Interdisciplinary Information Sciences Vol. 23, No. 2 (2017) 163–166

ISSN 1340-9050 print/1347-6157 online
DOI 10.4036/iis.2016.S.02

SHORT COMMUNICATION

Transforming a Matrix into a Standard Form

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We show that every matrix all of whose entries are in a fixed subgroup of the group of units of a commutative ring with identity is equivalent to a standard form. As a consequence, we improve the proof of Theorem 5 in D. Best, H. Kharaghani, H. Ramp [Disc. Math. 313 (2013), 855–864].

KEYWORDS: monomial matrix, weighing matrix, Hadamard matrix, permutation matrix

1. Introduction

Throughout this note, we let R be a commutative ring with identity. We fix a subgroup T of the group of units of R, and set $T_0 = T \cup \{0\}$. The set of $m \times n$ matrices with entries in T_0 is denoted by $T_0^{m\times n}$. If $T = \{z \in \mathbb{C} : |z| = 1\}$, then $W \in T_0^{n\times n}$ is called *a unit weighing matrix of order n with weight w* provided that $WW^* = wI$ where W^* is the transpose conjugate of W. Unit weighing matrices are introduced by D. Best, H. Kharaghani, and H. Ramp in [1, 2]. Moreover, a unit weighing matrix is known as a unit Hadamard matrix if $w = n$ (see [3]). A unit weighing matrix in which every entry is in $\{0, \pm 1\}$ is called a *weighing matrix*. We refer the reader to [4] for an extensive discussion of weighing matrices, and to [5] for more information on applications of weighing matrices.

The study on the number of inequivalent unit weighing matrices was initiated in [1]. Also, observing the number of weighing matrices in standard form leads to an upper bound on the number of inequivalent unit weighing matrices [1]. In this work, we will introduce a standard form of an arbitrary matrix in $T_0^{m\times n}$ and show that every matrix in $T_0^{m\times n}$ is equivalent to a matrix in standard form.

We equip T_0 with a total ordering $\prec$ satisfying $\min(T_0) = 1$ and $\max(T_0) = 0$. Moreover, let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ be arbitrary row vectors with entries in T_0. If k is the smallest index such that $a_k \neq b_k$, then we write $\boldsymbol{a} < \boldsymbol{b}$ provided $a_k \prec b_k$. We write $\boldsymbol{a} \le \boldsymbol{b}$ if $\boldsymbol{a} < \boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{b}$. If $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ are row vectors of a matrix $A \in T_0^{m\times n}$ and $\boldsymbol{a}_1 < \cdots < \boldsymbol{a}_m$, then we say that the rows of A are in *lexicographical order*.

Definition 1.1. We say that a matrix in $T_0^{m\times n}$ is in *standard form* if the following conditions are satisfied:
(S1) The first non-zero entry in each row is 1.
(S2) The first non-zero entry in each column is 1.
(S3) The first row is ones followed by zeros.
(S4) The rows are in lexicographical order according to $\prec$.

The subset of $T_0^{m\times m}$ consisting of permutation matrices, nonsingular diagonal matrices and monomial matrices, are denoted respectively, by $\mathbb{P}_m$, $\mathbb{D}_m$ and $\mathbb{M}_m$. Then $\mathbb{M}_m = \mathbb{P}_m\mathbb{D}_m$.

Definition 1.2. For $A, B \in T_0^{m\times n}$, we say that A is *equivalent* to B if there exist monomial T_0-matrices M_1 and M_2 such that $M_1AM_2 = B$.

We will restate the proof of [1, Theorem 5] as the following algorithm.

Algorithm 1.3. Let W be an arbitrary unit weighing matrix.
(1) We multiply each ith row of W by r_i^{-1} where r_i is the first non-zero entry in ith row. Denote the obtained matrix by $W^{(1)}$.
(2) Let c_j be the first non-zero entry in jth column of $W^{(1)}$. Let $W^{(2)}$ obtained from $W^{(1)}$ by multiplying each jth column by c_j^{-1}.
(3) Permute the columns of $W^{(2)}$ so that the first row has w ones. Denote the resulting matrix by $W^{(3)}$.
(4) Let $W^{(4)}$ be a matrix obtained from $W^{(3)}$ by sorting the rows of $W^{(3)}$ lexicographically with the ordering $\prec$.

