

Periodicity for the Hadamard Walk on Cycles

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The present paper treats the period T_N of the Hadamard walk on a cycle C_N with N vertices. Dukes (2014) considered the periodicity of more general quantum walks on C_N and showed $T_2 = 2$, $T_4 = 8$, $T_8 = 24$ for the Hadamard walk case. We prove that the Hadamard walk does not have any period except for his case, i.e., $N = 2, 4, 8$. Our method is based on a path counting and cyclotomic polynomials which is different from his approach based on the property of eigenvalues for unitary matrix that determines the evolution of the walk.

KEYWORDS: Quantum walk, Hadamard walk, periodicity, cycle

1. Introduction

The quantum walk (QW) is a quantum counterpart of the classical random walk. QWs have been widely investigated for the last decade, mainly in connection with quantum information science. The reviews and books on QWs are, for example, Kempe [1], Kendon [2], Venegas-Andraca [3, 4], Konno [5], Cantero *et al.* [6], Manouchehri and Wang [7], Portugal [8]. The properties of QWs on graphs including cycles were studied by Aharonov *et al.* [9]. In this paper, we consider two-state QWs on a cycle C_N with N vertices, where $C_N = \{0, 1, \dots, N - 1\}$. In particular, we focus on the periodicity of the Hadamard walk on C_N .

From now on we present a brief definition of the general two-state QWs on C_N , which includes the Hadamard walk as a special case. The discrete-time QW is a quantum version of the classical random walk with additional degree of freedom called chirality. The chirality takes values left and right, and it means the direction of the motion of the walker. At each time step, if the walker has the left chirality, it moves one step to the left, and if it has the right chirality, it moves one step to the right. Let us define

$$|L\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |R\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where L and R refer to the left and right chirality states, respectively.

The time evolution of the walk is determined by $U \in U(2)$, where $U(n)$ be the set of $n \times n$ unitary matrices and

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To define the dynamics of our model, we divide U into two matrices:

$$P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix},$$

with $U = P + Q$. The important point is that P (resp. Q) represents that the walker moves to the left (resp. right) at any position at each time step.

The QW considered here is

$$U = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (1.1)$$

This model is called *the Hadamard walk* which has been extensively and deeply investigated in the study of QWs.

Let Ψ_n denote the amplitude at time n of the QW on C_N :

$$\begin{aligned}
\Psi_n &= {}^T[\Psi_n^L(0), \Psi_n^R(0), \Psi_n^L(1), \Psi_n^R(1), \dots, \Psi_n^L(N-1), \Psi_n^R(N-1)], \\
&= {}^T[\Psi_n(0), \Psi_n(1), \dots, \Psi_n(N-1)], \\
&= {}^T\left[\begin{bmatrix} \Psi_n^L(0) \\ \Psi_n^R(0) \end{bmatrix}, \begin{bmatrix} \Psi_n^L(1) \\ \Psi_n^R(1) \end{bmatrix}, \dots, \begin{bmatrix} \Psi_n^L(N-1) \\ \Psi_n^R(N-1) \end{bmatrix}\right] \in (\mathbb{C}^2)^N,
\end{aligned}$$

where \mathbb{C} denote the set of complex numbers, T means the transposed operation, and $\Psi_n(x) = {}^T[\Psi_n^L(x), \Psi_n^R(x)]$ ($x \in C_N$) is the amplitude at time n and position x .

Now we introduce the following $2N \times 2N$ unitary matrix:

$$U_N^{(s)} = \begin{bmatrix} O & P & O & O & \cdots & O & Q \\ Q & O & P & O & \cdots & O & O \\ O & Q & O & P & \cdots & O & O \\ O & O & Q & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \cdots & O & P \\ P & O & O & O & \cdots & Q & O \end{bmatrix} \quad \text{with } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $N = 2$, following Dukes [10], we put

$$U_2^{(s)} = \begin{bmatrix} O & U \\ U & O \end{bmatrix}.$$

Then the state of the QW at time n is given by

$$\Psi_n = (U_N^{(s)})^n \Psi_0, \tag{1.2}$$

for any $n \geq 0$.

