walks and alternate quantum walks

In this subsection, we treat discrete-time quantum walks introduced by C. Di Franco et al. [9], [10]. It is called *alternate quantum walks*. We consider relationships between alternate quantum walks and our quantum walks, that is the quantum walk defined by the coin matrix A_1 given in (1.3.2).

In the standard model of the discrete-time quantum walks on the square lattice, a particle moves in four directions, right, left, up, down in one step. On the other hand, in the case of alternate quantum walks, a particle is permitted to move in the x-axis direction or in the y-axis direction of the square lattice lattice alternately. Namely, at first a particle moves either one unit to the right or one unit to the left and next moves either one unit to the up or one unit to the down. A particle moves from the origin to the four sites $(1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$. A particle repeats this motion on the square lattice. An alternate quantum walk on the square lattice is defined by a unitary operator on the Hilbert space $\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$.

To explain the set-up of an alternate quantum walk, we consider the Hilbert space $\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$ with the inner product defined by (1.2.1). We denote the shift operator on $\ell^2(\mathbb{Z}^2,\mathbb{C}^4)$ by τ_i, on $\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$ by σ_i.

Definition 1.4.1 (Shift operator). *For $f \in \ell^2(\mathbb{Z}^2,\mathbb{C}^2)$ and $(x,y) \in \mathbb{Z}^2$, the shift operators on the Hilbert space $\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$ σ_1, σ_2 are defined by*

$$(\sigma_1 f)(x,y) = f(x-1,y), \quad (\sigma_2 f)(x,y) = f(x,y-1).$$

Now, we prepare a two-by-two unitary matrix C in order to define an alternate quantum walk. Let $C = (c_{i,j})_{i,j=1,2}$ be a two-by-two unitary matrix. Decompose the matrix C as

$$C = Q_1 + Q_2,$$

where Q_i is defined by

$$Q_1 = \begin{pmatrix} c_{11} & c_{12} \\ 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 \\ c_{21} & c_{22} \end{pmatrix}. \tag{1.4.1}$$

By using the shift operators σ_1, σ_2 and a two-by-two unitary matrix C, we define an alternate quantum walk on the Hilbert pace $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$.

Definition 1.4.2 (Alternate quantum walk). *Alternate quantum walks are described by the unitary operators* $W_{C,1}, W_{C,2} : \ell^2(\mathbb{Z}^2, \mathbb{C}^2) \longrightarrow \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ *defined by*

$$W_{C,1} = (Q_1\sigma_2 + Q_2\sigma_2^{-1})(Q_1\sigma_1 + Q_2\sigma_1^{-1}), \quad W_{C,2} = (Q_1\sigma_1 + Q_2\sigma_1^{-1})(Q_1\sigma_2 + Q_2\sigma_2^{-1}).$$

Let φ_H (*resp.* φ_V) $\in \mathbb{C}^2$ with $|\varphi_H|_{\mathbb{C}^2} = 1$ (*resp.* $|\varphi_V|_{\mathbb{C}^2} = 1$) be an initial state of the alternate quantum walk $W_{C,1}$ (*resp.* $W_{C,2}$). To explain the relationship between alternate quantum walks and our quantum walks, we take a four-by-four unitary matrix of the form

$$A = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix},$$

where 0 is a two-by-two zero matrix and C_i $(i = 1, 2)$ is a two-by-two unitary matrix.

Theorem B ([19])**.** *When $C_1 = C_2 = C$, we have the followings.*

(1) *Suppose that $\varphi_3 = \varphi_4 = 0$. Let $\varphi_V = {}^{\mathrm{t}}(\varphi_1, \varphi_2)$ be a unit vector in $\mathbb{C}^2$. The following holds.*

$$(\pi_1 U_A^{2n})(\delta_{(0,0)} \otimes \varphi)(x, y) = W_{C,2}^n(\delta_{(0,0)} \otimes \varphi_V)(x, y).$$

(2) *Suppose that $\varphi_1 = \varphi_2 = 0$. Let $\varphi_H = {}^{\mathrm{t}}(\varphi_3, \varphi_4)$ be a unit vector in $\mathbb{C}^2$. The following holds.*

$$(\pi_2 U_A^{2n})(\delta_{(0,0)} \otimes \varphi)(x, y) = W_{C,1}^n(\delta_{(0,0)} \otimes \varphi_H)(x, y).$$

Here $\pi_1 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ (resp. $\pi_2 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$) denotes the orthogonal projection onto the two-dimensional subspace $\mathbb{C}\eta_1 + \mathbb{C}\eta_2$ (resp. $\mathbb{C}\eta_3 + \mathbb{C}\eta_4$) in $\mathbb{C}^4$, where $\{\eta_1, \cdots, \eta_4\}$ denotes the standard basis on $\mathbb{C}^4$.