Received October 24, 2016; Accepted November 8, 2016; J-STAGE Advance published November 28, 2016
2010 Mathematics Subject Classification: Primary 15B34, Secondary 05B20.
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Then $W^{(4)}$ is in standard form.

The steps (1)–(4) in Algorithm 1.3 was used in order to prove Theorem 5 in [1]. However, we provide a counterexample to show that this algorithm does not produce a standard form.

Counterexample 1.4. The matrix

$$W = \begin{bmatrix} 1 & -i & i & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & i & i \\ 1 & 0 & 0 & -1 & -i & i \\ 1 & 0 & 0 & -1 & i & -i \\ 0 & 1 & 1 & 0 & -i & -i \\ 1 & i & -i & 1 & 0 & 0 \end{bmatrix}$$

is a unit weighing matrix, where i is a 4th root of unity in $\mathbb{C}$. Also, we equip the set $\{0, \pm i, \pm 1\}$ with a total ordering $\prec$ defined by $1 \prec -1 \prec i \prec -i \prec 0$. Since the first nonzero entry in each row of W is one, $W^{(1)} = W$. Applying step (2), we obtain

$$W^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & i & -i & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & i & -i & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Notice that the first row of $W^{(2)}$ is all ones followed by zeros. So, $W^{(3)} = W^{(2)}$. Finally, by applying the last step of the algorithm, we have

$$W^{(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & i & -i & 0 & 1 & 1 \\ 0 & i & -i & 0 & -1 & -1 \end{bmatrix}.$$

We see that $W^{(4)}$ is not in standard form. So, we conclude that the algorithm does not produce a matrix in standard form as claimed.

This counterexample shows that the additional steps are needed to complete the proof of Theorem 5 in [1]. In the next section, we will prove a more general theorem than [1, Theorem 5] by showing that every matrix in $T_0^{m\times n}$ is equivalent to a matrix that is in standard form.

2. Main Theorem

In addition to the conditions (S1)–(S4) in Definition 1.1, we will consider the following condition:
(S3)′ The first nonzero row is ones followed by zeros.

Note that (S3)′ is weaker than (S3). The condition (S3)′ is crucial in the proof of Lemma 2.1, where we encounter a matrix whose first row consists entirely of zeros.

Lemma 2.1. *Let*

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \in T_0^{m\times(n_1+n_2)},$$

where $A_i \in T_0^{m\times n_i}$, $i = 1, 2$. *Then there exist* $P \in \mathbb{P}_m$ *and* $M \in \mathbb{M}_{n_2}$ *such that* PA_2M *satisfies* (S2) *and* (S3)′, *and* $[PA_1 \quad PA_2M]$ *satisfies* (S4).

Proof. Without loss of generality, we may assume A_1 satisfies (S4). Then there exist row vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ of A_1 such that $\boldsymbol{a}_1 < \cdots < \boldsymbol{a}_k$, and positive integers $m_1, \dots, m_k$ such that

$$A_1 = \begin{bmatrix} \mathbf{1}_{m_1}^\top & & \\ & \ddots & \\ & & \mathbf{1}_{m_k}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_k \end{bmatrix},$$

where $\sum_{i=1}^{k} m_i = m$. Write

$$A_2 = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix},$$

where $B_i \in T_0^{m_i \times n_2}$ for $i = 1, 2, \ldots, k$. We may assume $B_1 \neq 0$, since otherwise the proof reduces to establishing the assertion for the matrix A with the first m_1 rows deleted. Let $\boldsymbol{b}$ be a row vector of B_1 with maximum number of nonzero components. Then there exists $M \in \mathbb{M}_{n_2}$ such that the vector $\boldsymbol{b}M$ constitutes ones followed by zeros. Moreover, for each $i \in \{1, \ldots, k\}$, there exists $P_i \in \mathbb{P}_{m_i}$ such that the rows of $P_i B_i M$ are in lexicographic order. It follows that $\boldsymbol{b}M$ is the first row of $P_1 B_1 M$, that is also the first row of PA_2M. Set $P = \operatorname{diag}(P_1, \ldots, P_k)$. Then PA_2M satisfies (S3). Since $PA_1 = A_1$, we see that $[PA_1 \quad PA_2M]$ satisfies (S4).

