論文タイトル: シリコンの粒子輸送の限界値についてーEinstein方程式による解析ー

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Threshold of Sediment Particle Transport by the Lift Force due to H. Einstein’s Equation

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Abstract Using the lift equation of Hans Einstein, the threshold of a spherical ball transport by a current was examined. As a result, the criterion for the instantaneous suspension of balls on a nonuniform bed was determined by Einstein’s lift equation as follows:

$$\theta = \frac{\tau_{0}}{g(p_{s} - \rho)D} \approx 0.07$$

where $\theta$ is the dimensionless shear stress, $\tau_{0}$ is the shear stress, $g$ is the gravitational acceleration, $p_{s}$ is the sediment density, $\rho$ is the fluid density, and $D$ is the diameter of balls.

Key words: Hans Einstein, lift, spherical ball, nonuniform bed, shear stress

1. Introduction

The lift force removing a sediment particle from a river bed was first studied by Hans Albert Einstein (1950, pp. 30-31). His idea on the bed-load motion was as follows:

1. The probability of a given sediment particle being moved by the flow from the bed surface depends on the particle’s size, shape, and weight and on the flow pattern near the bed, but not on its previous history.
2. The particle moves if the instantaneous hydrodynamic lift force overcomes the particle weight.

Although this idea was not confirmed strictly by experiments yet, it is very charming in its basic concept that the critical condition for the entrainment of a sediment particle does not depend on the configuration of bed surface. This paper intends to reexamine Einstein’s equation.

2. Average lift pressure on the rough surface

Einstein and El-Samni (1949) measured the dynamic forces which were exerted on

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the individual protrusions of a rough wall by a turbulent flow. They selected plastic spherical balls, 6.86 cm in diameter, as one type of roughness to make the surface hydraulically rough in their flume experiments. In order to prevent the seepage of water between these balls and the bottom of the flume, only the upper half of the spheres were used by gluing the hemispheres in a hexagonal pattern onto the bottom of the flume. The horizontal and vertical component of the average dynamic force on this extremely rough surface then were determined separately.

The average lift force on the hemispheres was measured directly as a pressure difference $\Delta p$. It was measured by means of micrometer-point-gages, reading to 1/10,000 ft. and applied to the free water surfaces in pressure wells connected alternately to pressure taps in the flume side wall above the hemispheres and to other taps in the flume floor between the hemispheres. The pressure between hemispheres was always found to be higher than that in the side wall. This pressure difference $\Delta p$ was described in the form:

$$\Delta p = C_L (\rho/2) u^2$$  \hspace{1cm} (1)

where $\rho$ is the fluid density, and the dimensionless lift coefficient $C_L$ was found to have a constant value of 0.178 if the flow velocity $u$ was measured at a distance of 0.35 diameters from the theoretical wall (Fig. 1).

The theoretical wall was defined as follows: a large number of vertical velocity distributions at a center section were measured, and it was found that, irrespective of their location, they are plotted as straight lines if the theoretical wall was assumed to be at a distance of 0.2 $K_s$ below a plain tangent to the top of spheres whose diameter is $K_s$ (Fig. 2). The resulting equation for the distribution of the average velocity for hydraulically rough boundaries, previously noted by Keulegan (1938), is a function of the distance from the wall as

$$u_y/u_* = 5.75 \log_{10}(30.1 y/K_s)$$  \hspace{1cm} (2)

where $y$ is the distance from the wall, $u_y$ is the average point velocity at a distance $y$ from the wall, and $u_*$ is the shear velocity. The hydraulics of uniform flow include basically the description of the frictional loss for turbulent flow. The shear velocity of open channel flows is given by

$$u_* = (\tau_b/\rho)^{1/2} = (gRl_e)^{1/2}$$  \hspace{1cm} (3)

where $\tau_b$ is the shear stress on the bed, $g$ is the gravitational acceleration, $R$ is the hydraulic radius and $l_e$ is the slope of the energy grade line.

According to Einstein and El-Samni (1949), observation of the behavior of sediment particles in motion suggests that large instantaneous variations of the lift force must exist; the measurement of these fluctuations was very important. Therefore, as a pressure recording instrument, a Trimount pressure cell with AC-generator and
amplifier and a Hathaway Magnetic oscillograph were available. The pressure fluctuations at the top and the bottom of the spheres were measured individually. Unfortunately, the fluctuations of the pressure difference itself could not be recorded directly for reasons of instrumentation.

The average pressure \( h_B \) between and below the hemispheres is, however, something like a stagnation pressure, because the average pressure \( h_I \) on the top of the hemisphere as recorded through a large number of holes covering the top 60° segment of the hemisphere often approaches but never surpasses \( h_B \). The small fluctuations of the pressure \( h_B \) can be neglected. In order to describe the fluctuations of \( h_I \) more in detail, a duration curve of pressures was constructed for several pressure records and plotted on probability paper. As a result, these curves all were perfectly straight, proving that the pressures at this wall are statistically distributed according to the normal error law.