Put $\mathbb{R}_+ = [0, \infty)$. Here we introduce a map $\phi : (\mathbb{C}^2)^N \rightarrow \mathbb{R}_+^N$ such that if

$$\Psi = {}^T\left[\begin{bmatrix} \Psi^L(0) \\ \Psi^R(0) \end{bmatrix}, \begin{bmatrix} \Psi^L(1) \\ \Psi^R(1) \end{bmatrix}, \dots, \begin{bmatrix} \Psi^L(N-1) \\ \Psi^R(N-1) \end{bmatrix}\right] \in (\mathbb{C}^2)^N,$$

then

$$\phi(\Psi) = {}^T[|\Psi^L(0)|^2 + |\Psi^R(0)|^2, |\Psi^L(1)|^2 + |\Psi^R(1)|^2, \dots, |\Psi^L(N-1)|^2 + |\Psi^R(N-1)|^2] \in \mathbb{R}_+^N.$$

That is, for any $x \in C_N$,

$$\phi(\Psi)(x) = |\Psi^L(x)|^2 + |\Psi^R(x)|^2.$$

Sometimes we identify $\phi(\Psi(x))$ with $\phi(\Psi)(x)$. Moreover we define the measure of the QW at position x by

$$\mu(x) = \phi(\Psi(x)) \quad (x \in C_N).$$

The probability that quantum walker at time n , $X_n = X_n^\varphi$, starting from 0 exists at position $x \in \mathbb{Z}$ is defined by

$$P(X_n = x) = P(X_n^\varphi = x) = \phi((U^{(s)})^n \Psi_0^\varphi)(x).$$

Here the initial state Ψ_0^φ is given by

$$\Psi_0^\varphi = {}^T\left[\begin{bmatrix} \Psi_0^L(0) \\ \Psi_0^R(0) \end{bmatrix}, \begin{bmatrix} \Psi_0^L(1) \\ \Psi_0^R(1) \end{bmatrix}, \dots, \begin{bmatrix} \Psi_0^L(N-1) \\ \Psi_0^R(N-1) \end{bmatrix}\right] = {}^T\left[\varphi, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right],$$

where $\varphi = {}^T[\alpha, \beta] \in \mathbb{C}^2$ with $|\alpha|^2 + |\beta|^2 = 1$.

We put

$$\mathcal{N} = \{n \geq 1 : (U_N^{(s)})^n = I_{2N}\}.$$

If $\mathcal{N} \neq \emptyset$, the period $T_N (< \infty)$ is defined by $T_N = \min \mathcal{N}$. If $\mathcal{N} = \emptyset$, then we say that the QW does not have any period and write $T_N = \infty$.

Let eigenvalues of $U_N^{(s)}$ be $\{\lambda_k : k = 0, 1, \dots, 2N-1\}$. Remark that $(U_N^{(s)})^n = I_{2N}$ if and only if $\lambda_k^n = 1$ ($k = 0, 1, \dots, 2N-1$).

Dukes [10] studied periodicity of a class of two-state QWs on C_N by using the property of eigenvalues λ_k ($k = 0, 1, \dots, 2N-1$) of $U_N^{(s)}$: if period T_N is finite, then $\lambda_j^{T_N} = 1$ for any j . As for the Hadamard walk case, he showed $T_2 = 2$, $T_3 > 30$, $T_4 = 8$, $T_8 = 24$. So we prove that $T_N = \infty$ except for $N = 2, 4, 8$ (Theorem 2.6).

The rest of this paper is organized as follows. In Sect. 2, we present results on our model. Sections 3 and 4 are devoted to proofs of Lemma 2.4 and Theorem 2.6, respectively. In Sect. 5, we summarize our result and give a future problem.

