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Rotating Waves in a Model of Delayed Feedback Optical System with Diffraction

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We study a delayed parabolic functional differential equation on a circle that is coupled with an initial value problem for the Schrodinger equation. Such equations arise as models of nonlinear optical systems with a time-delayed feedback loop, when diffusion of molecular excitation and diffraction are taken into account. The goal of this paper is to prove the existence of spatially inhomogeneous rotating-wave solutions bifurcating from homogeneous equilibria. We pass to a rotating coordinate system and seek an inhomogeneous solution to an ordinary functional differential equation. We find the solution in the form of a small parameter expansion and explicitly compute the first-order coefficients. We also provide examples of parameters that satisfy the constraints imposed throughout the analysis.

KEYWORDS: delay, parabolic functional differential equation, rotating wave, Kerr medium, diffraction

1. Introduction

Nonlinear optics is one of the areas where self-organization occurs [1]. A typical nonlinear optical system with rich spatio-temporal dynamics comprises a thin layer of nonlinear Kerr medium and a ring cavity (feedback loop). Phase modulation in the nonlinear Kerr slice within an aperture $Q \subset \mathbb{R}^2$ – real-valued function $u(\mathbf{r}, t)$ – is described by the following parabolic equation [2]:

$$u_t(\mathbf{r}, t) + u(\mathbf{r}, t) - D\Delta u(\mathbf{r}, t) = |A_{FB}|^2, \quad \mathbf{r} \in Q, \quad t > 0, \tag{1.1}$$

augmented by boundary conditions on ∂Q and by an initial condition. Here, $D > 0$ is the diffusion coefficient; A_{FB} is the complex amplitude of the light field after it has passed the feedback loop. Equation (1.1) includes local coupling caused by diffusion of molecular excitation in the nonlinear Kerr layer. Depending on the configuration of the feedback loop, expression for the complex amplitude A_{FB} can bring nonlocal interactions into (1.1): time delay and/or spatial nonlocality (see [3, 4]). Among the natural physical phenomena that can be taken into account in the mathematical model are interference of the input and feedback light fields [6], and free propagation diffraction in the feedback loop [5]. In the most general case, (1.1) is a delayed nonlinear partial functional differential equation [6].

The magnitude of nonlocalities together with the input light field intensity form an effective toolkit for controlling the dynamics of the system, which is crucial for applications (see [7, 8]).

Whether Eq. (1.1) admits shape-preserving solutions is a matter of research. In [6], a periodic boundary value problem for Eq. (1.1) was shown to possess rotating-wave solutions, when time delay and rotation of spatial arguments are present. A similar equation was studied in case of Neumann boundary conditions and time delay [9]: only spatially homogeneous rotating waves were proved to exist.

In the present paper we study a mathematical model of an optical system with a thin ring aperture, time-delay device, interference, and diffraction. Our goal is to show that a one-dimensional periodic boundary value problem for Eq. (1.1) on a circle admits spatially inhomogeneous rotating-wave solutions bifurcating from spatially homogeneous equilibria. We prove the existence in a rotating coordinate system, seeking an inhomogeneous solution to an ordinary functional differential equation. We find the solution in the form of a small parameter expansion and explicitly compute the leading coefficients. We also provide examples of parameters that satisfy the constraints imposed throughout the analysis.

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2. Boundary Value Problem

2.1 Problem statement

We consider a periodic boundary value problem for a nonlinear parabolic functional differential equation on a circle:

$$\begin{aligned} &u_t + u = Du_{xx} + K|1 + \gamma A(x, z_0; e^{iu(t-T)})|^2, \quad u = u(x,t), \quad x \in (0, 2\pi), \quad t > 0, \\ &u|_{x=0} = u|_{x=2\pi}, \quad u_x|_{x=0} = u_x|_{x=2\pi}. \end{aligned} \tag{2.1}$$

Equation (2.1) is an implementation of (1.1) for $A_{FB} = \sqrt{K}(1 + \gamma A(x, z_0; e^{iu(t-T)}))$, where $K > 0$ is the nonlinearity coefficient that is proportional to the input light field intensity, $0 < \gamma < 1$ is the interference visibility, $T > 0$ is the time delay in the feedback loop, and $z_0 > 0$ is the free propagation distance of the light field. Here, $A(x, z; A_0)$ denotes the solution to a periodic initial-boundary value problem for the linear Schrodinger equation that describes propagation of light waves in the paraxial approximation:

$$\begin{aligned} &A_z + iA_{xx} = 0, \\ &A|_{z=0} = A_0(x), \\ &A|_{x=0} = A|_{x=2\pi}, \quad A_x|_{x=0} = A_x|_{x=2\pi}. \end{aligned} \tag{2.2}$$

For convenience, we shall introduce a linear operator B that propagates its input over the distance z_0 according to (2.2):

$$B : H(\mathbb{C}) \ni A_0(x) \mapsto A(x, z; A_0)|_{z=z_0} \in H(\mathbb{C}), \quad A_0(x) \in H^2_{2\pi}(\mathbb{C}). \tag{2.3}$$

We now introduce the notation for functional spaces to be used in the sequel: H is the Lebesgue space $L^2(0, 2\pi)$ of real-valued functions with the usual inner product; H^2 is the Sobolev space $W^{2,2}(0, 2\pi)$ of real-valued functions with the following inner product and norm

$$\langle u, v\rangle_{H^2} = \int_0^{2\pi} (uv + u''v'')\, dx, \quad \|u\|_{H^2} = \sqrt{\langle u, u\rangle_{H^2}};$$

$H^2_{2\pi} = \{u \in H^2 : u(0) = u(2\pi), u'(0) = u'(2\pi)\}$ is a closed linear subspace of $H^2(\mathbb{C})$ consisting of periodic functions. Their complex-valued counterparts will be denoted by $H(\mathbb{C})$, $H^2(\mathbb{C})$, and $H^2_{2\pi}(\mathbb{C})$, respectively; complex inner products are defined as $\langle u, v\rangle_{\mathbb{C}} = \langle u, \overline{v}\rangle$.

Lemma 2.1.

(1) *The linear operator B defined by (2.3) has a complete orthogonal set of eigenfunctions $\{e^{inx}\}_{n\in\mathbb{Z}}$. The corresponding eigenvalues are $\lambda_n(B) = e^{in^2 z_0}$.*
(2) *It is an isometry of $H(\mathbb{C})$.*
(3) *It is an isometry of $H^2_{2\pi}(\mathbb{C})$ if treated as $B : H^2_{2\pi}(\mathbb{C}) \longrightarrow H^2_{2\pi}(\mathbb{C})$.*

Proof.

