Diagrammatical reasoning with numerical operations

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Diagrams themselves can serve as syntactical objects in reasoning, in spite of the common belief that diagrams easily cause misapplication so they should not be used in reasoning (especially in geometry). It is necessary for using diagrams in reasoning (1) to figure out the notion of well-formed diagrams and a set of legitimate operations on them; (2) to give semantics; and (3) to show completeness.

In this paper a system of Venn diagrams with addition and subtraction is introduced. It is an extension of the system Venn 1[1]. A finite completeness proof will be sketched after proof examples. Decidability and interpolation will be obtained as corrolaries.

1 Syntax

Basic notions are defined in the same way as Hammer[3]. *A Venn diagram with numbers* consists of the following primitive symbols:

1. a rectangle, which may contain symbols other than the rectangle.
2. closed curves which are drawn in a rectangle.
3. name labels for closed curves.
4. symbol x.
5. integers associated with one of $\geq, \leq, =$. We call a pair *a number label*.
6. lines with a number label, which connect x's. We call a chain of lines a number link. More specifically, a $\langle \geq n \rangle$-sequence (respectively, a $\langle \leq n \rangle$-sequence, a $\langle = n \rangle$-sequence) is a chain of lines with a number label $\langle \geq n \rangle$ ($\langle \leq n \rangle$, $\langle = n \rangle$) connecting x's.

A node of a number link is an x in the sequence. *The length of a number link* is the number of its nodes. We take each integer n as an abbreviation of the

[1] Shin[4], Hammer[3]

$\langle = n \rangle$-sequence of length 1. A *well-formed diagram* is recursively defined as follows:

1. The emply rectangle with no curves nor labels in it is a well-formed diagram.

2. A rectangle with only a closed curve labeled by a name label is a well-formed diagram.

3. Given a well-formed diagram D with no number label, the result of adding one new curve to D tagged by a new label is a diagram, provided the addition is carried out according to both the following rules:

   (a) The new curve must be added to the diagram so that it overlaps a proper part of every minimal region of D.

   (b) The new curve must be added so that no combination of curves occurs twise.

4. Given well-formed diagram D with no number link, the addition of a number link of length 0 to any of its minimal regions results in another well-formed diagram.

5. Given well-formed diagram D, the addition of a number link results in a well-formed diagram, provided that each node of the link falls entirely within some minimal region. To avoid redundancy, we require that no two nodes of a single link are in the same minimal region.

6. Only what is formed by the above rules is a well-formed diagram.

A region is said to be *determined to be* n if a region which contains $\langle = n \rangle$ and no $\langle = m \rangle$ for any $m \neq n$. We denote the set of all name labels appearing in D by $Name_D$. *A counterpart relation* is defined in Hammer[3].

Proposition 1. A region in D is minimal if and only if the set of name labels that construct the region is equal to $Name_D$.

Proposition 2.

1. Let r and s be counterparts. Then there are closed curves $C_1, \cdots, C_m$ which r and s fall within, and $C_{m+1}, \cdots, C_n$ which r and s are outside of.

2. Let D and D' be well-formed diagrams. A region r in D has its counterpart in D' if and only if, if r is falling within $C_1, \cdots, C_m$ and outside of $C_{m+1}, \cdots, C_n$, then the set $\{C_1, \cdots, C_n\}$ of name labels for curves is a subset of $Name_D \cap Name_{D'}$.

2 Semantics

We intend to interpret the number label as a representation of the number of members in the region. Hence, though we allowed number labels with a negative integer, diagrams with any number link with such a label must be false in every model.

A set assignment is (U, f) where U is a finite set, and f is a function which assigns a subset of U to some regions including V, with the following restrictions:

1. $f(V) = U$
2. For a basic region within curve A if A is in $dom(f)$,$f(A)$ is the subset of U named A.
3. $f(r \cup s) = f(r) \cup f(s)$ if $f(r)$ and $f(s)$ are defined.
4. $f(\overline{r}) = U - f(r)$ if $f(r)$ is defined.
5. If there is any number link with label $\langle \geq n \rangle$ (resp. $\langle \leq n \rangle$,$\langle = n \rangle$) exactly falling in a region r in $dom(f)$, the set assigned to r has members of the number equal to or greater than (resp. equal to or less than, equal to) n.