2. Results

This section gives our results. We begin with $N = 3$ case. Then

$$U_3^{(s)} = \begin{bmatrix} O & P & Q \\ Q & O & P \\ P & Q & O \end{bmatrix}.$$

So we have

$$(U_3^{(s)})^2 = \begin{bmatrix} PQ + QP & Q^2 & P^2 \\ P^2 & PQ + QP & Q^2 \\ Q^2 & P^2 & PQ + QP \end{bmatrix},$$

$$(U_3^{(s)})^3 = \begin{bmatrix} P^3 + Q^3 & PQP + QP^2 + P^2Q & PQ^2 + QPQ + Q^2P \\ PQ^2 + QPQ + Q^2P & P^3 + Q^3 & PQP + QP^2 + P^2Q \\ PQP + QP^2 + P^2Q & PQ^2 + QPQ + Q^2P & P^3 + Q^3 \end{bmatrix}.$$

Let $A(k, l)$ ($1 \leq k, l \leq m$) denote the (k, l) component of a $2m \times 2m$ matrix A . For example, $(U_3^{(s)})^3(1, 2) = PQP + QP^2 + P^2Q$.

In order to compute $(U_N^{(s)})^n(k, l)$, we use nice relations: $P^2 = aP$, $Q^2 = dQ$. Moreover we introduce the following 2×2 matrices, R and S :

$$R = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}.$$

Then we obtain the next table of products of matrices, P , Q , R , and S :

| | P | Q | R | S |
|-----|------|------|------|------|
| P | aP | bR | aR | bP |
| Q | cS | dQ | cQ | dS |
| R | cP | dR | cR | dP |
| S | aS | bQ | aQ | bS |

where $PQ = bR$, for example. In particular, for the Hadamard walk case, we have

| | P | Q | R | S |
|-----|--------------|---------------|--------------|---------------|
| P | $P/\sqrt{2}$ | $R/\sqrt{2}$ | $R/\sqrt{2}$ | $P/\sqrt{2}$ |
| Q | $S/\sqrt{2}$ | $-Q/\sqrt{2}$ | $Q/\sqrt{2}$ | $-S/\sqrt{2}$ |
| R | $P/\sqrt{2}$ | $-R/\sqrt{2}$ | $R/\sqrt{2}$ | $-P/\sqrt{2}$ |
| S | $S/\sqrt{2}$ | $Q/\sqrt{2}$ | $Q/\sqrt{2}$ | $S/\sqrt{2}$ |

This path counting method was introduced and intensively studied by [11, 12]. Using this relation, for example, we compute

$$(U_3^{(s)})^3(1, 2) = PQP + QP^2 + P^2Q = bcP + abR + acS = \left(\frac{1}{\sqrt{2}}\right)^2 (P + R + S).$$

Similarly we have

$$(U_3^{(s)})^4(1, 2) = Q^3P + P^4 + PQ^3 + QPQ^2 + Q^2PQ$$

$$= a^3P + bd^2R + (cd^2 + bcd + bcd)S = \left(\frac{1}{\sqrt{2}}\right)^3 \{P + R + (1 - 1 - 1)S\}.$$

Moreover we write the number of paths which contribute to $(U_N^{(s)})^n(k, l)$ by $w(N, n; (k, l))$, e.g.,

$$w(3, 2; (1, 2)) = 1, w(3, 3; (1, 2)) = 3, w(3, 4; (1, 2)) = 5.$$

It is easy to see the following property of the Hadamard walk on cycles.