The hemispheres studied thus could be considered as an idealized sediment deposit,
but some measurements on actual sediment particles were necessary to verify the applicability of the results to natural sediment. The gravel used in Einstein and El-Samni's experiments had about the same average size as the spheres, but had a considerable spread of grain sizes as revealed by a sieve-size analysis. The gravel was placed in a flume 20.3 cm thick with the surface made as smooth as possible, to a depth of 15.3 cm. Only flows which did not cause particle motion were tested.

Again, it was observed that a plane could be found as that which contains all points to make the velocity distributions appear as straight lines on a semilogarithmic graph. This plane was found to be located 0.04 ft. below the top points of the gravel as determined by laying a board on the bed. If this distance was again assumed to represent 20 percent of the representative grain size, the size composing the theoretical bed is 0.2 ft. (6.1 cm). From the mechanical analysis it was found that 67 percent of the particles (by weight) are smaller than this size: $D_{57}$. Einstein (1950, p. 12) referred to $D_{35}$ as the roughness $K_s$.

The average dynamic lift on the bed particles was measured by a similar method to that used in the studies with the hemispheres; that is, the difference between pressures in tap holes of the channel wall above the bed and in perforated tubes under the bed was measured by means of micro-point-gauges. The average lift pressure was found to be again

$$\Delta p = 0.178 (\rho/2) u^2$$

(4)

if the velocity $u$ again was measured 0.35 grain diameters from the theoretical wall, and the sieve size of that grain size was introduced as the diameter of $D_{35}$ (35 percent...
of the mixture was finer by weight).

3. Pressure difference on a uniform bed for rough boundaries

The vertical velocity distribution for hydraulically rough boundaries is given by Eq. (2). It is, however, generally known that the wall acts hydraulically smooth if $\delta/K_s > 3$. For hydraulically smooth boundaries, the velocity profile equation is different from that of rough boundaries. The thickness $\delta$ of the laminar sublayer along a smooth boundary has been given as:

$$\delta = \frac{11.6 \nu}{U_*}$$  \hspace{1cm} (5)

where $\nu$ is the kinematic viscosity of the water.

Einstein (1950, p. 8) combined the transition between hydraulically rough and smooth conditions in the form:

$$u_\tau/U_* = 5.75 \log_{10}(30.1 \frac{x y}{K_s})$$  \hspace{1cm} (6)

where $x$ is the corrective parameter given as a function of $K_s/\delta$. When $K_s/\delta > 10$, $x$ is constant and unity. Next, he gives the apparent roughness of the surface: $\delta' = K_s/x$. Thus, Eq. (6) becomes

$$u_\tau/U_* = 5.75 \log_{10}(30.1 \frac{y}{\delta'})$$  \hspace{1cm} (7)

Einstein (1950, p. 35) assumed the characteristic grain size of mixture $X$; the velocity acting on all particles of a mixture must be measured at a distance 0.35X from the theoretical bed. Setting $y = 0.35X$, Eq. (7) becomes

$$u/U_* = 5.75 \log_{10}(10.54X/\delta')$$  \hspace{1cm} (8)

where

$$X = 0.77 \delta' \quad \text{if } \delta'/\delta > 1.80$$
$$X = 1.39 \delta \quad \text{if } \delta'/\delta < 1.80$$

For gravel-bed rivers, $\delta'/\delta$ is always larger than 1.80; $X = 0.77 \delta$. Thus, Eq. (8) becomes

$$u/U_* = 5.23$$  \hspace{1cm} (9)

Then, the substitution of the preceding result into Eq. (4) yields the following expression:

$$\Delta p = 0.178 \left(\frac{\rho}{2}\right) (5.23 U_*)^2 = 2.435 \tau_s$$  \hspace{1cm} (10)

Thus, the pressure difference $\Delta p$ is directly proportional to the shear stress $\tau_s$. 
4. Lift force acting on the particle for a nonuniform bed

According to Einstein (1950, p. 35), the particle smaller than $X$ seem to hide between the other particles or in the laminar sublayer, respectively, and their lift must thus be corrected by division with a parameter $\xi$ which itself is a function of $D/X$: $D$ is the diameter of balls. When $D/X > 1.5$, $\xi$ is constant and unity.

An additional correction factor $Y$ was found to describe the change of the lift coefficient $C_l$ in mixtures with various roughness conditions (Einstein, 1950, p. 35). The correction $Y$ is a function of $K_s/\delta$, $Y$ being unity for uniform sediment (Fig. 1). When $K_s/\delta > 4$, however, $Y$ becomes a constant, i.e., 0.53 (Fig. 3).

As already noted, according to Einstein (1950, p. 35), the probability of a particle being eroded from the bed means the probability that the dynamic lift $L$ acting on the particle is larger than its weight $W'$ under water. He described that the weight of the particle under water is

$$W' = g(\rho_s - \rho)A_2D^3$$  \hspace{1cm} (11)

where $\rho_s$ is the sediment density and $A_2$ is the constant of particle volume, while the average lift force may be expressed as

\[ Y = 0.53 \]

Fig. 3  Schematic diagram of lift force acting on the ball for a nonuniform bed
where $A_1$ is the constant of particle area.