A model on $Name$ is a set assignment (U, I) where I is defined for all regions constructed from basic regions with name labels in $Name$. We simply say *a model* without specifying the name label set, when the name label set contains all name labels that we want to mention. A well-formed diagram D is *true in a model* (U, I) (written $(U, I) \models D$) if, for every region r exactly within which a $\langle \geq n \rangle$(resp. $\langle \leq n \rangle$, $\langle = n \rangle$)-sequence falls, the size of $I(r)$ is equal to or greater than (resp. equal to or less than, equal to) n. A well-formed diagram D *entails* D' (or D' *is a logical consequence of* D, and written $D \models D'$) if D' is true in every model of D. A set of well-formed diagrams Δ *entails* D' (or D' is *a logical consequence of* Δ, and written $\Delta \models D'$) if D' is true in every model in which all diagrams in Δ are true.

The following are trivial, though used later in the completeness proof.

Proposition 1. For each model (U, I), there is a unique g such that assigns a subset of U to minimal regions of D.

Proposition 2. For each set assignment (U, g) to minimal regions of $Name$, there is a unique model (U, I) on $Name$ with I is an extension of g.

Proposition 3. If regions r, s are counterparts, and (U, I) is a model, then $I(r) = I(s)$.

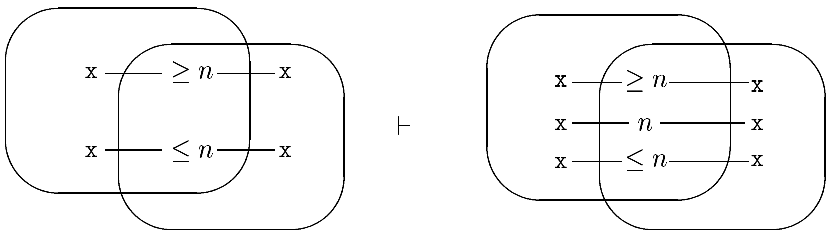
Proposition 4. If a diagram D has $\langle \leq n \rangle$-sequence or $\langle = n \rangle$-sequence for any negative n, there is no model (U, I) such that $(U, I) \models D$.

3 Rules of Inference; Soundness

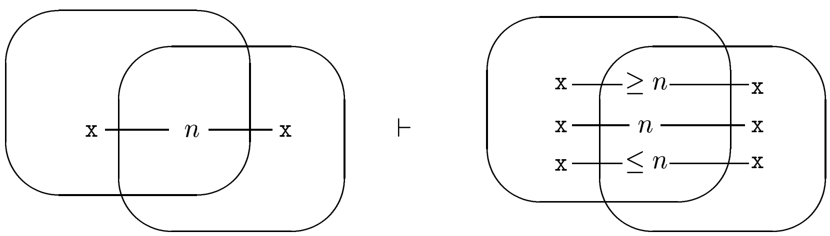
The aim of this section is to present twelve rules of inference as truth preserving operations of diagrams.

Rule: Non-negativeness. D' is obtained by adding of $\langle \geq 0 \rangle$-sequence of any length to any region in D.

Rule: Equality. If a region r in D has both $\langle \geq n \rangle$-sequence and $\langle \leq n \rangle$-sequence exactly within r, D' is obtained by adding $\langle = n \rangle$-sequence into r.

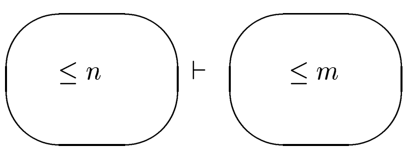


Rule: Number weakening for equality. If a region r in D has $\langle = n \rangle$-sequence, D' is obtained by adding of $\langle \geq n \rangle$-sequence $\langle \leq n \rangle$-sequence in r.



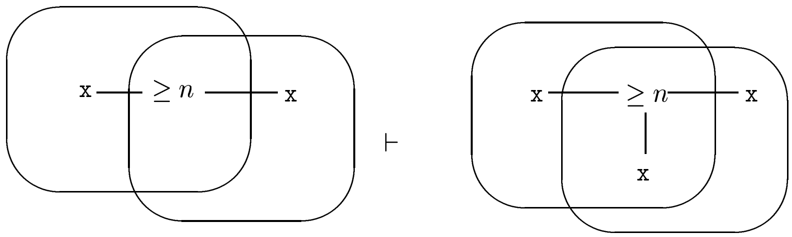
Rule: Number weakening for inequalities.

1. If a region r in D has $\langle \geq n \rangle$-sequence, D' is obtained by adding $\langle \geq m \rangle$-sequence for an arbitrary $m < n$ in r.

