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Short Note

Derivation of the Three-Parameter Model of Frequency Distribution of Earthquake Magnitude

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Abstract: On the basis of a simple seismicity model, we theoretically derive a distribution function which represents frequency distribution of earthquake magnitude. The model is constructed by superposing earthquake sub-sets of which magnitude-frequency distribution follows a band-limited exponential law with decay constant of b'. The largest earthquake of each sub-set is assumed to follow another band-limited exponential distribution with decay constant of q'. When $b' \approx q'$, the theoretical distribution function asymptotes to the three-parameter formula of magnitude-frequency distribution that was derived for main shock-aftershock sequences. The three-parameter model is useful for estimating the size of potentially largest earthquake in the respective region.

1. Introduction

It is well known that the number of earthquakes occurring in a fixed space-time window follows the Gutenberg-Richter's relation (Gutenberg and Richter, 1944):

$$\log n(M\,;\,\Delta M) = a - bM, \tag{1}$$

where $n(M\,;\,\Delta M)$ is the earthquake frequency for a small magnitude range between $M-\Delta M$ and M, and a and b are constants. This relation is obtained by integrating the distribution function:

$$f(M) = A\mathrm{e}^{-b'M} \tag{2}$$

from $M-\Delta M$ to M, where

$$\begin{aligned} A &= b\,10^{a}/\{(10^{b\Delta M}-1)\log \mathrm{e}\}, \\ b' &= b/\log \mathrm{e}. \end{aligned} \tag{3}$$

The Gutenberg-Richter's relation has been successfully applied to various kinds of data sets of global to local scales, and b-values of around 1 are obtained for tectonic earthquakes. However, magnitude-frequency distribution of earthquakes sometimes exhibits significant deviation from equation (1). For improving the fitting with observational data, Utsu (1971) proposed a three-parameter equation:

$$\log n(M\,;\,\Delta M) = \alpha - \beta M + \log(\gamma - M), \tag{4}$$

where α, β and γ are constants.

The present study shows that equation (4) is derived from a simple model; superposition of band-limited exponential distributions.

2. Limitation of Gutenberg-Richter's Law

The Gutenberg-Richter's law of magnitude-frequency distribution must break at a small magnitude since the earthquake frequency predicted by equation (1) or (2) diverges when M approaches minus infinity. Indeed, Watanabe (1973) found that the number of ultramicro-earthquakes of $M \leqq -0.9$ was significantly smaller than that was expected from equation (1) for the Matsushiro earthquake swarm. It is also beyond doubt that there exists an upper limit of earthquake magnitude. The upper limit, being ultimately limited by the size of the earth, may differ from region to region depending on the tectonic condition.

Those conditions deviate the magnitude-frequency distribution of actual earthquakes from equation (1), in particular when the data cover a wide magnitude range. The curve of M versus log $n(M\,;\,\Delta M)$ plot is expcted to bend upward due to the limit at small and large magnitudes. Figure 1 shows an example of such an upward bending.

For fitting such a non-exponential distribution, several equations alternative to (1) are proposed (*e.g.*, Lomnitz, 1964; Riznichenko, 1964: Mogi, 1967; Sacuiu and Zorilescu, 1970; Utsu, 1971; Saito *et al.*, 1973). Among those is the three-parameter equation of Utsu (1971) which is expressed by equation (4). This equation was derived for an earthquake set that is composed of main shock-aftershock series by assuming an empirical relation between the magnitudes of main shock and largest aftershock. In the following sections, we derive a more general formula of magnitude-frequency distribution on the basis of a simple seismicity model. It will be shown that equation (4) is an asymptotic form of the general equation.

3. Model

We model a set of earthquakes by superposition of sub-sets of which earthquakes follow a magnitude-frequency relation of band-limited exponential distribution:

$$f(M\,;\,M_m)=\begin{cases} f_0\,\mathrm{e}^{-b'M}, & M \leqq M_m\,; \\ 0, & M > M_m. \end{cases} \tag{5}$$

In the equation, f_0 differs for each sub-set, but b' is assumed as a constant irrespective of sub-sets. M_m is the magnitude of the maximum earthquake in the sub-set. Lower limit of magnitude is not applied here since it is believed to be very small (*e.g.*, Watanabe, 1973).

We further assume that the maximum earthquakes of sub-sets follow another magnitude-frequency relation of band-limited exponential type:

$$g(M_m)=\begin{cases} g_0\,\mathrm{e}^{-q'M_m}, & M_m \leqq M_c\,; \\ 0, & M_m > M_c. \end{cases} \tag{6}$$

M_c is the critical magnitude that defines the upper limit of magnitude in the earthquake set.

By using equations (5) and (6), magnitude-frequency distribution for the whole earthquakes is given by

$$h(M)=\int_{M}^{M_c} f(M\,;\,M_m)g(M_m)dM_m. \tag{7}$$

4. Superposition of Sub-Sets

For conducting the integration of (7), a caution is needed that the scaling factor f_0 in (5) is not a constant but a function of M_m. It is because $f(M\,;\,M_m)$ must be so weighted that the maximum earthquake satisfies the condition of equation (6). We define the "maximum earthquake" belonging to a sub-set by an earthquake of which magnitude is between $M_m-\Delta M$ and M_m. A small magnitude interval ΔM is fixed for all the sub-sets. The number of maximum earthquake thus defined is

$$\int_{M_m-\Delta M}^{M_m} f(M\,;\,M_m)\,dM=f_0\,\mathrm{e}^{-b'M_m}(\mathrm{e}^{b'\Delta M}-1)/b'. \tag{8}$$

Scaling factor f_0 is given by putting the right-hand side of (8) at unity.

