Note on the Bending Vibration of Long Elastic Beam

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Bending vibration excited by deviation of one end of a long elastic beam is discussed for typical cases, where the solutions are given in compact forms. The method used is Laplace transformation, fitted to analyze transient phenomena. The solutions for several boundary conditions are given and discussed shortly.

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§1. Introduction

The bending vibration of a homogenuous, uniform and semi-infinite elastic beam is somewhat complicated, because that the equation of bending vibration is of the fourth order in derivatives along beam, and that there appear damping as well as progressive waves. The case of sinusoidal oscillation applied on the end early treated[1)] by Seiichi Higuchi and recently rediscussed,[1)] the method adopted being essentially separation of variables.

The problem is of time dependent character and in general has transient nature. For this reason, it may be adequate to apply the method of Laplace transform. This note is tried motivated by the work[1)] of S. Higuchi.

§2. Formulation

The end cross-section ($x = o$, say) of a homogenuous, uniform and long elastic beam is assumed to contain one of principal inertia axes named z-axis. On this cross-section we apply bending deviation for $t>o$ to excite elastic motion. The center line of the beam at equilibrium is taken x-axis, and the deviation is denoted as $u(x,t)$. This satisfies approximately[2)] for $o<x<\infty$

$$\rho \frac{\partial^2 u}{\partial t^2} = - E I \frac{\partial^4 u}{\partial x^4} \quad \text{or} \quad \ddot{u} + c^2 u'''' = o \quad \text{for} \quad t>o. \tag{1}$$

Here $c^2 = EI/\rho$ and ρ, E, I are density, Young modulus and the second moment of the cross-section about y-axis.

The boundary conditions for (1) are assumed, as the first example

$$u = f(t), \qquad \partial u/\partial x = o \qquad \text{at} \quad x = +o \tag{2}$$

and radiation condition at infinity. (3)

The initial conditions are, for $o<x<\infty$

$$u = o, \qquad \partial u/\partial t = o \qquad \text{at} \quad t = +o. \tag{4}$$

The problem is solved using Laplace transform; it is defined for $u(x,t)$

$$U(x,p) = \int_o^\infty dt\, e^{-pt}\, u(x,t), \quad \text{Re}\, p>o. \tag{5}$$

This satisfies

$$p^2 U + c^2 U'''' = o, \qquad \text{Re}\, p>o, \tag{6}$$

since partial integration of $\int_o^\infty dt\, \ddot{u}\, e^{-pt}$ under initial conditions (4) with Re $p>o$ gives just $\int_o^\infty dt\, u(x,t) e^{-pt} = U$.

The conditions (2) now clearly turn out

$$U = F = \int_o^\infty dt\, f(t)\, e^{-pt}, \quad dU/dx = o \qquad \text{at } x = +o. \tag{7}$$

These are boundary conditions for (6) ; if $F(p)$ is singular, a_1 , a_2 to be appeared in (13) are possibly so. As for the radiation conditions (3), remark will be given in below.

Assume $U(x,p) = \exp(-K(p)x)$, then Eq.(6) gives

$$p^2 + c^2 K^4 (p) = o, \tag{8}$$

as determining equation of $K(p)$ for any complex p with Re $p>o$.

§3. Solutions

Eq.(8) shows that $K(p)$ is four valued function for one complex p. Clearly $p=o$ and ∞ are branch points ; if branch cut $o\to-\infty$ is introduced, we have four one-valued functions $K_i(p)$, $i=1,2,3,4$. Thus the general solution of (6) is

$$U = \sum_{i=1}^{4} a_i \exp(-K_i(p)x), \tag{9}$$

and its inverse Laplace transform

$$u(x,t) = \frac{1}{2\pi i} \int_{p_o-i\infty}^{p_o+i\infty} dp\; U(x,p) e^{pt} \quad \text{for} \quad t>o \tag{10}$$

just gives general solutions for (1) with Re $p = p_o > o$, (p_o is larger than convergence coordinate of Laplace transform: in our case limit $p_o \to +o$ is allowed.) Hereafter the integration path in (10) is simply named C.

We must determine K of Eq.(8). Putting $p = re^{i\theta}$, we get from (8) generally

$$K = \left(\frac{r}{c}\right)^{1/2} \exp\left(i\left[\frac{\theta}{2} - \frac{\pi}{4} + \frac{2\pi}{4} n\right]\right), \quad n = \cdots, -1, o, 1, 2, \cdots \tag{11}$$

As noted in above, it suffices to have four K_i at most ; for $n = o,1,2,3$ we set K_i , $i = 1,2,3,4$. The condition in (6) or Re $p>o$ requires $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, thus only two of K's in (9) with $n=o,1$ are allowed and $a_3 = a_4 = o$; radiation condition (3) also is satisfied. To fit the last boundary conditions (7), we take

$$a_1 + a_2 = F, \quad K_1 a_1 + K_2 a_2 = o \quad \text{or} \quad a_1 = F\, e^{i\pi/4}/\sqrt{2}i, \quad a_2 = -F\, e^{-i\pi/4}/\sqrt{2}i. \tag{12}$$