With the above notation, we prove the assertion by induction on n_2. First we treat the case where $\boldsymbol{b}M = \mathbf{1}$. This in particular includes the case where $n_2 = 1$, the starting point of the induction. In this case, the first row of PA_2M is $\mathbf{1}$, hence PA_2M satisfies (S2). The other assertions have been proved already.

Next we consider the case where $\boldsymbol{b}M = [\mathbf{1}_{n_2-n_2'} \quad \mathbf{0}_{n_2'}]$, with $0 < n_2' < n_2$. Define $A_1' \in T_0^{m \times (n_1+n_2-n_2')}$ and $A_2' \in T_0^{m \times n_2'}$ by setting $[A_1' \quad A_2']$ to be the matrix obtained from $[A_1 \quad PA_2M]$ by deleting the first row. By inductive hypothesis, there exist $P' \in \mathbb{P}_{m-1}$ and $M' \in \mathbb{M}_{n_2'}$ such that $P'A_2'M'$ satisfies (S2) and (S3)$'$, and $[P'A_1' \quad P'A_2'M']$ satisfies (S4). By our choice of $\boldsymbol{b}$, the row vector $\boldsymbol{b}M$ is lexicographically the smallest member among the rows of P_1B_1M, and the same is true among the rows of the matrix P_1B_1M'', where

$$M'' = M \begin{bmatrix} I_{n_2-n_2'} & 0 \\ 0 & M' \end{bmatrix}.$$

It follows that the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix} \begin{bmatrix} A_1 & PA_2M'' \end{bmatrix} = \begin{bmatrix} * & 0 \\ P'A_1' & P'A_2'M' \end{bmatrix}$$

satisfies (S4). Set

$$P'' = \begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix} P.$$

Since $P'A_2'M'$ satisfies (S2), while the first row of $P''A_2M''$ is the same as that of PA_2M which is $[\mathbf{1}_{n_2-n_2'} \quad \mathbf{0}_{n_2'}]$, the matrix $P''A_2M''$ satisfies both (S2) and (S3)$'$. We have already shown that the matrix $[P''A_1 \quad P''A_2M]$ satisfies (S4). □

Lemma 2.2. *Under the same assumption as in Lemma 2.1, there exist $M_1 \in \mathbb{M}_m$ and $M_2 \in \mathbb{M}_{n_2}$ such that $[M_1A_1 \quad M_1A_2M_2]$ satisfies* (S1) *and* (S4)*, and $M_1A_2M_2$ satisfies* (S2) *and* (S3)$'$.

Proof. We will prove the assertion by induction on m. Suppose $m = 1$. It is clear that every single row vector always satisfies (S4). Also, every single row vector satisfying (S3)$'$ necessarily satisfies (S2). Now, if $A_1 = 0$ or $n_1 = 0$, then there exists $M_2 \in \mathbb{M}_{n_2}$ such that A_2M_2 satisfies (S3)$'$ and hence (S1) is satisfied. If $A_1 \neq 0$, then there exist $a \in T$ and $M_2 \in \mathbb{M}_{n_2}$ such that aA_1 satisfies (S1) and aA_2M_2 satisfies (S3)$'$.