(1) Let $A_0(x) = e^{inx}$. The Fourier method applied to (2.2) readily gives $A(x, z) = e^{inx}e^{in^2 z}$. The statement follows.
(2) Let $u \in H^2_{2\pi}(\mathbb{C})$ and $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$. According to the Parseval's identity,

$$\begin{aligned} \|Bu\|^2_{H(\mathbb{C})} &= \sum_{n=-\infty}^{+\infty} |\langle Bu, e_n\rangle_{H(\mathbb{C})}|^2 = \sum_{n=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} \langle u, e_k\rangle_{H(\mathbb{C})} \langle Be_k, e_n\rangle_{H(\mathbb{C})} \right|^2 \\ &= \sum_{n=-\infty}^{+\infty} |e^{in^2 z_0} \langle u, e_n\rangle_{H(\mathbb{C})}|^2 = \sum_{n=-\infty}^{+\infty} |\langle u, e_n\rangle_{H(\mathbb{C})}|^2 = \|u\|^2_{H(\mathbb{C})}. \end{aligned}$$

(3) Let $e_n = \frac{1}{\sqrt{2\pi(1+n^4)}} e^{inx}$. The proof is similar to (2). □

Boundary value problem (2.1) admits spatially homogeneous equilibria $W(x,t) \equiv W$ that can be found as roots of

$$F(W, K) \equiv W - K(1 + 2\gamma \cos W + \gamma^2) = 0. \tag{2.4}$$

Lemma 2.2. *Suppose a pair $(\hat{W}, \hat{K})$ satisfies Eq. (2.4), and let the following nondegeneracy condition hold*

$$1 + 2\hat{K}\gamma \sin \hat{W} \neq 0. \tag{2.5}$$

Then there exists a $\mu_0 > 0$ such that for $\mu \in (-\mu_0, \mu_0)$ a solution $(W(\mu), K(\mu))$ to Eq. (2.5) is defined and has the form

$$K(\mu) = \hat{K} + \mu, \quad W(\mu) = \hat{W} + \hat{W}_1\mu + \hat{W}_2\mu^2 + \ldots.$$

Proof. It follows from (2.5) that $\frac{\partial F}{\partial W}|_{(\hat{W},\hat{K})} \neq 0$. The statement is then a consequence of the analytic implicit function theorem. □

We now bring boundary value problem (2.1) to the local form in the vicinity of a spatially homogeneous equilibrium

$W(\mu)$ by setting $u(x,t) = W(\mu) + v(x,t)$:

$$\begin{aligned} &v_t + v = Dv_{xx} - W(\mu) + K(\mu)|1 + \gamma e^{iW(\mu)} B e^{iv(t-T)}|^2, \\ &v|_{x=0} = v|_{x=2\pi}, \quad v_x|_{x=0} = v_x|_{x=2\pi}. \end{aligned} \tag{2.6}$$

Extracting linear in v terms we can obtain the following representation of (2.6):

$$v_t + v = Dv_{xx} + L(\mu)v(t-T) + F(v(t-T), \mu), \tag{2.7}$$

where $L(\mu)$ and $F(\cdot, \mu)$ are operators of the form

$$L(\mu)w = -2\gamma K(\mu) Im[(\gamma + e^{iW(\mu)})Bw], \tag{2.8}$$

$$F(w, \mu) = \gamma K(\mu)\{2Re[(\gamma + e^{iW(\mu)})B(e^{iw} - iw - 1)] + \gamma |B(e^{iw} - 1)|^2\}. \tag{2.9}$$

Lemma 2.3. *Let the assumptions of Lemma 2.2 hold. Then the operator $L(\mu)$ can be expanded in the vicinity of $\mu = 0$ as*

$$L(\mu) = L_0 + \mu L_1 + \mu^2 L_2(\mu), \tag{2.10}$$

where

$$\begin{aligned} &L_0 w = -2\gamma \hat{K} Im[(\gamma + e^{i\hat{W}})Bw], \\ &L_1 w = -2\gamma\{\hat{K} Re[\hat{W}_1 e^{i\hat{W}} Bw] + Im[(\gamma + e^{i\hat{W}})Bw]\}, \\ &\|L_2(\mu)w\|_H \le C_0 \|w\|_H, \quad \forall w \in H^2_{2\pi}. \end{aligned}$$

Proof. Some algebra applied to (2.8) yields

$$\begin{aligned} L(\mu)w &= -2\gamma(\hat{K} + \mu) Im[(\gamma + e^{iW(\mu)})B(w)] \\ &= \{e^{iW(\mu)} = e^{i\hat{W}} + \mu i \hat{W}_1 e^{i\hat{W}} - e^{i\hat{W}} - \mu i \hat{W}_1 e^{i\hat{W}} + e^{iW(\mu)}\} \\ &= -2\gamma \hat{K} Im[(\gamma + e^{i\hat{W}})B(w)] - 2\gamma\mu\{\hat{K} Re[\hat{W}_1 e^{i\hat{W}} B(w)] + Im[(\gamma + e^{i\hat{W}})B(w)]\} \\ &\quad - 2\gamma\mu^2\{Re[\hat{W}_1 e^{i\hat{W}} B(w)] + \hat{K} Im[(-e^{i\hat{W}} - \mu i \hat{W}_1 e^{i\hat{W}} + e^{iW(\mu)})B(w)]\}, \end{aligned}$$

which proves expansion (2.10). To show that $L_2(\mu)$ is bounded it suffices to use inequalities

$$\|L_2(\mu)w\|_H \le C\|Bw\|_H, \quad w \in H^2_{2\pi},$$

where $C = 2\gamma\{|\hat{W}_1| + \hat{K}(2 + \mu_0|\hat{W}_1|)\}$, and recall that B is isometric according to Lemma 2.1. □

Lemma 2.4. *Let the assumptions of Lemma 2.2 hold. Then the operator $F(w, \mu)$ is analytic from a neighborhood of $(0,0) \in H^2_{2\pi} \times \mathbb{R}$ into H. The operator F and its Frechet derivatives $F_{w^n \mu^m}$ vanish at the origin $(0,0)$ for $n < 2$.*

Proof. The formal Taylor series expansion of $F(w, \mu)$ at the origin $(0,0)$ is

$$F(w, \mu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{n} \frac{1}{(n+m)!} F_{w^n \mu^m}(0,0) w^n \mu^m.$$

Expanding e^{iw} in its Taylor series and grouping the terms, one can show that $F_{w^n \mu^m}(0,0) w^n \mu^m = 0$ for $n < 2$. Hence, the expansion can be rewritten as

$$F(w, \mu) = \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} F_{w^n \mu^m}(0,0) w^n \mu^m.$$

To prove the convergence of the Taylor series we derive an upper bound for the following expressions

$$\frac{1}{k!(n-k)!} \|Bw^k Bw^{n-k}\|_{H(\mathbb{C})}, \quad w \in H^2_{2\pi}, \quad n \ge 2, \quad k = 0, 1, \ldots, n.$$