1.4.2 Alternate quantum walks on the triangular lattice

We discuss discrete-time quantum walks on the triangular lattice. Discrete-time quantum walks on the triangular lattice are studied in [14], [18] and [20]. We consider alternate quantum walks on the triangular lattice and relationships between quantum walks associated with the following unitary matrix A_2 and alternate quantum walks.

$$A_2 = \begin{pmatrix} 0 & 0 & c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & 0 & 0 & c_{21} & c_{22} \\ c_{11} & c_{12} & 0 & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To explain our result, let us give the notations. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be linearly independent vectors in $\mathbb{R}^2$ and we put $\boldsymbol{e}_3 = -(\boldsymbol{e}_1 + \boldsymbol{e}_2)$. We define the triangular lattice $G = (V, E)$ by

$$V = \Big\{ v = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 ; \alpha_1, \alpha_2 \in \mathbb{Z} \Big\}, \quad E = \Big\{ (v_1, v_2) \in V \times V : v_1 - v_2 \in \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3\} \Big\}.$$

Definition 1.4.3. *A discrete-time quantum walk on the triangular lattice is described by a unitary operator* $U_{A_2} : \ell^2(V, \mathbb{C}^6) \longrightarrow \ell^2(V, \mathbb{C}^6)$ *associated with the matrix* A_2 *defined by*

$$U_{A_2} = P_1 \tau_1 + P_2 \tau^{-1} + P_3 \tau_2 + P_4 \tau_2^{-1} + P_5 \tau_3 + P_6 \tau_3^{-1},$$

where τ_i *is a shift operator on the Hilbert space* $\ell^2(V, \mathbb{C}^6)$ *given by*

$$(\tau_i f)(v) = f(v - \boldsymbol{e}_i) \quad (f \in \ell^2(V, \mathbb{C}^6), \ v \in V).$$

Here P_i is defined by

$$P_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} & a_{i6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (i = 1, 2, 3, 4, 5, 6).$$

Definition 1.4.4 (Shift operator). *For $f \in \ell^2(V, \mathbb{C}^2)$ and $v \in V$, the shift operators on the Hilbert space $\ell^2(V, \mathbb{C}^2)$ σ_1, σ_2, σ_3 are defined by*

$$(\sigma_1 f)(v) = f(v - \boldsymbol{e}_1), \quad (\sigma_2 f)(v) = f(v - \boldsymbol{e}_2), \quad (\sigma_3 f)(v) = f(v - \boldsymbol{e}_3).$$

By using the shift operators σ_1, σ_2, σ_3 and a two-by-two unitary matrix C given by (1.4.1), we define alternate quantum walks on the triangular lattice.

Definition 1.4.5 (Alternate quantum walk on the triangular lattice). *Alternate quantum walks are described by the unitary operators $W_{C,1}, W_{C,2}, W_{C,3} : \ell^2(V, \mathbb{C}^2) \longrightarrow \ell^2(V, \mathbb{C}^2)$ defined by*

$$W_{C,1} = (Q_1\sigma_2 + Q_2\sigma_2^{-1}) \cdot (Q_1\sigma_3 + Q_2\sigma_3^{-1}) \cdot (Q_1\sigma_1 + Q_2\sigma_1^{-1}),$$

$$W_{C,2} = (Q_1\sigma_3 + Q_2\sigma_3^{-1}) \cdot (Q_1\sigma_1 + Q_2\sigma_1^{-1}) \cdot (Q_1\sigma_2 + Q_2\sigma_2^{-1}),$$

$$W_{C,3} = (Q_1\sigma_1 + Q_2\sigma_1^{-1}) \cdot (Q_1\sigma_2 + Q_2\sigma_2^{-1}) \cdot (Q_1\sigma_3 + Q_2\sigma_3^{-1}).$$

Let $\varphi_{W,i} \in \mathbb{C}^2$ with $|\varphi_{W,i}|_{\mathbb{C}^2} = 1$ $(i = 1, 2, 3)$ be an initial state of an alternate quantum walk $W_{C,i}$ and $\varphi \in \mathbb{C}^6$ with $|\varphi|_{\mathbb{C}^6} = 1$ be an initial state of the discrete-time quantum walk U_{A_2}. We can show the following Theorem C in a similar way to prove Theorem B.