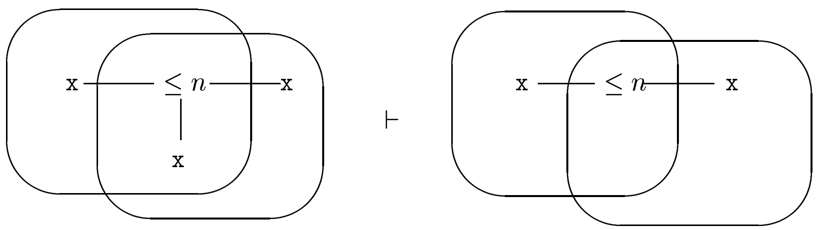


2. If a region r in D has $\langle \leq n \rangle$-sequence, D' is obtained by adding $\langle \leq m \rangle$-sequence for an arbitrary $m > n$ in r.

Rule: Extension of a number link. If a region r in D has $\langle = n \rangle$-sequence or $\langle \geq n \rangle$-sequence, D' is obtained by adding of $\langle \geq n \rangle$-sequence in any superregion r' of r.

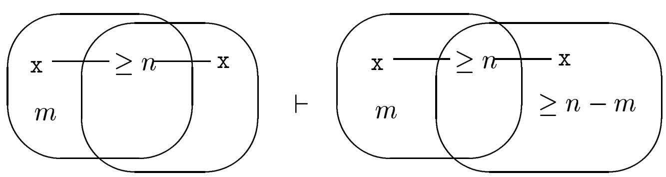


Rule: Erasure of part of a number link. If $\langle \leq n \rangle$-link is in r in D, D' is obtained from D by adding $\langle \leq n \rangle$-link in any subset of r.



Rule: Subtraction of numbers.

1. If there is $\langle = n \rangle$-sequence or $\langle \geq n \rangle$-sequence in a region r in D and there is a subregion r' of r with $\langle = m \rangle$-sequence or $\langle \leq m \rangle$-sequence, D' is obtained from D by drawing a new number link in $r \backslash r'$ with number label $\langle \leq n - m \rangle$.



2. If there is $\langle = n \rangle$-sequence or $\langle \leq n \rangle$-sequence in a region r in D and there is a subregion r' of r with $\langle = m \rangle$-sequence or $\langle \geq m \rangle$-sequence, D' is obtained from D by drawing a new number link in $r \backslash r'$ with number label $\langle \geq n - m \rangle$.

Rule: Addition of numbers. Let r, s be regions in D such that they do not overlap each other.

- If r has $\langle \leq n \rangle$-sequence or $\langle = n \rangle$-sequence, and if s has $\langle \leq m \rangle$-sequence, then D' is obtained from D by adding $\langle \leq n + m \rangle$-sequence in $r \cup s$.

- If r has $\langle \geq, n \rangle$-sequence or $\langle = n \rangle$-sequence, and if s has $\langle \geq m \rangle$-sequence, then D' is obtained from D by adding $\langle \geq n + m \rangle$-sequence in $r \cup s$.

Rule: Erasure. D' is obtained from D either by erasing a number link in D, erasing a curve, provided that

1. A region of D' has a number link if there is its counterpart in D with a number link of the same number label.
2. If the erasure of a curve eliminate the boundery between r and r' in D, and if r (r') has a node of number links which have no node in $r'(r)$, erase all those links together with the curve.

Rule: Introduction of a new curve. D' is obtained from D by adding a new closed curve provided that:

- the name label of the new curve is distinct from any name labels in D.
- if a region r in D has a number link then there exists in D' its counterpart with a number link of the same number label.
- if a region r in D' has a number link then there exists in D its counterpart with a number link of the same number label.

Rule: Unification. D' is obtained from D_1, D_2 if

- the set of name labels of D' is the union of the set of name labels of D_1 and that of D_2.
- a region r' in D' has a number link iff its counterpart either in D_1 or in D_2 has a number link of the same number label.

Rule: Inconsistency. If D satisfies the following condition, any diagram D' is obtained from D: There is a region r in D such that two number links, $\langle \geq n \rangle$-sequence and $\langle \leq m \rangle$-sequence, and $n > m$.

We write $D \vdash D'$ if D' is obtained from D by iterative applications of the above rules of inference finitely many times. We also write $\Delta \vdash D'$ if there is a sequence $D_1, \cdots, D_n = D'$ such that D_i is either in Δ or obtained from the previous ones by one of the above rules. Then, soundness is proved by tedious verification.

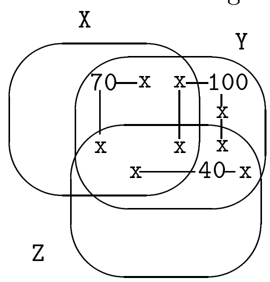
Theorem(Soundness). If $D \vdash D'$, then $D \models D'$.