The embedding theorems [10] show that

$$\begin{aligned} &\|fg\|_{H^2(\mathbb{C})} \le C_0 \|f\|_{H^2(\mathbb{C})} \|g\|_{H^2(\mathbb{C})}, \quad \forall f, g \in H^2(\mathbb{C}), \\ &\|w^m\|_{H^2} \le C_0^{m-1} \|w\|^m_{H^2}, \quad \forall w \in H^2. \end{aligned}$$

Therefore, recalling that B is isometric, we can obtain the following bound

$$\|Bw^k Bw^{n-k}\|_{H(\mathbb{C})} \le \|Bw^k Bw^{n-k}\|_{H^2(\mathbb{C})} \le C_0 \|Bw^k\|_{H^2(\mathbb{C})} \|Bw^{n-k}\|_{H^2(\mathbb{C})} = C_0 \|w^k\|_{H^2} \|w^{n-k}\|_{H^2} \le C_0^{n-1} \|w\|^n_{H^2}.$$

This leads to a majorant for the Taylor series term corresponding to w^n:

$$2\gamma(\gamma+1)(\hat{K} + \mu) \frac{C_0^{n-1} \|w\|^n_{H^2}}{n!} + \gamma^2 (\hat{K} + \mu) \sum_{k=0}^{n} \frac{C_0^{n-1} \|w\|^n_{H^2}}{k!(n-k)!} = (\hat{K} + \mu)(2\gamma(\gamma+1) + 2^n \gamma^2) \frac{C_0^{n-1}}{n!} \|w\|^n_{H^2},$$

which means that the Taylor series expansion of $F(w,\mu)$ converges for any $w \in H^2_{2\pi}$ and $|\mu| < \mu_0$ (as $W(\mu)$ is undefined elsewhere). □

Direct computation gives a few Frechet derivatives:

$$F_{ww}(0,0)w^2 = -2\gamma\hat{K}\{Re[(\gamma + e^{i\hat{W}})Bw^2] - \gamma|Bw|^2\},$$
$$F_{www}(0,0)w^3 = 2\gamma\hat{K}\{Im[(\gamma + e^{i\hat{W}})Bw^3] - 3\gamma Im[\overline{Bw}Bw^2]\},$$
$$F_{ww\mu}(0,0)w^2\mu = 2\mu\gamma\{\hat{K}Im[\hat{W}_1 e^{i\hat{W}}Bw^2] - Re[(\gamma + e^{i\hat{W}})Bw^2] + \gamma|Bw|^2\}.$$

Hence, $F(w,\mu)$ can be represented as

$$F(w,\mu) = \frac{1}{2}F_{ww}(0,0)w^2 + \frac{1}{6}F_{www}(0,0)w^3 + \frac{1}{2}F_{ww\mu}(0,0)w^2\mu + F_4(w,\mu), \tag{2.11}$$

where $F_4(w,\mu)$ is analytic and contains terms $w^n\mu^m$, $n \geq 2$, $n+m > 3$.

2.2 Rotating coordinate system

We approach the construction of rotating-wave solutions by passing to a rotating coordinate system. To this end, we introduce a family of rotation operators

$$R_\alpha f(x) = f((x+\alpha) \bmod 2\pi), \quad \alpha \in \mathbb{R}, \quad f \in H(\mathbb{C})$$

and seek the solution $v(x,t)$ in the form

$$v(x,t) = R_{-\Omega t}v(x), \quad \Omega \in \mathbb{R}, \quad v(x) \in H^2_{2\pi}. \tag{2.12}$$

Lemma 2.5.
(1) *Eigenfunctions of the rotation operator R_α are the complex exponents $\{e^{inx}\}_{n\in\mathbb{Z}}$. The corresponding eigenvalues are $\lambda_n(R_\alpha) = e^{in\alpha}$.*
(2) *The operators B and R_α commute.*

Proof. Direct computation proves the statement. □

Ansatz (2.12) transforms boundary value problem (2.6) into

$$\begin{aligned} &Dv'' + \Omega v' - v + L(\mu)R_{\Omega T}v + F(R_{\Omega T}v,\mu) = 0,\\ &v(0) = v(2\pi), \quad v'(0) = v'(2\pi). \end{aligned} \tag{2.13}$$

3. Existence of Rotating Waves

3.1 Linearized operator

To pursue the study of boundary value problem (2.13) we now turn to the linear part of the equation:

$$A_\Omega : H \ni v \mapsto Dv'' + \Omega v' - v + L_0 R_{\Omega T}v \in H, \quad D(A_\Omega) = H^2_{2\pi}. \tag{3.1}$$

The H-adjoint operator of A_Ω is

$$A^*_\Omega : H \ni u \mapsto Du'' - \Omega u' - u + R_{-\Omega T}L^*_0 u \in H, \quad D(A^*_\Omega) = H^2_{2\pi}, \tag{3.2}$$

with L^*_0 defined as

$$L^*_0 u = 2\gamma\hat{K}Im[(\gamma + e^{-i\hat{W}})B^*u].$$

Here, the operator B^* acts according to

$$B^* : E_{z_0}(x) \mapsto E(x,z;E_{z_0})|_{z=0}, \quad E_{z_0} \in H^2_{2\pi}(\mathbb{C}),$$

where $E(x,z;E_{z_0})$ is the solution to a periodic initial-boundary value problem for the adjoint Schrodinger equation

$$\begin{aligned} &E_z - iE_{xx} = 0,\\ &E|_{z=z_0} = E_{z_0}(x),\\ &E|_{x=0} = E|_{x=2\pi}, \quad E_x|_{x=0} = E_x|_{x=2\pi}. \end{aligned}$$

Along with A_Ω we shall consider its complexification:

$$A^{\mathbb{C}}_\Omega u = A_\Omega f + iA_\Omega g, \quad (A^{\mathbb{C}}_\Omega)^* u = (A^*_\Omega)^{\mathbb{C}} u = A^*_\Omega f + iA^*_\Omega g, \quad u = f + ig \in H^2_{2\pi}(\mathbb{C}).$$

Lemma 3.1. *The linear operators $A^{\mathbb{C}}_\Omega$ and $A^{\mathbb{C}*}_\Omega$ possess a complete orthogonal in $H(\mathbb{C})$ system of eigenfunctions $\{e^{inx}\}_{n\in\mathbb{Z}}$. The corresponding eigenvalues are*

$$\begin{aligned}\lambda_n(A_\Omega^{\mathbb{C}}) &= -Dn^2 - 1 + in\Omega - 2\gamma\hat{K}e^{in\Omega T}\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\},\\ \lambda_n(A_\Omega^{\mathbb{C}*}) &= -Dn^2 - 1 - in\Omega - 2\gamma\hat{K}e^{-in\Omega T}\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\}.\end{aligned} \tag{3.3}$$