Lemma 2.1. *For any $N \geq 2$, $n \geq 1$, $w(N, n; (k, k))$ is even for $k \in \{1, 2, \dots, N\}$ and $w(N, n; (1, l)) = w(N, n; (1, N - (l - 2)))$ for $l \in \{2, 3, \dots, [(N/2) + 1]\}$, where $[x]$ is the integer part of real number x .*

We should note that $P, Q, R,$ and S form an orthogonal basis of the vector space of 2×2 matrices with respect to the trace inner product $\langle A|B \rangle = \text{tr}(A^*B)$. Thus if there exist $c_p, c_q, c_r, c_s \in \mathbb{C}$ such that

$$c_p P + c_q Q + c_r R + c_s S = O_2,$$

then $c_p = c_q = c_r = c_s = 0$. It implies that for any $N \geq 2, n \geq 1$ and $k \in \{1, 2, \dots, N\}$, if $w(N, n; (k, l))$ is odd, then $(U_N^{(s)})^n(k, l) \neq O_2$. Combining this property with Lemma 2.1, we obtain the following lemma which is one of the key results of our method.

Lemma 2.2. *If there exists $n \geq 1$ such that $(U_N^{(s)})^n = I_{2N}$, then $w(N, n; (k, l))$ is even for any $k, l \in \{1, 2, \dots, N\}$.*

To count the number of paths, we introduce the adjacency matrix A_N of C_N :

$$A_N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For example, in $N = 3$ case, we have

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (A_3)^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad (A_3)^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}.$$

In general, $(A_N)^n(k, l)$ is the number of paths from k to l with length n .

Moreover we introduce another $N \times N$ matrix $B_N^{(n)}$ whose component $B_N^{(n)}(k, l)$ is equal to $(A_N)^n(k, l) \pmod{2}$. For $N = 3$ case, we get

$$B_3^{(1)} = B_3^{(2)} = B_3^{(3)} = \cdots = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

By using notation $B_N^{(n)}(k, l)$, Lemma 2.2 can be rewritten as

Lemma 2.3. *If there exists $n \geq 1$ such that $(U_N^{(s)})^n = I_{2N}$, then $B_N^{(n)}(k, l) = 0$ for any $k, l \in \{1, 2, \dots, N\}$.*

On the other hand, we have the following result.

Lemma 2.4. *For any odd number $N(\geq 3)$, we have $B_N^{(n)}(k, l) = 1$ for some distinct $k, l \in \{1, 2, \dots, N\}$.*

The proof will appear in Sect. 3. Combining Lemma 2.3 with Lemma 2.4 immediately gives

Proposition 2.5. *For any odd number $N(\geq 3)$, we have $T_N = \infty$.*

By using Proposition 2.5 and the property of cyclotomic polynomials, we obtain the following main result.

Theorem 2.6. *For any N except for $N = 2, 4, 8$, we have $T_N = \infty$.*

The proof will appear in Sect. 4. We should remark that Higuchi *et al.* [13] investigated the periodicity of the Szegedy walk on graphs, e.g., the complete graphs, by using a method based on the property of cyclotomic polynomials. On the other hand, we consider the periodicity of the Hadamard walk on cycles by using not only cyclotomic polynomials but also the path counting for the walk.

Combining Dukes' result, $T_2 = 2, T_4 = 8, T_8 = 24$, with our Theorem 2.6 gives immediately

Theorem 2.7. *For any $N \geq 2$,*

$$T_N = \begin{cases} 2, & (N = 2), \\ 8, & (N = 4), \\ 24, & (N = 8), \\ \infty, & (N \neq 2, 4, 8). \end{cases}$$

We should note that for the classical random walk in which the walker moves one step to the left with probability p and to the right with probability q with $p + q = 1 (p, q \in [0, 1])$, the eigenvalues $\{\lambda_k : k = 0, 1, \dots, N - 1\}$ of the corresponding transition matrix are given by

$$\lambda_k = \cos\left(\frac{2k\pi}{N}\right) + i(q-p)\sin\left(\frac{2k\pi}{N}\right) \quad (k = 0, 1, \dots, N-1).$$

Therefore we see that for any $N \geq 3$,

$$T_N = \begin{cases} N, & (p = 0, 1), \\ \infty, & (p \in (0, 1)), \end{cases}$$

since $\lambda_k = e^{2\pi ik/N}$ ($p = 0$), $e^{-2\pi ik/N}$ ($p = 1$), and $|\lambda_1| < 1$ ($0 < p < 1$). Note that $T_2 = 2$ for all $p \in [0, 1]$.