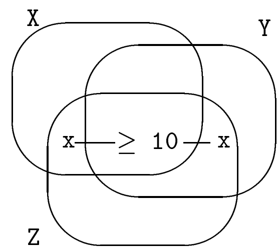
Corollary Let Δ be a finite set of well-formed diagrams. If $\Delta \vdash D'$, then $\Delta \models D'$.

3.1 A example of diagrammatical proof

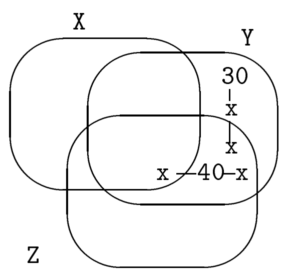
Problem: Suppose 100 Ys to exist: then if 70 Xs be Ys, and 40 Zs be Ys, it follows that 10 Xs (at least) are Zs.(De Morgan(1872))

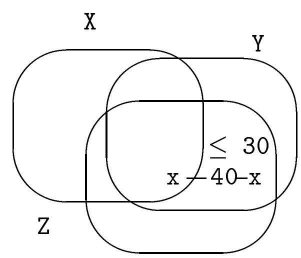
We want to deduce the right diagram from the left diagram:



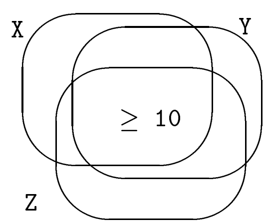


A proof looks like this. First, we apply subtraction to the premiss diagram to get the left below. Then, we can get the right by erasure of part of a number link.





Apply substaction again, and we get this:



Then, apply extension of a number link, we can prove the desired diagram from the first one.

4 Finite Completeness Result

The aim of this section is to extend Shin's theorem to our system of Venn diagrams with numerical operations so as to prove a finite completeness result.

Theorem. If $D \models D'$, then $D \vdash D'$.

Proof. Our basic strategy is the same as Shin[4]. It gives a procedure to construct a proof of D' from D. The key notion is the 'maximal diagram', which contains all pieces of information deducible from D.

Now, suppose $D \models D'$. First, consider the case in which D is not satisfiable. Then $D \models D'$ trivially holds. But $D \vdash D^\dagger$ for any arbitrary $D^\dagger$, and in particular, $D \vdash D'$.

Thus, we can assume that D is satisfiable in the following argument. As the first step, we will add new closed curves to D so that all the regions in D' has its counterpart in the resulting diagram D^+. Let us add new closed curves to D so that all the regions in D' so that has its counterpart in the resulting diagram D^+. Then add a $\langle \geq 0 \rangle$-sequence to each region of the resulting diagram, so that all regions must be assigned a number more than 0. Call the resulting diagram D^+. Clearly $D \vdash D^+$.

Then, as the second step we can construct the maximal diagram D^+_{max} by a finite sequence of diagrams $D^+ = D_0, \cdots, D_m = D^+_{max}$. This step is devided into six substeps. Let $D_0 = D^+$.

1. The $6n + 1$-th substep is to check there is any region with both $\geq$ and $\leq$ sequences of the same number. If there is, apply equality to put an $=$-sequence. If not, we just go on.

2. The $6n + 2$-th substep is to apply number weakening. If there is a region within which there is an $=$-sequence but there is no $\geq$-sequence or $\leq$-sequence, put the sequence that does not appear in the region so that there are both $\geq$- and $\leq$-sequence in the region as well as the $=$-sequence.

3. The $6n+3$-th substep: If there are any two number links in disjoint regions, and if the resulting number link does not cause redundancy, apply the rule of addition of numbers. If there are not such links, just go on.

4. The $6n+4$-th substep is to apply subtraction if possible. That is, if there is any number link to which the rule of substraction of numbers is applicable, apply it. If not, just go on.

5. The $6n + 5$-th substep is to shorten a number link if possible. That is, if there is any number link to which the rule of erasure of part of a number link is applicable, apply it. If not, just go on.

6. The $6n + 6$-th substep is to apply the rule of extension of a number link.

Iterate the above procedure until there is no number link to which none of the rules above is applicable. The procedure described above is applicable only a finite number of times (let m be the number of steps). It is because each diagram which appears in each step has finitely many regions and finitely many links, and we excluded the redundancy that the same number link should not appear more than once in a single region.