Proof. According to Lemma 2.1 and Lemma 2.5,

$$B(R_{\Omega T}\cos nx) = e^{in^2 z_0}R_{\Omega T}\cos nx,$$

$$B(R_{\Omega T}\sin nx) = e^{in^2 z_0}R_{\Omega T}\sin nx.$$

Hence,

$$L_0 R_{\Omega T}\cos nx + iL_0 R_{\Omega T}\sin nx = -2\gamma\hat{K}Im[(\gamma + e^{i\hat{W}})e^{in^2 z_0}]e^{in\Omega T}e^{inx}$$

and, thus,

$$A_\Omega^{\mathbb{C}}e^{inx} = (-Dn^2 - 1 + in\Omega - 2\gamma\hat{K}e^{in\Omega T}Im[(\gamma + e^{i\hat{W}})e^{in^2 z_0}])e^{inx}.$$

The case of the adjoint operator is similar. □

Lemma 3.2. *Let the assumptions of Lemma 2.2 hold. Then dimensions of $N(A_\Omega^{\mathbb{C}})$ and $N(A_\Omega^{\mathbb{C}*})$ are even.*

Proof. According to (3.3), if $\lambda_n(A_\Omega^{\mathbb{C}}) = 0$ then $\lambda_{-n}(A_\Omega^{\mathbb{C}}) = 0$. So, only $n = 0$ can make dimensions odd. But condition (2.5) states that

$$Re\lambda_0(A_\Omega^{\mathbb{C}}) = -(1 + 2\gamma\hat{K}\sin\hat{W}) \neq 0.$$

Hence, dimensions are always even. □

We now impose the main constraint on the parameters of the model.

Condition 3.3. Let $\Omega = \Omega_*$ be such that the system of equations

$$\begin{aligned}2\gamma\hat{K}\cos(n\Omega T)\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\} &= -Dn^2 - 1,\\ 2\gamma\hat{K}\sin(n\Omega T)\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\} &= \Omega n\end{aligned}$$

has exactly two solutions $n = \pm n_*$, $n_* \in \mathbb{N}$.

Lemma 3.4. *Let Condition 3.3 hold. Then $N(A_{\Omega_*}^{\mathbb{C}}) = N(A_{\Omega_*}^{\mathbb{C}*}) = \operatorname{span}\{e^{in_*x}, e^{-in_*x}\}$.*

Proof. According to Lemma 3.1, Condition 3.3 means that $\lambda_n(A_{\Omega_*}^{\mathbb{C}}) = 0$ only for $n = \pm n_*$. The statement follows. □

Lemma 3.5. *Let Condition 3.3 hold. Then $R(A_{\Omega_*}) = N(A_{\Omega_*}^*)^\perp$.*

Proof. (1) We start with the $R(A_{\Omega_*}) \subseteq N(A_{\Omega_*}^*)^\perp$ inclusion. Let $f \in R(A_{\Omega_*})$; then a function u exists in $D(A_{\Omega_*})$ such that

$$A_{\Omega_*}u = f, \tag{3.4}$$

and for all $g \in N(A_{\Omega_*}^*)$ the following equality holds:

$$\langle f, g\rangle_H = \langle A_{\Omega_*}u, g\rangle_H = \langle u, A_{\Omega_*}^* g\rangle_H = 0.$$

(2) To prove the inverse inclusion we consider an element $f \in N(A_{\Omega_*}^*)^\perp$. We expand f into its Fourier series

$$f = \sum_{n=-\infty}^{\infty} f_n e^{inx}, \quad f_{-n} = \overline{f}_n, \quad f_{n_*} = f_{-n_*} = 0$$

and seek a solution $u \in D(A_{\Omega_*})$ of (3.4) as a formal series

$$u = \sum_{n=-\infty}^{\infty} u_n e^{inx}. \tag{3.5}$$

From (3.4) we obtain equations

$$u_n\lambda_n(A_{\Omega_*}) = f_n,$$

that are uniquely solvable for $n \neq \pm n_*$:

$$u_n = \frac{f_n}{-Dn^2 - 1 + in\Omega - 2\gamma\hat{K}e^{in\Omega T}\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\}}, \quad u_{-n} = \overline{u}_n.$$

Setting $u_{n_*} = u_{-n_*} = 0$, we check that the element u defined by (3.5) indeed lies in $D(A_{\Omega_*})$. Parseval's identity gives

$$\|u\|_{H^2}^2 = \|u\|_H^2 + \|u''\|_H^2 = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(1+n^4)|u_n|^2 = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}c(n)|f_n|^2 \le C\|f\|_H^2,$$

with

$$c(n) = \frac{1+n^4}{|-Dn^2 - 1 + in\Omega - 2\gamma\hat{K}e^{in\Omega T}\{\gamma\sin(n^2 z_0) + \sin(\hat{W} + n^2 z_0)\}|^2} \le C.$$

Hence, $u \in H^2$. To show periodicity it is sufficient to note that (3.5) and its termwise derivative converge uniformly on $[0, 2\pi]$.

□

Corollary 3.6. $R(A_{\Omega_*}) = N(A_{\Omega_*})^{\perp}$.

Corollary 3.7. $R(A_{\Omega_*})$ *is closed in* H.

Corollary 3.8. *Let* $\mathcal{P} : H_{2\pi}^2 \longrightarrow R(A_{\Omega_*})$ *be the projection operator. Then* $A_{\Omega_*} : \mathcal{P}H_{2\pi}^2 \longrightarrow R(A_{\Omega_*})$ *is continuously invertible.*

3.2 Existence theorem

In Section 2 we introduced two additional parameters that led us to boundary value problem (2.13): the small perturbation μ of the nonlinearity parameter $\hat{K}$ and the rotation speed Ω of the sought rotating-wave solution. These are now genuinely included in the following definition.

Definition 3.9. Let $S = (v, \Omega, \mu) \in H_{2\pi}^2 \times \mathbb{R} \times \mathbb{R}$. We call the triplet S a solution to the boundary value problem (2.13) if S turns (2.13) into identity.

It is evident that $S_0 = (0, \Omega_*, 0)$ is a solution. We will be considering the nontrivial perturbations of S_0.