Theorem C ([20]).

(1) *Suppose that $\varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$. Let $\varphi_{W,3} = {}^{\mathrm{t}}(\varphi_1, \varphi_2)$ be a unit vector in $\mathbb{C}^2$. The following holds.*

$$(\pi_1 U_{A_2}^{3n})(\delta_{(0,0)} \otimes \varphi)(v) = W_{C,3}^n(\delta_{(0,0)} \otimes \varphi_{W,3})(v).$$

(2) *Suppose that $\varphi_1 = \varphi_2 = \varphi_5 = \varphi_6 = 0$. Let $\varphi_{W,1} = {}^{\mathrm{t}}(\varphi_3, \varphi_4)$ be a unit vector in $\mathbb{C}^2$. The following holds.*

$$(\pi_2 U_{A_2}^{3n})(\delta_{(0,0)} \otimes \varphi)(v) = W_{C,1}^n(\delta_{(0,0)} \otimes \varphi_{W,1})(v).$$

(3) *Suppose that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$. Let $\varphi_{W,2} = {}^{\mathrm{t}}(\varphi_5, \varphi_6)$ be a unit vector in $\mathbb{C}^2$. The following holds.*

$$(\pi_3 U_{A_2}^{3n})(\delta_{(0,0)} \otimes \varphi)(v) = W_{C,2}^n(\delta_{(0,0)} \otimes \varphi_{W,2})(v).$$

Here $\pi_1 : \mathbb{C}^6 \longrightarrow \mathbb{C}^2$ (resp. $\pi_2, \pi_3 : \mathbb{C}^6 \longrightarrow \mathbb{C}^2$) denotes the orthogonal projection onto the two-dimensional subspace $\mathbb{C}\eta_1 + \mathbb{C}\eta_2$ (resp. $\mathbb{C}\eta_3 + \mathbb{C}\eta_4$, $\mathbb{C}\eta_5 + \mathbb{C}\eta_6$) in $\mathbb{C}^6$, where $\{\eta_1, \cdots, \eta_6\}$ denotes the standard basis on $\mathbb{C}^6$.

1.5 Localization of discrete-time quantum walks on the integer lattice

In this section, we focus on a localization phenomenon of the transition probability for periodic unitary transition operators. The localization phenomenon is one of typical properties of the quantum walks which is not seen for usual classical random walks. We investigate this phenomenon for discrete-time quantum walks on the d-dimensional integer lattices.

The first simulation discovering the localization was reported by T. D. Mackay et al. [26] by using the *Grover walk* on the square lattice. A quantum walk defined by the following D-by-D unitary matrix (coin matrix) G is called the Grover walk.

$$G = 2\phi\phi^* - I, \quad \phi = \frac{1}{\sqrt{D}}{}^t(1, 1, \cdots, 1). \tag{1.5.1}$$

This coin matrix is introduced by L. K. Grover in the quantum algorithm [12]. Quantum algorithms are intensively studied in connection with quantum computing. One of the most famous quantum algorithms is Grover's search algorithm. L. K Grover used the above coin matrix G in Grover's search algorithm. The discrete-time quantum walk on the square lattice is used by A. Ambainis-J. Kempe-A. Rivosh [6] in order to improve Grover's search algorithm in 2005. The proof of the localization for the Grover walk on the square lattice was given by N. Inui et al. in 2004 [15].

Theorem 1.5.1 (T. Tate [35], 2014). *For a periodic unitary transition operator U on $\ell^2(\mathbb{Z}^d, \mathbb{C}^D)$, we have the following.*

(1) *U has an eigenvalue ω if and only if the matrix-valued function $\hat{U}(z)$ on T^d has an eigenvalue ω for all points z in T^d.*

(2) *U has a localization for an initial state $\varphi \in \mathbb{C}^D$ at a point $x \in \mathbb{Z}^d$ if and only if $spec(U)_p \neq \emptyset$, where $spec(U)_p$ is the set of eigenvalues of U.*

By using Theorem 1.5.1, we can investigate whether or not the discrete-time quantum walk has a localization.

1.5.1 The general form of the Grover walk

We give the definition of the general form of the Grover walk and a model of discrete-time quantum walks with localization. In this subsection, we assume that a periodic unitary transition operator has the following condition (S_1) for the set of steps S.