From now on we briefly review previous results on the Hadamard walk on C_N . To do so, we define the time-averaged measure $\bar{\mu}_n$ at time n and the limit measure $\bar{\mu}_\infty$ for the Hadamard walk on C_N by

$$\bar{\mu}_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} P(X_k = x),$$

$$\bar{\mu}_\infty(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P(X_k = x)$$

for any $x \in C_N$. Aharonov *et al.* [9] proved that the time-averaged limit measure $\bar{\mu}_\infty$ is uniform for odd N , that is, $\bar{\mu}_\infty(x) = 1/N$ ($x \in C_N$), independent of the initial state.

Bednarska *et al.* [14] considered the Hadamard walk on C_N with even N . They obtained the eigenvalues and eigenvectors of $U_N^{(s)}$ and gave an explicit formula of $\bar{\mu}_\infty$ starting from a single vertex for any N . By using the formula, they showed that $\bar{\mu}_\infty$ is uniform for $N = 2, 4$. Moreover they found that $\bar{\mu}_\infty$ is very sensitive to the arithmetic properties of N .

Bednarska *et al.* [15] reported examples for three different kinds of behaviour of the total variation distance between a uniform measure and the time-averaged measure $\bar{\mu}_n$ for the Hadamard walk on C_N with even N .

From Theorem 2.7, we have

Corollary 2.8. For $N = 2, 4, 8$,

$$\bar{\mu}_\infty = \frac{1}{T_N} \sum_{n=0}^{T_N-1} \mu_n.$$

3. Proof of Lemma 2.4

Before we move to the proof, we consider $N = 5$ case. In this case, $B_5^{(1)} (= A_5)$ is given by

$$B_5^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then we would like to show that for any $n \geq 1$, there exists (k, l) with $k \neq l$ such that $B_5^{(n)}(k, l) = 1$ by induction.

When $n = 1$, we see immediately $B_5^{(1)}(1, 2) = 1$.

Next we consider $n = m$ case. By Lemma 2.1, we put

$$[B_5^{(m)}(1, 1), B_5^{(m)}(1, 2), B_5^{(m)}(1, 3), B_5^{(m)}(1, 4), B_5^{(m)}(1, 5)] = [0, c_1, c_2, c_2, c_1]. \quad (3.1)$$

We assume that when $n = m$, the statement holds. That is, $(c_1, c_2) \neq (0, 0)$. We consider $n = m + 1$. We assume that

$$[B_5^{(m+1)}(1, 1), B_5^{(m+1)}(1, 2), B_5^{(m+1)}(1, 3), B_5^{(m+1)}(1, 4), B_5^{(m+1)}(1, 5)] = [0, 0, 0, 0, 0]. \quad (3.2)$$

Then Eq. (3.1) implies

$$[B_5^{(m+1)}(1, 1), B_5^{(m+1)}(1, 2), B_5^{(m+1)}(1, 3), B_5^{(m+1)}(1, 4), B_5^{(m+1)}(1, 5)] = [0, c_2, c_1 + c_2, c_1 + c_2, c_2]. \quad (3.3)$$

Combining Eq. (3.2) with Eq. (3.3) gives

$$c_2 = c_1 + c_2 = 0.$$

Thus we have $c_1 = c_2 = 0$. This contradicts the assumption for $n = m$, i.e., $(c_1, c_2) \neq (0, 0)$. Therefore we see that there exists $(1, l)$ with $l \in \{2, 3, 4, 5\}$ such that $B_5^{(m+1)}(1, l) = 1$. So Lemma 2.4 is valid for $N = 5$.