The result of integration is

$$\left.\begin{aligned} h(M)&=K\mathrm{e}^{-q'M}\,[1-\mathrm{e}^{-(q'-b')(M_c-M)}],\\ K&=g_0\,[b'/(q'-b')]/(\mathrm{e}^{b'\Delta M}-1). \end{aligned}\right\} \tag{9}$$

For small magnitude of $M\ll M_c$, $h(M)$ asymptotically approaches a simple exponential distribution with a decay constant of q'.

By puttig ΔM at a small value, $\Delta M\ll 1/b'$, $1/q'$, the frequency of earthquakes for the magnitude range from $M-\Delta M$ to M is

$$n(M\,;\,\Delta M)=\int_{M-\Delta M}^{M} h(M)\,dM=g_0/(q'-b')\mathrm{e}^{-q'M}[1-\mathrm{e}^{-(q'-b')(M_c-M)}]. \tag{10}$$

If $(q'-b')\,(M_c-M)\ll 1$, equation (10) reduces to

$$n(M\,;\,\Delta M)=g_0\,\mathrm{e}^{-q'M}\,(M_c-M). \tag{11}$$

This is equivalent to equation (4), where $\alpha=\log g_0$, $\beta=q'\log\mathrm{e}$, and $\gamma=M_c$. Therefore, the three-pamameter equation (4) is the asymptotic form of more general equation (10) for $b'\approx q'$.

5. Discussion and Conclusion

We formulated the distribution function of earthquake magnitude, equation (10), on the basis of band-limited exponential distributions. The approximation form of (11) coincides with equation (4) by Utsu (1971). Equation (11), however, was derived without any assumption on the nature of aftershocks so that the application is not limited to main shock-aftershock sequences.

M_c defined in equation (6) represents the possible largest earthquake in the respective region. Therefore, the three-parameter fitting to magnitude-frequency distribution may provide an valuable tool to estimate the size of potential "ultimate earthquake". Utsu (1974) dividing whole Japan into eighteen regions computed the γ-value, equivalent to M_c, for each region by using the observation data of the Japan Meteorological Agency. The results shows a clear regionality of γ-value, ranging from 5.7 to ∞. Such an analysis including recent earthquake data will be of great value to evaluate long-term earthquake risk.

The present model is easily pictured by thinking a region where active faults of various lengths distribute. Suppose each of active faults generates earthquakes which follow the magnitude-frequency distribution of exponential type similar to the Gutenber-

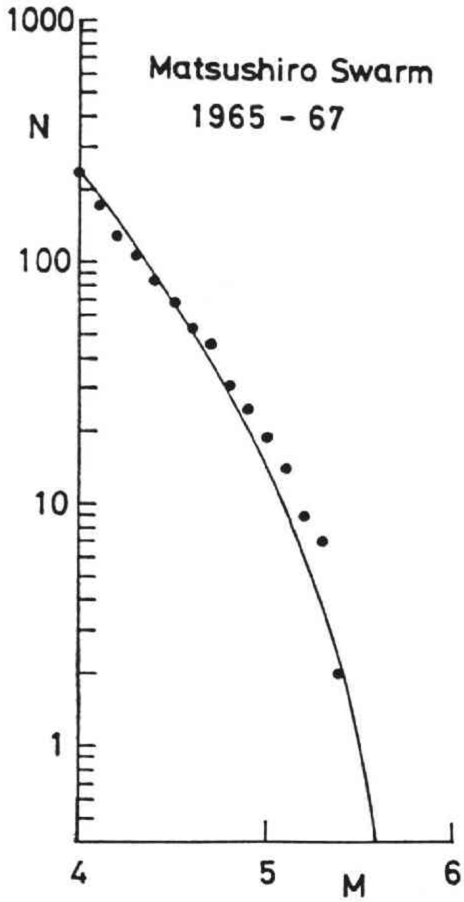

Fig. 1. Non-exponential frequency distribution of earthquake magnitude observed for the Matsushiro earthquake swarm [after Utsu (1974)]. Solid curve indicates the best fit of equation (4), where $\beta=0.64$ and $\gamma=5.7$. Note that the ordinate is cumulative frequency summing up the earthquakes of magnitude M or larger.

g-Richter's law. This "intrinsic" distribution must be band-limited as was shown by equation (5), since earthquakes from an active fault have an upper limit of magnitude (maximum earthquake) corresponding to the fault length.

For the maximum earthquakes, Ohtake (1993) suggested the validity of equation (6) showing that the frequency of active faults in Japan is in proportion to a negative power of fault length. The power-type distribution of fault length is converted to an exponential type distribution of earthquake magnitude by assuming the scaling law of fault parameters by Kanamori and Anderson (1975), and the relation between magnitude and seismic moment by Kanamori (1977).

It, however, is not clear so far whether the condition of $b' \approx q'$ is satisfied or not. When this condition in not satisfied, observational data will further deviate from the three-parameter equation (4) even though the present model is valid. For testing the validity of our model, quantitative studies of active faults are needed.

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