Condition 1.5.1. (S_1) $\alpha \in S \Longrightarrow -\alpha \in S$.

Let A be a $D \times D$ unitary matrix A satisfied with $A^2 = I$. The Grover matrix (1.5.1) is the special case of A. We consider the general form of the Grover walks on the d-dimensional integer lattice. A unitary operator $H_A : \ell^2(\mathbb{Z}^d, \mathbb{C}^D) \longrightarrow \ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ is given by

$$H_A = \sum_{\alpha \in S} P_\alpha A \tau^\alpha, \tag{1.5.2}$$

where $\{P_\alpha\}_{\alpha \in S}$ is a spectral resolution on $\mathbb{C}^D$.

Since the matrix A is a unitary self-adjoint matrix, the matrix A has the eigenvalue 1 and the eigenvalue -1. Let ε_1 (*resp.* ε_{-1}) be the eigenspace of the eigenvalue 1 (*resp.* -1) of A.

Theorem D (**[21]**). *Let H_A be a periodic unitary transition operator given by* (1.5.2)*. If $|P_\alpha \phi|^2_{\mathbb{C}^D} = |P_{-\alpha}\phi|^2_{\mathbb{C}^D}$ for any $\phi \in \varepsilon_1$, the transition probability for the quantum walk defined by the unitary transition operator H_A is localized at some point.*

1.5.2 Two-step of the general form of the Grover walk

We treat a discrete-time quantum walk given by a product of two general forms of the Grover walk. In this subsection, we do not assume the condition (S_1) for the set of steps S. Let A be a $D \times D$ unitary matrix A satisfied with $A^2 = I$.

Definition 1.5.1. *A product of two general forms of the Grover walk* $U_A : \ell^2(\mathbb{Z}^d, \mathbb{C}^D) \longrightarrow \ell^2(\mathbb{Z}^d, \mathbb{C}^D)$ *is given by*

$$U_A = \left(\sum_{\alpha \in S} P_\alpha A \tau^{-\alpha}\right) \cdot \left(\sum_{\alpha \in S} P_\alpha A \tau^{\alpha}\right),$$

where $\{P_\alpha\}_{\alpha \in S}$ *is a spectral resolution on* $\mathbb{C}^D$.

Theorem E ([**21**]). *Let* U_A *be a periodic unitary transition operator given by Definition 1.5.1. If dim* $\varepsilon_1 <$ *dim* ε_{-1}, *the transition probability for the quantum walk defined by the unitary transition operator* U_A *is localized at some point.*

By using Theorem D or Theorem E, we can construct a model of discrete-time quantum walks with localization.

Bibliography

[1] D. Aharonov, A. Ambainis, J. Kempe and U. Vazirani, Quantum walks on graphs, Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing (2001), ACM, Heraklion, 50-59.

[2] Y. Aharonov, L. Davidovich and N. Zagury, Quantum random walks, Phys. Rev. A **48** (1993), 1687-1690.

[3] A. Ahlbrecht, A. Alberti, D. Meschede, V. B. Scholz, A. H. Werner and R. F. Werner, Bound molecules in an interacting quantum walk, arXiv:1105.1051 (2011).

[4] A. Ambainis, Quantum walk algorithm for element distinctness, SIAM J. Comput. **37** (2007), 210-239.

[5] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath and J. Watrous, One-dimensional quantum walks, Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing (2001), ACM, Heraklion, 37-49.

[6] A. Ambainis, J. Kempe and A. Rivosh, Coins make quantum walks faster, Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (2005), ACM-SIAM, Vancouver, 1099-1108.

[7] Z. Bian, J. Li, H. Qin, X. Zhan, R. Zhang, B. C. Sanders and P. Xue, Realization of single-qubit positive-operator-valued measurement via a one-dimensional photonic quantum walk, Phys. Rev. Lett. **114** (2015), 203602/1-5.

[8] E. B. Davies, Quantum theory of open systems, Academic Press Inc., London, (1976).

[9] C. Di Franco, M. Mc Gettrick and Th. Busch, Mimicking the probability distribution of a two-dimensional Grover walk with a single-qubit coin, Phys. Rev. Lett. **106** (2011), 080502/1-4.

[10] C. Di Franco, M. Mc Gettrick, T. Machida and Th. Busch, Alternate two-dimensional quantum walk with a single-qubit coin, Phys. Rev. A **84** (2011), 042337/1-8.