Clearly, for every diagram D_i appearing in each step in the above procedure, $D \vdash D_i$. By the soundness result, we also have $D \models D_i$ for all $n \leq m$. Moreover, since $D \vdash D_m$, $D_m \models D'$ holds.

Then, apply the rule of number weakening to all number links of inequality labels in D_m (1) with the integer in $\leq$-sequences incremented by 1 up to the largest number appearing in D'; (2)with the integer in $\geq$-sequences decremented by 1 until 0. Since D' has only finitely many symbols, this step is iterated at most finitely many times. Let us call the resulting diagram D^+_{max}. Then clearly, $D \vdash D^+_{max}$. By the soundness result, we also have $D \models D^+_{max}$ for all $n \leq m$. Moreover, $D^+_{max} \models D'$ holds.

Claim. If there is a number link of a number label in a region r' in D', then there is a number link of the same number label in its counterpart r in D^+_{max}.

With the claim established, we would be able to erase all the curves and number links in D^+_{max} that do not appear in D' so that we would have $D \vdash D'$.

Proof of the claim. Suppose there is r' in D' such that r' has a number sequence but its counterpart r in D^+_{max} does not have any corresponding sequence. Before examining what it means by 'any corresponding sequence', note a fact: as we applied the rule of number weakening of inequalities,

1. if r has a $\langle \leq n \rangle$-sequence, then it also has $\langle \leq m \rangle$-sequences for all $m > n$ less than or equal to the largest number in D'.
2. if r has a $\langle \geq n \rangle$-sequence, then it also has $\langle \geq m \rangle$-sequences for all $0 \leq m < n$.

And if r has a $\langle = n \rangle$-sequence, it has both a $\langle \leq n \rangle$-sequence and a $\langle \geq n \rangle$-sequence, too. Hence, such r has $\langle \leq m \rangle$-sequences for all $m > n$ less than or equal to the largest number in D' and $\langle \geq m \rangle$-sequences for all $0 \leq m < n$.

There are problematic cases but we can construct a countermodel for them. Let us enumerate all the pairs of counterparts such that r_i has no corresponding sequence with r'_i. There are at most a finite number of such pairs, since both D^+_{max} and D' are well-formed. Define a partial assignment f so as to assign to r a set of size adequate for the number link(s) in r: in any cases above, we can take a natural number so as not to support D'. In fact,

1. Take $f(r) = m$ for an arbitrary $m \neq n$, if
   (a) r' has a $\langle = n \rangle$-sequence, and r has no sequence,
   (b) r' has a $\langle \leq n \rangle$-sequence, and r has no sequence, or
   (c) r' has a $\langle \geq n \rangle$-sequence, and r has no sequence.
2. Take $f(r)$ as the largest m, if
   (a) r' has a $\langle = n \rangle$-sequence, and r has $\langle \leq m \rangle$-sequences and the largest m is less than n, or
   (b) r' has a $\langle \geq n \rangle$-sequence, and r has $\langle \leq m \rangle$-sequences and the largest m is less than n.

3. Take $f(r)$ as the smallest m, if
   (a) r' has a $\langle = n\rangle$-sequence, and r has $\langle \geq m\rangle$-sequences and the smallest m is greater than n, or
   (b) r' has a $\langle \leq n\rangle$-sequence, and r has $\langle \geq m\rangle$-sequences and the smallest m is greater than n.
4. Take $f(r) = m$, if
   (a) r' has a $\langle \leq n\rangle$-sequence, and r has a $\langle = m\rangle$-sequence for $m > n$, or
   (b) r' has a $\langle \geq n\rangle$-sequence, and r has $\langle = m\rangle$-sequences for $m < n$.

Extend f to I whose domain is all the regions so as to support D^+_{max}. That is, $(U, I) \models D^+_{max}$. At the same time, $(U, I) \not\models D'$. But it contradicts our assumption that $D^+_{max} \models D'$.

From the proof, we can extract finite completeness and interpolation as corrolaries.

Corrolary. Let Δ be a finite set of diagrams. If $\Delta \models D'$, then $\Delta \vdash D'$.

Proof. Apply unification to Δ before the construction of the theorem so as to get a single diagram.

Corrolary. For any D, D' such that $D \models D'$, there is D° such that $D \models D^\circ$ and $D^\circ \models D'$.

Proof. Take D^+_{max} in the proof as D°.

References

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[3] Hammer, Eric M.(1995) *Logic and Visual Information*. CSLI publications.

[4] Shin, Sun-Joo (1994) *The Logical Status of Diagrams*. Cambridge UP.