We can extend this argument to general odd number $N = 2M + 1$ as follows.

When $n = 1$, we easily see $B_N^{(1)}(1, 2) = 1$. Next we consider $n = m$. In a similar way, we put

$$[B_N^{(m)}(1, 1), B_N^{(m)}(1, 2), \dots, B_N^{(m)}(1, N)] = [0, c_1, c_2, \dots, c_{M-1}, c_M, c_M, c_{M-1}, \dots, c_2, c_1]. \quad (3.4)$$

We assume that when $n = m$, the statement holds. That is, $(c_1, c_2, \dots, c_M) \neq (0, 0, \dots, 0)$. We consider $n = m + 1$. We assume that

$$[B_N^{(m+1)}(1, 1), B_N^{(m+1)}(1, 2), \dots, B_N^{(m+1)}(1, N)] = [0, 0, \dots, 0]. \quad (3.5)$$

Then Eq. (3.4) gives

$$\begin{aligned} & [B_N^{(m+1)}(1, 1), B_N^{(m+1)}(1, 2), \dots, B_N^{(m+1)}(1, N)] \\ &= [0, c_2, c_1 + c_3, c_2 + c_4, \dots, c_{M-2} + c_M, c_{M-1} + c_M, c_{M-1} + c_M, c_{M-2} + c_M, \dots, c_2 + c_4, c_1 + c_3, c_2]. \end{aligned} \quad (3.6)$$

From Eq. (3.5) and Eq. (3.6), we obtain

$$c_2 = c_1 + c_3 = c_2 + c_4 = \dots = c_{M-2} + c_M = c_{M-1} + c_M = 0.$$

Thus we have $c_1 = c_2 = \dots = c_M = 0$. This contradicts the assumption for $n = m$, i.e., $(c_1, c_2, \dots, c_M) \neq (0, 0, \dots, 0)$. Therefore we see that there exists $(1, l)$ with $l \in \{2, \dots, N\}$ such that $B_N^{(m+1)}(1, l) = 1$, and the proof is completed.

4. Proof of Theorem 2.6

First we introduce cyclotomic polynomials: $F_1(\lambda) = \lambda - 1$, and for $n \geq 2$,

$$F_n(\lambda) = \prod_{\substack{1 \leq k \leq n-1: \\ \gcd(k, n)=1}} \left(\lambda - \exp\left(\frac{2\pi i k}{n}\right) \right),$$

where $\gcd(n_1, n_2, \dots, n_k)$ denotes the greatest common divisor of (n_1, n_2, \dots, n_k) .

Before we move to a proof of Theorem 2.6, we give another proof of Dukes' result, $T_2 = 2$, $T_4 = 8$, $T_8 = 24$, by using cyclotomic polynomials. By definition of $U_N^{(s)}$, we have

$$\det(\lambda I_{2N} - U_N^{(s)}) = \prod_{k=0}^{N-1} \left\{ \lambda^2 + i\sqrt{2} \sin\left(\frac{2\pi k}{N}\right) \lambda - 1 \right\}, \quad (4.1)$$

see [14, 15], for example. From Eq. (4.1), we compute

$$\begin{aligned} \det(\lambda I_4 - U_2^{(s)}) &= F_1(\lambda)^2 F_2(\lambda)^2, \\ \det(\lambda I_8 - U_4^{(s)}) &= F_1(\lambda)^2 F_2(\lambda)^2 F_8(\lambda), \\ \det(\lambda I_{16} - U_8^{(s)}) &= F_1(\lambda)^2 F_2(\lambda)^2 F_8(\lambda) F_{12}(\lambda)^2. \end{aligned}$$

Then we have the desired conclusion:

$$T_2 = \text{lcm}(1, 2) = 2, \quad T_4 = \text{lcm}(1, 2, 8) = 8, \quad T_8 = \text{lcm}(1, 2, 8, 12) = 24,$$

where $\text{lcm}(n_1, n_2, \dots, n_k)$ denotes the least common multiple of (n_1, n_2, \dots, n_k) .