Condition 3.10. The following inequality holds:

$$-2\gamma n_*^2[\hat{K}\hat{W}_1\cos(\hat{W} + n_*^2 z_0) + \sin(\hat{W} + n_*^2 z_0) + \gamma\sin(n_*^2 z_0)][2\gamma\hat{K}T\{\sin(\hat{W} + n_*^2 z_0) + \gamma\sin(n_*^2 z_0)\} - \cos(n_*\Omega_* T)] \neq 0.$$

Theorem 3.11. *Let the assumptions of Lemma 2.2, Condition 3.3 and Condition 3.10 hold. Then an* $\varepsilon_0 > 0$ *exists such that for* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *a twice continuously differentiable with respect to* ε *branch of solutions*

$$S_\varepsilon = (v(x;\varepsilon), \Omega_* + \omega(\varepsilon), \mu(\varepsilon)) \in H_{2\pi}^2 \times \mathbb{R} \times \mathbb{R}$$

is defined. Moreover, $S_{\varepsilon=0} = S_0$.

Proof. Consider a triplet $S = (v(\varepsilon), \Omega + \omega, \mu)$, where

$$v(\varepsilon) = \varepsilon(\varphi + \xi), \quad \varphi = \frac{1}{\sqrt{\pi}}\cos(n_* x) \in N(A_{\Omega_*}), \quad \xi \in D(A_{\Omega_*}) \cap N(A_{\Omega_*})^{\perp}. \tag{3.6}$$

Plugging S into (2.13) and dividing by $\varepsilon \neq 0$, we can get

$$A_{\Omega_*}\xi + \omega(\varphi' + \xi') + L(\mu)R_{\Omega T}(\varphi + \xi) - L_0 R_{\Omega_* T}(\varphi + \xi) + \tfrac{1}{\varepsilon}F(R_{\Omega T}\varepsilon(\varphi + \xi), \mu) = 0. \tag{3.7}$$

Subspaces $N(A_{\Omega_*}) = \operatorname{span}\{\varphi, \varphi'\}$ and $R(A_{\Omega_*}) = N(A_{\Omega_*})^{\perp}$ are invariant under the operators $d/dx, L(\mu), L_0$ (since φ and φ' are eigenfunctions of B). Therefore, we can project (3.7) onto the kernel and range:

$$\begin{aligned}
&A_{\Omega_*}\xi + \omega\xi' + L(\mu)R_{\Omega T}\xi - L_0 R_{\Omega_* T}\xi + \tfrac{1}{\varepsilon}\mathcal{P}F(R_{\Omega T}v(\varepsilon), \mu) = 0 \in R(A_{\Omega_*}),\\
&\omega\langle\phi', \phi\rangle_H + \langle L(\mu)R_{\Omega T}\phi, \phi\rangle_H - \langle L_0 R_{\Omega_* T}\phi, \phi\rangle_H + \tfrac{1}{\varepsilon}\langle F(R_{\Omega T}v(\varepsilon), \mu), \phi\rangle_H = 0 \in \mathbb{R},\\
&\omega\langle\phi', \phi'\rangle_H + \langle L(\mu)R_{\Omega T}\phi, \phi'\rangle_H - \langle L_0 R_{\Omega_* T}\phi, \phi'\rangle_H + \tfrac{1}{\varepsilon}\langle F(R_{\Omega T}v(\varepsilon), \mu), \phi'\rangle_H = 0 \in \mathbb{R}.
\end{aligned} \tag{3.8}$$

System (3.8) is equivalent to (3.7). It produces a nonlinear operator equation

$$\mathcal{F}(\xi, \omega, \mu, \varepsilon) = 0, \tag{3.9}$$

where the operator

$$\mathcal{F} : \mathcal{P}H_{2\pi}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow R(A_{\Omega_*}) \times \mathbb{R} \times \mathbb{R}$$

is twice continuously Frechet differentiable in the vicinity of $(0, 0, 0, 0)$.

We now seek an implicit function $(\xi, \omega, \mu)(\varepsilon)$ from (3.9). To this end, we compute the Frechet derivative of $\mathcal{F}$ at the origin $(0, 0, 0, 0)$ with respect to (ξ, ω, μ):

$$\nabla F = \begin{pmatrix} A_{\Omega_*} & 0 & 0 \\ 0 & \langle\phi', \phi\rangle + T\langle L_0 R_{\Omega_* T}\phi', \phi\rangle & \langle L_1 R_{\Omega_* T}\phi, \phi\rangle \\ 0 & \langle\phi', \phi'\rangle + T\langle L_0 R_{\Omega_* T}\phi', \phi'\rangle & \langle L_1 R_{\Omega_* T}\phi, \phi'\rangle \end{pmatrix}.$$

According to Corollary 3.8, the linear operator $A_{\Omega_*} : \mathcal{P}H^2_{2\pi} \longrightarrow R(A_{\Omega_*})$ is continuously invertible. Thus, the linear operator

$$\nabla F : \mathcal{P}H^2_{2\pi} \times \mathbb{R} \times \mathbb{R} \longrightarrow (A_{\Omega_*}) \times \mathbb{R} \times \mathbb{R}$$

is continuously invertible whenever the following matrix is nondegenerate:

$$\nabla \mathcal{F}_0 = \begin{pmatrix} \langle \phi', \phi \rangle + T\langle L_0 R_{\Omega_* T}\phi', \phi\rangle & \langle L_1 R_{\Omega_* T}\phi, \phi\rangle \\ \langle \phi', \phi' \rangle + T\langle L_0 R_{\Omega_* T}\phi', \phi'\rangle & \langle L_1 R_{\Omega_* T}\phi, \phi'\rangle \end{pmatrix}.$$

It follows from Condition 3.10 that $\det \nabla\mathcal{F}_0 \neq 0$. The statement of the theorem is then a consequence of the twice continuously Frechet differentiable implicit operator theorem [11]. □

3.3 Expansion coefficients

In the proof of Theorem 3.11 we constructed a twice continuously differentiable triplet $(\xi, \omega, \mu)(\varepsilon)$. It can be expanded in powers of ε as follows:

$$\begin{aligned} \xi(\varepsilon) &= \varepsilon\xi_1 + \varepsilon^2\xi_2 + \overline{o}(\varepsilon^2) \in \mathcal{P}H^2_{2\pi}, \\ \omega(\varepsilon) &= \varepsilon\omega_1 + \varepsilon^2\omega_2 + \overline{o}(\varepsilon^2) \in \mathbb{R}, \\ \mu(\varepsilon) &= \varepsilon\mu_1 + \varepsilon^2\mu_2 + \overline{o}(\varepsilon^2) \in \mathbb{R}. \end{aligned} \tag{3.10}$$

We then plug expansion (3.10) into Eq. (3.7) to get the identity:

$$\begin{aligned} &A_{\Omega_*}\xi(\varepsilon) + \omega(\varepsilon)\frac{d}{dx}(\phi + \xi(\varepsilon)) + L_0[R_{\Omega_* T+\omega(\varepsilon)T}(\phi + \xi(\varepsilon)) - R_{\Omega_* T}(\phi + \xi(\varepsilon))] + \mu(\varepsilon)L_1 R_{\Omega_* T+\omega(\varepsilon)T}(\phi + \xi(\varepsilon)) \\ &\quad + \mu^2(\varepsilon)L_2(\mu(\varepsilon))R_{\Omega_* T+\omega(\varepsilon)T}(\phi + \xi(\varepsilon)) + \frac{1}{\varepsilon}F(R_{\Omega_* T+\omega(\varepsilon)T}\varepsilon(\phi + \xi(\varepsilon)), \mu(\varepsilon)) \equiv 0. \end{aligned} \tag{3.11}$$