Inserting (13) into U (9), we get the solution (10) :

$$u(x,t) = \frac{-1}{2\pi\sqrt{2}} \int_C dp\, e^{pt}\, F(p)\, [\exp(-K_1(p)x + i\tfrac{\pi}{4}) - \exp(-K_2(p)x - i\tfrac{\pi}{4})]\,. \tag{13}$$

Referring (7) and changing integration order, we obtain

$$U(x,t) = \int_o^\infty d\tau\; f(\tau)\; G(x,t-\tau) \tag{14}$$

where

$$G(x,t-\tau) = \frac{-1}{\sqrt{2}\;2\pi} \int_C dp\, e^{p(t-\tau)}\, [\exp(-K_1(p)x + i\tfrac{\pi}{4}) - \exp(-K_2(p)x - i\tfrac{\pi}{4})] \tag{15}$$

§4. Properties of G(x,t)

Before explicit evaluation of G(x,t), we check that U(x,t) thus obtained satisfies the boundary conditions (2). Firstly G(x,t) is zero for $t<o$, as easily seen. Nextly for $t>o$, we must integrate literally along C. Shifting C up to the real axis ($p_o \to o$), we get, with $p = ir$(for pos. imag. axis)

$$G(x,\tau) = \frac{1}{\sqrt{2}\pi} \int_o^\infty dr\, [\exp(-\sqrt{\tfrac{r}{c}}x)\, \sin(r\tau + \tfrac{\pi}{4}) - \sin(r\tau - \sqrt{\tfrac{r}{c}}x - \tfrac{\pi}{4})]\,. \tag{16}$$

This is, for $x \to +o$

$$G(+o,\tau) = \frac{1}{\pi} \int_o^\infty dr \cos r\tau = \delta(\tau), \tag{17}$$

where $\delta(t)$ is Dirac δ-function; thus the first condition of (2) holds:

$$u(+o,t) = \int_o^\infty d\tau\, f(\tau)\, \delta(t-\tau) = f(t).$$

The solution obtained in (15) also satisfies the second condition or

$$(\partial u(x,t)/\partial x)_{x\to +o} = o, \quad \text{because} \quad \lim_{x\to +o} \partial G(x,\tau)/\partial x = o.$$

We proceed to give the explicit expression for G(x,t), Eq.(16). After changing-variable r to ξ^2, the required integrals can be obtained.[4)] C,S Fresnel integrals appear many times, but finally cancel out and we get a compact result :

$$G(x,\tau) = \frac{1}{\tau}\, \frac{x}{\sqrt{2\pi c\tau}}\, \sin \frac{x^2}{4c\tau}\;. \tag{18}$$

Omitting $1/\tau$, the essential part of (22) is roughly $g(\xi) = \xi \sin \xi^2$, $\xi^2 = x^2/4c\tau$; it, as a whole, increases linearly with ξ, but due to more and more rapid fluctuations local compensation occurs between neighbors. The first and main maximum arises at about $\xi_o \approx 1{,}35 (\tan \xi_o + 2\xi_o^2 = o)$; it resembles wave packet extending $\xi^2 \approx 1 \sim 3$.

Inserting (22) in (5), we arrive at the final result :

$$u(x,t) = \int_o^t d\tau \frac{f(\tau)}{t-\tau} \frac{x}{\sqrt{2\pi c(t-\tau)}} \sin \frac{x^2}{4c(t-\tau)} \qquad (19)$$

This integral may be evaluated for simple $f(\tau)$. If we put $t - \tau = x^2/4cs$ and s or $\eta = s^2$, etc. in (23), then

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_{x^2/4ct}^{\infty} ds \frac{\sin s}{\sqrt{s}} f(t - \frac{x^2}{4cs}) = 2\sqrt{\frac{2}{\pi}} \int_{(x^2/4ct)^{1/2}}^{\infty} d\eta \sin \eta^2 f(t - \frac{x^2}{4c\eta^2})$$

$$= 2\sqrt{\frac{2}{\pi}} (\frac{x^2}{4c})^{1/2} \int_{t^{-1/2}}^{\infty} d\zeta \sin \frac{x^2}{4c\zeta^2} f(t - \zeta^{-2}) \qquad (20)$$

The choice between these forms depends on the form of f. For example, if f(t) in (2) is constant fo, we may use the second form and

$$u(x,t) = 2 f_o (\int_o^{\infty} - \int_o^{(x^2/2\pi ct)^{1/2}}) d(\sqrt{\frac{2}{\pi}} \eta) \sin[\frac{\pi}{2}(\sqrt{\frac{2}{\pi}} \eta)^2]$$

$$= 2 f_o [\frac{1}{2} - S((\frac{x^2}{2\pi ct})^{1/2})], \qquad (21)$$

where $S(z) = \int_o^z dt \sin(\frac{\pi}{2} t^2)$ is Fresnel integral.[4] For this simple case $u(x,t)$ tends to a <u>stationary</u> value $2 f_o \frac{1}{2}$ for $t \to \infty$ thus the <u>transient</u> term being proportional to $-S((\frac{x^2}{2\pi ct})^{1/2})$, which is $-\frac{1}{2}$ for small t, nextly oscillates more and more slowly and largely with t up to the last stage where it suddenly dies out to o ($-S = -o.7$ for $(x^2/2\pi ct)^{1/2} = 1.5$).