Assume the assertion is true up to $m - 1$. First, we consider the case where $A_1 = 0$ or $n_1 = 0$. Without loss of generality, we may assume $A_2 \neq 0$. Furthermore, we may assume that the first row and the first column of A_2 are ones followed by zeros. Then there exists $P' \in \mathbb{P}_{n_2}$ such that

$$A_2P' = \left[\begin{array}{cccc|c} 1 & \mathbf{1} & 0 & & 0 \\ \mathbf{1}^T & B_1 & B_2 & & 0 \\ 0 & & C_1 & & C_2 \end{array}\right]$$

where $B_2 \in T_0^{m_1 \times t}$ has no zero column. By Lemma 2.1, there exist $P \in \mathbb{P}_{m_1}$ and $M \in \mathbb{M}_t$ such that PB_2M satisfies (S2) and (S3)$'$ and $[PB_1 \quad PB_2M]$ satisfies (S4). Let

$$C_1' = C_1 \begin{bmatrix} I_{n_2-n_2'-t-1} & 0 \\ 0 & M \end{bmatrix}.$$

By inductive hypothesis, there exist $M_1' \in \mathbb{M}_{m-m_1-1}$, and $M_2' \in \mathbb{M}_{n_2'}$ such that $[M_1'C_1' \quad M_1'C_2M_2']$ satisfies (S1) and (S4), and $M_1'C_2M_2'$ satisfies (S2) and (S3)$'$. By setting

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & M_1' \end{bmatrix}, \quad M_2 = P' \begin{bmatrix} I_{n_2-n_2'-t} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M_2' \end{bmatrix},$$

the matrix $M_1A_2M_2$ satisfies (S1)–(S4).

Next we consider the case $A_1 \neq 0$. Without loss of generality, we may assume that the first nonzero column in A_1 is ones followed by zeros. Write

$$A_1 = \begin{bmatrix} 0_{m\times t} & \begin{matrix} \mathbf{1}^T & B_1 \\ 0 & D_1 \end{matrix} \end{bmatrix}$$

for some $t < n_1$, with $B_1 \in T_0^{m_1\times(n_1-t-1)}$ and $D_1 \in T_0^{m_2\times(n_1-t-1)}$ for some m_1, m_2 with $m_1 + m_2 = m$ and $m_2 < m$. Then there exists $P' \in \mathbb{P}_{n_2}$ such that

$$A_2P' = \begin{bmatrix} B_2 & 0_{m_1\times n_2'} \\ D_2 & C_2 \end{bmatrix}$$

for some $n_2' \geq 0$, where $B_2 \in T_0^{m_1\times(n_2-n_2')}$ has no zero column. By Lemma 2.1, there exist $P \in \mathbb{P}_{m_1}$ and $M \in \mathbb{M}_{n_2-n_2'}$ such that PB_2M satisfies (S2) and (S3)′ and $[PB_1 \quad PB_2M]$ satisfies (S4). Let $C_1 = [D_1 \quad D_2M]$. Then by inductive hypothesis, there exist $M_1' \in \mathbb{M}_{m_2}$ and $M_2' \in \mathbb{M}_{n_2'}$ such that $[M_1'C_1 \quad M_1'C_2M_2']$ satisfies (S1) and (S4), and $M_1'C_2M_2'$ satisfies (S2) and (S3)′. By setting

$$M_1 = \begin{bmatrix} P & 0 \\ 0 & M_1' \end{bmatrix}, \quad M_2 = P'\begin{bmatrix} M & 0 \\ 0 & M_2' \end{bmatrix},$$

the proof is complete. □

Theorem 2.3. *Every matrix in $T_0^{m\times n}$ is equivalent to a matrix that is in standard form.*

Proof. Let $W \in T_0^{m\times n}$. Setting $A_1 = \varnothing$ and $A_2 = W$ in Lemma 2.2, we see that W is equivalent to a matrix that is in standard form. □

Corollary 2.4. *Every unit weighing matrix is equivalent to a unit weighing matrix that is in standard form.*

Proof. Setting $T = \{z \in \mathbb{C} : |z| = 1\}$, the proof is immediate from Theorem 2.3. □

Acknowledgments

The second author would like to acknowledge the Hitachi Global Foundation for providing a scholarship grant.

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