[11] G. Grimmett, S. Janson and P. F. Scudo, Weak limits for quantum random walks, Phys. Rev. E **69** (2004), 026119/1-6.

[12] L. K. Grover, A fast quantum mechanical algorithm for database search, Proceedings of the Twenty-eight Annual ACM Symposium on Theory of Computing (1996), ACM, Philadelphia, 212-219.

[13] L. K. Grover, Quantum mechanics helps in searching for a needle in a haystack, Phys. Rev. Lett. **79** (1997), 325-328.

[14] Y. Higuchi, N. Konno, I. Sato and E. Segawa, Spectral and asymptotic properties of Grover walks on crystal lattices, J. Funct. Anal. **267** (2014), 4197-4235.

[15] N. Inui, Y. Konishi and N. Konno, Localization of two-dimensional quantum walks, Phys. Rev. A **69** (2004), 052323/1-9.

[16] N. Inui, N. Konno and E. Segawa, One-dimensional three-state quantum walk, Phys. Rev. E **72** (2005), 056112/1-7.

[17] T. Kitagawa, M. S. Rudner, E. Berg and E. Demler, Exploring topological phases with quantum walks, Phys. Rev. A **82** (2010), 033429/1-9.

[18] B. Kollár, M. Štefaňák, T. Kiss and I. Jex, Recurrences in three-state quantum walks on a plane, Phys. Rev. A **82** (2010), 012303/1-7.

[19] T. Komatsu, Limiting distributions of quantum walks on the square lattice, to appear in Yokohama Math. J..

[20] T. Komatsu, Localization of the Grover walk on the triangular lattice with six-state, in preparation.

[21] T. Komatsu and T.Tate, Localization of the general form of the Grover walk on the d-dimensional integer lattice, in preparation.

[22] N. Konno, A new type of limit theorems for the one-dimensional quantum random walk, J. Math. Soc. Japan **57** (2005), 1179-1195.

[23] P. Kurzyński and A. Wójcik, Quantum walk as a generalized measuring device, Phys. Rev. Lett. **110** (2013), 200404/1-5.

[24] T. Machida, C. M. Chandrashekar, N. Konno and Th. Busch, Limit distributions for different forms of four-state quantum walks on a two-dimensional lattice, Quantum Inf. Comput. **15** (2015), 1248-1258.

[25] M. Mohseni, A. M. Steinberg and J. A. Bergou, Optical realization of optimal unambiguous discrimination for pure and mixed quantum states, Phys. Rev. Lett. **93** (2004), 200403/1-4.

[26] T. D. Mackay, S. D. Bartlett, L. T. Stephenson and B. C. Sanders, Quantum walks in higher dimensions, J. Phys. A: Math. Gen. **35** (2002), 2745-2753.

[27] P. J. Mosley, S. Croke, I. A. Walmsley and S. M. Barnett, Experimental realization of maximum confidence quantum state discrimination for the extraction of quantum information, Phys. Rev. Lett. **97** (2006), 193601/1-4.

[28] A. Nayak and A. Vishwanath, Quantum walk on the line, arXiv: quant-ph/0010117 (2000).

[29] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information, Cambridge University Press, New York, (2000).

[30] E. Roldán, C. Di Franco, F. Silva and G. J. de Valcárcel, N-dimensional alternate coined quantum walks from a dispersion-relation perspective, Phys. Rev. A **87** (2013), 022336/1-5.

[31] N. Shenvi, J. Kempe and K. B. Whaley, Quantum random-walk search algorithm, Phys. Rev. A **67** (2003), 052307/1-11.

[32] T. Sunada and T. Tate, Asymptotic behavior of quantum walks on the line, J. Funct. Anal. **262** (2012), 2608-2645.

[33] T. Sunada, Quantum walks in view of discrete geometric analysis, Unpublished article.

[34] M. Szegedy, Quantum speed-up of Markov chain based algorithms, Proceedings of the Forty-five Annual IEEE Symposium on Foundations of Computer Science (2004), IEEE, Rome, 32-41.

[35] T. Tate, Eigenvalues, absolute continuity and localizations for periodic unitary transition operators, arXiv:1411.4215 (2014).

[36] K. Watabe, N. Kobayashi, M. Katori and N. Konno, Limit distributions of two-dimensional quantum walks, Phys. Rev. A **77** (2008), 062331/1-9.