§6. Other Boundary Conditions

We briefly treat the cases the boundary conditions differ from (2). As the first variant we take $u = o$, $\partial u / \partial x = g(t)$ at $x = +o$. (2')
In this case in (9) also only $a_1 \neq o$, $a_2 \neq o$ and $a_1 = -a_2 = G_1(p)/(\frac{p}{c})^{1/2} (i\sqrt{2})$. We only write down the result :

$$u(x,t) = \int_o^{\infty} d\tau \, g(\tau) G_1(x, t-\tau),$$

$$G_1(x,\tau) = \frac{1}{2\pi} \frac{1}{\sqrt{2}} \int_C dp \, e^{p\tau} (\frac{c}{p})^{1/2} [-\exp(-K_1(p)x) + \exp(-K_2(p)x)]. \qquad (22)$$

Moreover $G_1(+o,\tau) = o$, $(\partial G_1/\partial x)_{x=+o} = \delta(\tau)$.

If g(t) in (2') is δ-function $B\delta(\omega t)$ for t>o, we get[4)]

$$u(x,t) = -\frac{B}{2\pi\omega}\int_o^\infty \frac{dq}{\sqrt{q}}\,[\cos((\frac{q}{c})^{1/2}x - qt) - \sin((\frac{q}{c})^{1/2}x - qt)$$

$$+ \exp(-(\frac{q}{c})^{1/2}x)\,(\cos qt + \sin qt)]. \tag{22}$$

Here appear damping wave as well as progressive wave. The former decays and after one period $T = 2\pi/q$ decreases by a factor $\exp(-(\frac{q}{c})^{1/2}\Delta x) = e^{-2\pi} = o.oo17$ for $\Delta x = \lambda \equiv 2\pi\ (c/q)^{1/2}$.

The integral (22) are easily evaluated, giving

$$u(x,t) = -\frac{B}{2\pi\omega}\cdot 2(\frac{2\pi}{t})^{1/2}\cos\frac{x^2}{4ct}, \tag{23}$$

which rapidly oscillates for small t, and decays out after long time, showing that it is transient.

Finally we take the case that boundary conditions are changed to

$$u(x,t) = o,\ \partial^2u/\partial x^2 = h(t) \quad \text{at} \quad x{=}{+}o. \tag{24}$$

Laplace transform U(x,p) defined by (5) must satisfy

$$U = o, \quad \partial^2U/\partial x^2 = \int_o^\infty dt\, e^{-pt}\,h(t) = H \quad \text{at} \quad x = +o. \tag{25}$$

The procedure to solve is similar as in §2,3; (22) and radiation condition give

$$a_3 = a_4 = o, \quad a_1 = -a_2 = H/(K_1^2 - K_2^2) = -\frac{c}{p}H/2i. \tag{26}$$

Using (26), we get

$$u(x,t) = \int_o^\infty d\tau\, h(\tau)\, h(\tau)\, G_2(x,t-\tau), \tag{27}$$

where

$$G_2(x,\tau) = \frac{1}{4\pi}\int_C dp\, e^{p\tau}\,\frac{c}{p}\,[\exp(-K_1(p)x) - \exp(-K_2(p)x)]. \tag{28}$$

The integral along C may be evaluated similarly. (The integrand of (27) has no pole at $p = o$.). We note $G_2(x,\tau) = o$ for $\tau<o$. On the other hand, for $\tau>o$, it turns out

$$G_2(x,\tau) = \frac{1}{2\pi}\int_o^\infty dr\,\frac{c}{r}\,[-\cos(r\tau - (\frac{r}{c})^{1/2}x) + \exp(-(\frac{r}{c})^{1/2}x)\cos r\tau]. \tag{29}$$

It is evidently zero for $x\to +o$, whereas $(\partial^2G(x,\tau)/\partial x^2)_{x\to+o} = \delta(\tau)$. These certify, similarly in the former example, that (29) matches the boundary conditions (24).

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References

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2) e.g. S.P. Timoshenko, Vibrating Problems in Engineering, 3rd Ed. 1955, Chapt. 5, §58.

3) Handb. of Math. Func., etc. Ed. by M. Abramowitz and I.A. Segan, Dover Publ, Inc., N.Y., pp.297-304. It contains numerical values and cusves of S, C func.