According to Einstein (1950, p. 36), at any instant the lift force may be described by:

$$L = |1 + \eta| \Delta \rho A_1 D^2$$

where $\eta$ is a parameter varying with time. The value of $\eta$ may be either positive or negative. In both cases the lift is actually positive and must, therefore, be understood on an absolute basis.

A particle is eroded when $L > W'$ (Einstein, 1950, p. 36). Thus,

$$1 > W'/L$$

$$= (1/|1 + \eta|) [((\rho_e - \rho)/\rho)(D/RI_e)(2A_2/0.178A_1 5.75^2)][1/\log_{10}(10.54X/L)]$$

Introducing the correction factors $\xi$ and $Y$ according to the previously quoted assumptions, Einstein (1950, p. 36) generalized Inequality (14):

$$|1 + \eta| > \xi Y \cdot B \frac{\rho}{\rho_e} \beta_x$$

Introducing the abbreviations:

$$\Psi = ((\rho_e - \rho)/\rho) \cdot (D/RI_e)$$

$$B = 2A_2/(0.178A_1 5.75^2)$$

$$\beta_x = \log_{10}(10.54X/L)$$

By removing the brackets of Inequality (15) from Eq.(16), the inequality becomes

$$\frac{1}{\xi Y} \cdot |1 + \eta| \cdot 0.178 \cdot A_1 5.75^2 \cdot [\log_{10}(10.54X/L)]^2 \cdot (\rho/2) \cdot RI_e > (\rho_e - \rho) \cdot A_2 D$$

By using Eq. (8), Inequality (17) becomes

$$\frac{1}{\xi Y} \cdot |1 + \eta| \cdot A_1 0.178 \cdot (\rho/2) \cdot (u/U_e)^2 \cdot RI_e > (\rho_e - \rho) \cdot A_2 D$$

And the substitution of Eq. (4) into Inequality (18) then yields

$$\frac{1}{\xi Y} \cdot |1 + \eta| \cdot A_1 \cdot \Delta \rho \cdot (gRI_e/U_e)^2 > g(\rho_e - \rho) \cdot A_2 D$$

From Eq. (3): $\tau_0 = \rho U_e^2 = \rho gRI_e$, Inequality (19) becomes

$$\frac{1}{\xi Y} \cdot |1 + \eta| \cdot \Delta \rho \cdot A_1 D^2 > g(\rho_e - \rho) \cdot A_2 D^3$$

5. Einstein's criterion for the instantaneous suspension of balls on a nonuniform bed

Setting a nonuniform bed in hydraulically rough conditions, the quantities $\xi$ and $Y$ are constants: $\xi = 1$ and $Y = 0.53$ (Fig. 3). Therefore, the substitution of Eq.(10) into Inequality (20) yields
\[ |1 + \eta| \cdot 4.595 \cdot \tau_0 \cdot A_1 D^2 > g(\rho_s - \rho) \cdot A_2 D^3 \]  
(21)

The left side of Inequality (21) is the lift force \( L \) acting on a particle, while the right side of Inequality (21) is the weight of a particle \( W' \) under the water:

\[ L = 4.595 \cdot \tau_0 \cdot |1 + \eta| \cdot A_1 D^2 \]  
(22)

and

\[ W' = g(\rho_s - \rho) \cdot A_2 D^3 \]  
(23)

where \( A_2 = \frac{\pi}{6} \), because of the ball volume.

\( \eta \) is a parameter which varies with the normal error law: the standard deviation is 0.364 of the average lift (Einstein, 1950, p. 31). Inokuchi and Takayama (1973) gives \( \eta' \), i.e., three times \( \eta \) as the upper limit of fluctuations: 1.092. Substituting \( \eta' \) into \( \eta \), the maximum lift force \( L_{\text{max}} \) is given by

\[ L_{\text{max}} = 4.595 \cdot \tau_0 \cdot |1 + \eta'| \cdot A_1 D^2 \]  
(24)

In Eq. (24), \( A_1 = \frac{\pi}{4} \), where \( A_1 \) denotes the constant of the largest projected area of a spherical ball that may be subject to the lift force (Inokuchi and Takayama, 1973).

Setting \( L_{\text{max}} = W' \), thus one obtains

\[ 9.613 \cdot \tau_0 \cdot \left(\frac{\pi}{4}\right) D^2 = g(\rho_s - \rho) \cdot \frac{1}{2} \cdot \frac{\pi}{6} D^3 \]

Therefore, the dimensionless shear stress \( \theta \) of Shields (1936) is

\[ \theta = \tau_0 \cdot g(\rho_s - \rho) D \approx 0.07 \]  
(25)

The Einstein's criterion \( \theta \) (=0.07) for the instantaneous suspension of balls on a nonuniform bed was found to be close to Shields' criterion \( \theta_e \) (=0.06) for the incipient motion of sediment particles.

6. Conclusion

The Einstein's criterion for the instantaneous suspension of balls on a nonuniform bed is given by Eq. (25).

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Note

1) In Inequality (14), Einstein (1950, p. 36) used the hydraulic radius $R'_a$ with respect to the bed and grain instead of the hydraulic radius $R$. The hydraulic radius $R'_a$ is considered to be almost equal to $R$ in the experimental conditions of this study.

References