From (3.10) we can deduce that

$$R_{\Omega_* T+\omega(\varepsilon)T}y = R_{\Omega_* T}y + \omega(\varepsilon)TR_{\Omega_* T}y' + \overline{o}(w(\varepsilon)T) = R_{\Omega_* T}y + \varepsilon\omega_1 TR_{\Omega_* T}y' + \overline{o}(\varepsilon). \tag{3.12}$$

The Taylor series (2.11) of the operator $F(w, \mu)$ together with (3.12) allow us to expand the nonlinear term of (3.11) in powers of ε:

$$F(R_{\Omega_* T+\omega(\varepsilon)T}\varepsilon(\phi + \xi(\varepsilon)), \mu(\varepsilon)) = \varepsilon^2 F_2 + \varepsilon^3 F_3 + \varepsilon^4 F_4(\varepsilon), \tag{3.13}$$

where

$$\begin{aligned} &F_2 = \frac{1}{2}F_{ww}(0,0)R_{\Omega_* T}[\phi^2], \\ &F_3 = F_{ww}(0,0)R_{\Omega_* T}[\phi\xi_1] + \omega_1 TF_{ww}(0,0)R_{\Omega_* T}[\phi\phi'] + \frac{1}{6}F_{w^3}(0,0)R_{\Omega_* T}[\phi^3] + \frac{1}{2}F_{w^2\mu}(0,0)R_{\Omega_* T}[\phi^2\mu_1], \\ &\|F_4(\varepsilon)\|_H \leq C(\Sigma), \quad \varepsilon \in \Sigma, \quad \forall\Sigma \Subset (-\varepsilon_0, \varepsilon_0). \end{aligned}$$

We can now proceed with computing the first-order expansion coefficients ξ_1, ω_1, and μ_1. To this end, we collect the linear in ε terms in (3.11):

$$A_{\Omega_*}\xi_1 = -[\omega_1(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_1 L_1 R_{\Omega_* T}\phi + F_2]. \tag{3.14}$$

By virtue of Corollary 3.6, Eq. (3.14) has a solution if and only if the right hand side is orthogonal to $N(A^*_{\Omega_*})$; if a solution $\xi_1 \in \mathcal{P}H^2_{2\pi}$ exists, it is unique.

By definition, $F_2 \in \mathrm{span}\{1, cos(2n_*x), sin(2n_*x)\}$ and is orthogonal to $N(A^*_{\Omega_*})$ as a consequence. We, thus, get the following solvability condition:

$$\begin{aligned} \langle \omega_1(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_1 L_1 R_{\Omega_* T}\phi, \phi\rangle_H = 0, \\ \langle \omega_1(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_1 L_1 R_{\Omega_* T}\phi, \phi'\rangle_H = 0. \end{aligned}$$

This is a system of two linear equations; its matrix coincides with $\nabla\mathcal{F}_0$ from Theorem 3.11, which is nondegenerate according to Condition 3.10. Therefore, Eq. (3.14) has a solution if and only if

$$\omega_1 = 0, \quad \mu_1 = 0. \tag{3.15}$$

The solution $\xi_1 \in \mathcal{P}H^2_{2\pi}$ can be sought in the form

$$\xi_1 = c_0 + c_c\cos(2n_*x) + c_s\sin(2n_*x).$$

This leads to the following expression:

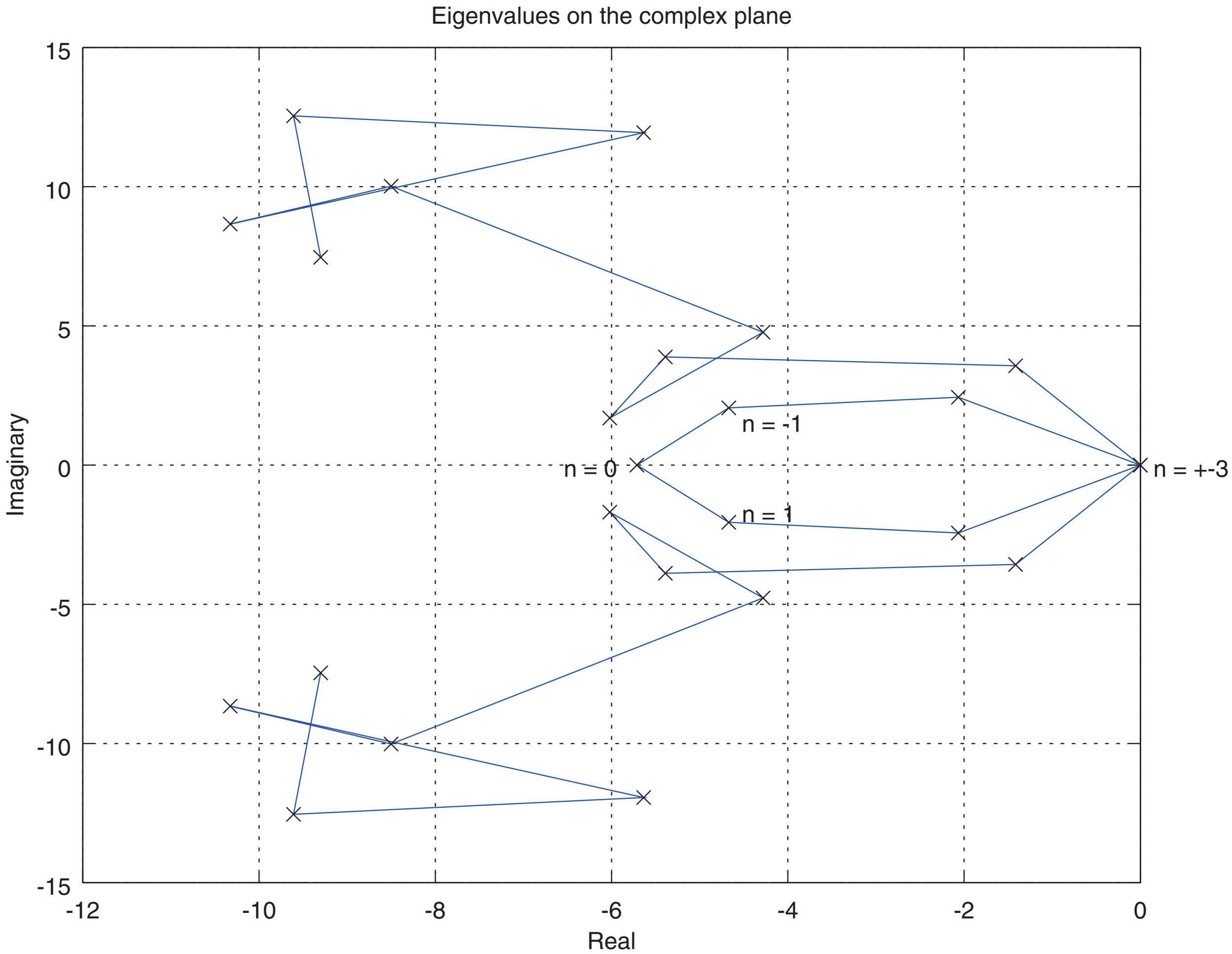

Fig. 1. Eigenvalues $\lambda_n(A_{\Omega_*})$ plotted for $n \in [-12, 12]$. The parameters are $n_* = 3$, $\Omega_* = 0.96191$, $\hat{W} = 2.3562$, $\hat{K} = 4.7124$, $\gamma = 0.70711$, $D = 0.074009$, $T = 0.72577$, and $z_0 = 0.087266$.

$$\xi_1 = \frac{a_0}{\lambda_0(A_{\Omega_*})} + Re\left[\frac{a_2}{\lambda_{2n_*}(A_{\Omega_*})} R_{\Omega_* T} e^{i2n_* x}\right],$$

which is well defined due to Condition 3.3. Here,

$$a_0 = \frac{1}{2\pi}\gamma\hat{K}\cos\hat{W}, \quad a_2 = \frac{1}{2\pi}\gamma\hat{K}[\gamma(\cos(4n_*^2 z_0) - 1) + \cos(\hat{W} + 4n_*^2 z_0)].$$

To find second-order coefficients, we shall update expansion (3.12) of $R_{\Omega T}$ using (3.15):

$$R_{\Omega_* T + \omega(\varepsilon)T} y = R_{\Omega_* T} y + \varepsilon^2 \omega_2 T R_{\Omega_* T} y' + \overline{o}(\varepsilon^2).$$

We can now collect the ε^2 terms in (3.11):

$$A_{\Omega_*}\xi_2 = -[\omega_2(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_2 L_1 R_{\Omega_* T}\phi + F_3]. \tag{3.16}$$

Equation (3.16) has a solution if and only if ω_2 and μ_2 satisfy the following system of linear equations:

$$\langle \omega_2(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_2 L_1 R_{\Omega_* T}\phi, \phi\rangle_H = -\langle F_3, \phi\rangle_H,$$

$$\langle \omega_2(\phi' + TL_0 R_{\Omega_* T}\phi') + \mu_2 L_1 R_{\Omega_* T}\phi, \phi'\rangle_H = -\langle F_3, \phi'\rangle_H.$$

The matrix $\nabla\mathcal{F}_0$ is nondegenerate, so ω_2 and μ_2 can be found. To find the unique ξ_2, one can solve

$$A_{\Omega_*}\xi_2 = -\mathcal{P}F_3.$$

4. Examples of Parameters

Below we present a few examples of parameters that satisfy the conditions of Lemma 2.2, Condition 3.3, and Condition 3.10.

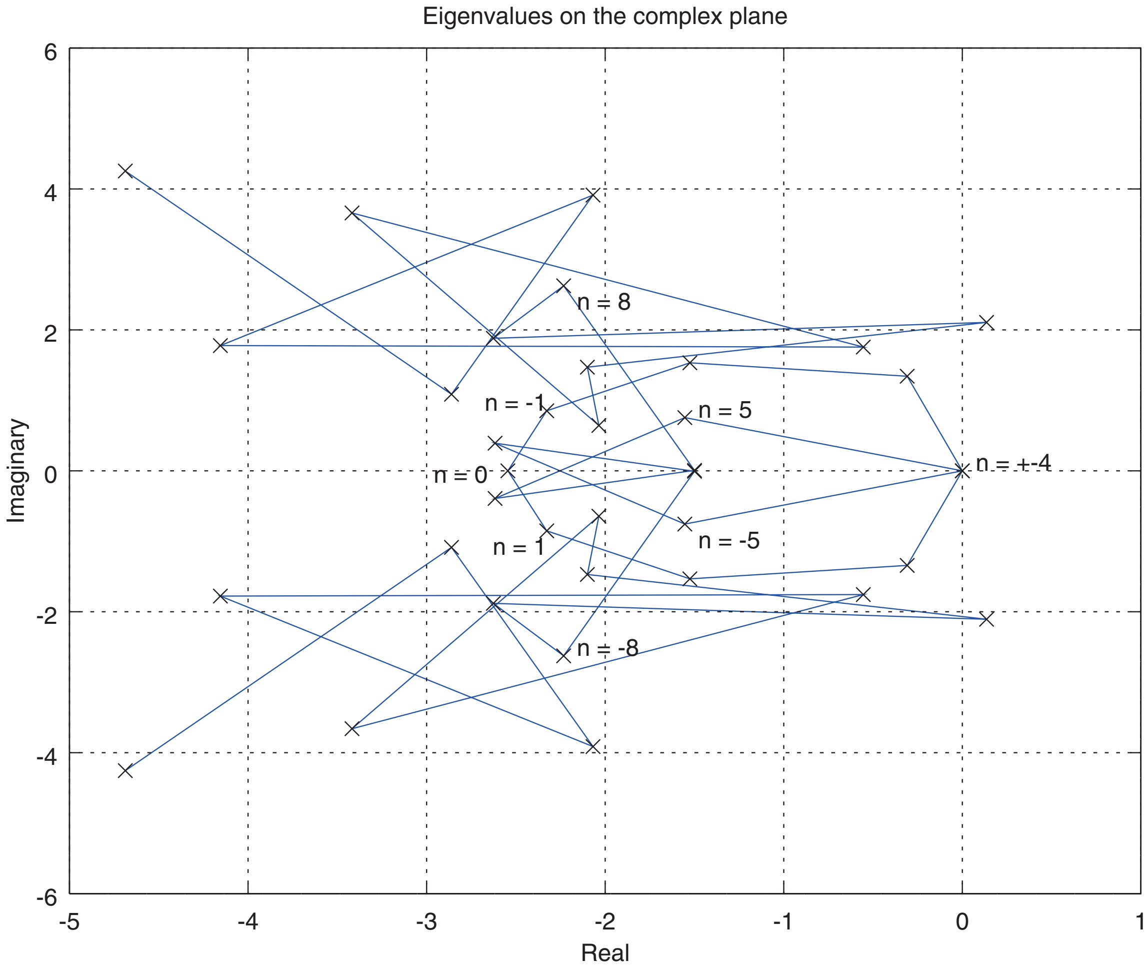

Fig. 2. Eigenvalues $\lambda_n(A_{\Omega_*})$ plotted for $n \in [-18, 18]$. The parameters are $n_* = 4$, $\Omega_* = 0.16094$, $\hat{W} = 1.5708$, $\hat{K} = 0.92703$, $\gamma = 0.83333$, $D = 0.007190$, $T = 4.0667$, and $z_0 = 0.098175$.

- To show that $\pm n_*$ is the unique pair of roots in Condition 3.3 it is sufficient to check only for $|n| \le 2\gamma\hat{K}(\gamma + 1.1)/\Omega_*$, since for bigger values we have $|Im\lambda_n| > 0.2\gamma\hat{K}$.
- The expression in Condition 3.10 involves $\hat{W}_1$. It can be computed as $\hat{W}_1 = \frac{1+2\gamma\cos\hat{W}+\gamma^2}{1+2\hat{K}\gamma\sin\hat{W}}$

4.1 Example 1

n_*	Ω_*	$\hat{W}$	$\hat{K}$	γ	D	T	z_0
3	0.96191	2.3562	4.7124	0.70711	0.074009	0.72577	0.087266

- Expression in condition of Lemma 2.2 equals to $5.721 \neq 0$.
- Sufficient to check for $|n| \le 2\gamma\hat{K}(\gamma + 1.1)/\Omega_* = 11.896$. Figure 1 illustrates that Condition 3.3 is satisfied.
- Expression in Condition 3.10 equals to $-3.2513 \neq 0$.

4.2 Example 2

n_*	Ω_*	$\hat{W}$	$\hat{K}$	γ	D	T	z_0
4	0.16094	1.5708	0.92703	0.83333	0.007190	4.0667	0.098175

- Expression in condition of Lemma 2.2 equals to $2.545 \neq 0$.
- Sufficient to check for $|n| \le 2\gamma\hat{K}(\gamma + 1.1)/\Omega_* = 17.696$. Figure 2 illustrates that Condition 3.3 is satisfied.
- Expression in Condition 3.10 equals to $-35.170 \neq 0$.

4.3 Example 3

n_*	Ω_*	$\hat{W}$	$\hat{K}$	γ	D	T	z_0
1	5.2477	7.864	6.882	0.3878	0.009241	0.335536	0.003142

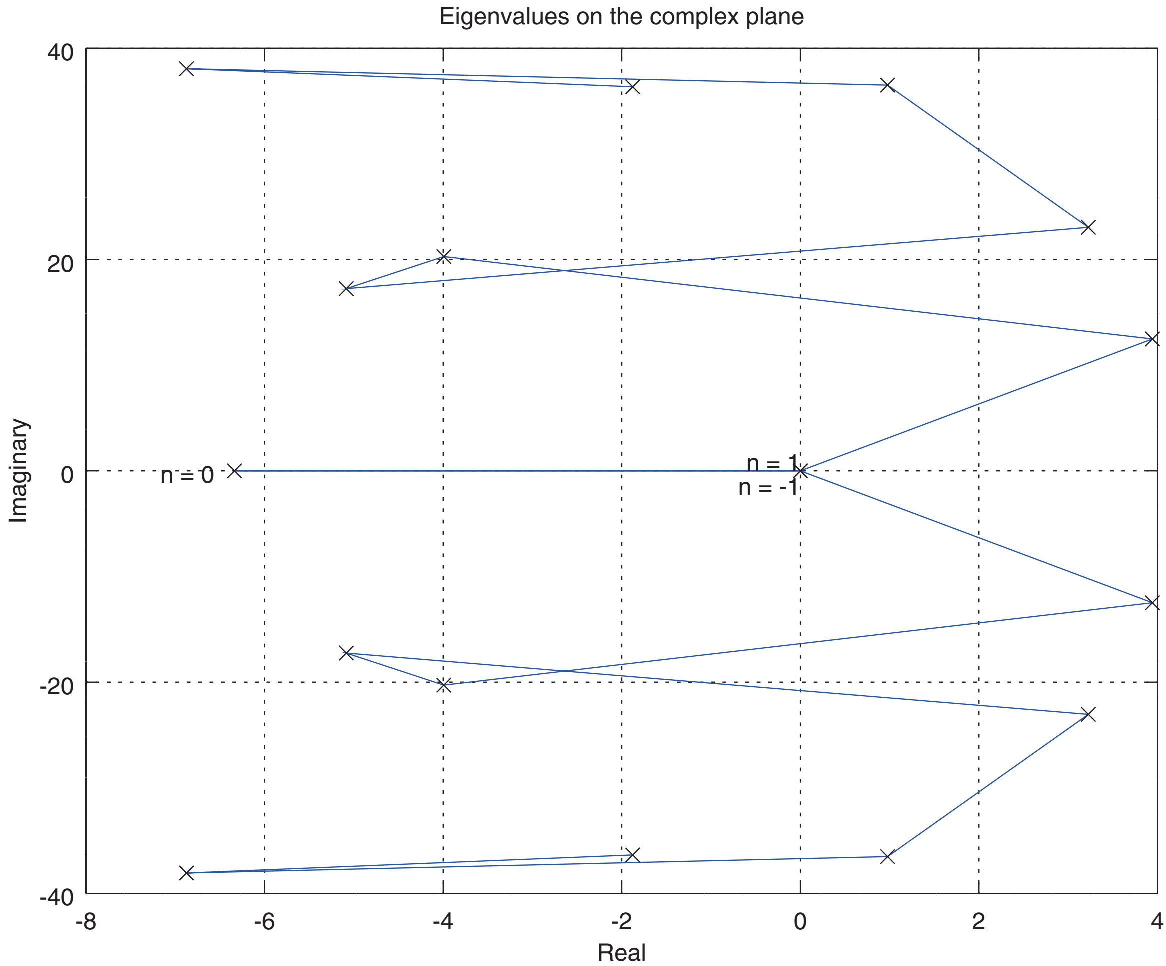

Fig. 3. Eigenvalues $\lambda_n(A_{\Omega_*})$ plotted for $n \in [-8, 8]$. The parameters are $n_* = 1$, $\Omega_* = 5.2477$, $\hat{W} = 7.864$, $\hat{K} = 6.882$, $\gamma = 0.3878$, $D = 0.009241$, $T = 0.335536$, and $z_0 = 0.003142$.

- Expression in condition of Lemma 2.2 equals to $6.337 \neq 0$.
- Sufficient to check for $|n| \leq 2\gamma\hat{K}(\gamma + 1.1)/\Omega_* = 1.422$. Figure 3 illustrates that Condition 3.3 is satisfied.
- Expression in Condition 3.10 equals to $-1.5138 \neq 0$.

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