Next we give another proof of $N = 3$ case of Proposition 2.5 by using cyclotomic polynomials. That is, we prove $T_3 = \infty$. From Eq. (4.1), we calculate

$$\det(\lambda I_6 - U_3^{(s)}) = F_1(\lambda) F_2(\lambda) G(\lambda),$$

where

$$G(\lambda) = \lambda^4 - \frac{\lambda^2}{2} + 1.$$

On the other hand, it is known that there are only four cyclotomic polynomials with degree 4 as follows:

$$\begin{aligned} F_5(\lambda) &= \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1, & F_8(\lambda) &= \lambda^4 + 1, \\ F_{10}(\lambda) &= \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1, & F_{12}(\lambda) &= \lambda^4 - \lambda^2 + 1. \end{aligned}$$

Thus, we confirm that $G(\lambda)$ is not a cyclotomic polynomial and conclude that $T_3 = \infty$.

From now on, we move to a proof of Theorem 2.6. First we consider odd N case. Then we have

Proposition 4.1. *For any odd N , there exist $m(N), r_1, r_2, \dots, r_{m(N)} \geq 1$ such that*

$$\det(\lambda I_{2N} - U_N^{(s)}) = \prod_{j=1}^{m(N)} F_{r_j}(\lambda) \times G(\lambda),$$

where $G(\lambda)$ is not a cyclotomic polynomial.

The proof is that if we do not have such a $G(\lambda)$, then $T_N = \text{lcm}(r_1, r_2, \dots, r_{m(N)}) < \infty$ and this contradicts Proposition 2.5.

Now we consider $N = 2^n \times M$ case, where $n \geq 1$ and M is an odd number. By Eq. (4.1), we see that there exists a polynomial $H(\lambda)$ such that

$$\begin{aligned} \det(\lambda I_{2N} - U_N^{(s)}) &= \prod_{k=0}^{2^n \times M - 1} \left\{ \lambda^2 + i\sqrt{2} \sin\left(\frac{2\pi k}{2^n \times M}\right) \lambda - 1 \right\} \\ &= \prod_{k'=0}^{M-1} \left\{ \lambda^2 + i\sqrt{2} \sin\left(\frac{2\pi \times 2^n \times k'}{2^n \times M}\right) \lambda - 1 \right\} \times H(\lambda) = \det(\lambda I_{2M} - U_M^{(s)}) \times H(\lambda). \end{aligned}$$

From Proposition 4.1, we see that there exist $m(M), r_1, r_2, \dots, r_{m(M)} \geq 1$ such that

$$\det(\lambda I_{2N} - U_N^{(s)}) = \prod_{j=1}^{m(M)} F_{r_j}(\lambda) \times G_M(\lambda) \times H(\lambda),$$

where $G_M(\lambda)$ is not a cyclotomic polynomial. So we have $T_N = \infty$ for $N = 2^n \times M$, where $n \geq 1$ and M is an odd number.

Therefore it is enough to deal with $N = 2^n$ ($n \geq 4$) cases, since $T_2 = 2, T_{2^2} = 8, T_{2^3} = 24$. For $N = 2^4 = 16$ case, we obtain

$$\det(\lambda I_{2^5} - U_{2^4}^{(s)}) = F_1(\lambda)^2 F_2(\lambda)^2 F_8(\lambda) F_{12}(\lambda)^2 \times \left(\lambda^4 - \frac{\lambda^2}{2} + 1 \right)^2 \left(\lambda^4 - \frac{3\lambda^2}{2} + 1 \right)^2. \quad (4.2)$$

Since the right hand side of (4.2) contains a factor $\lambda^4 - \lambda^2/2 + 1$, we obtain $T_{2^4} = \infty$ as in the case of $N = 3$. For $N = 2^n$ ($n \geq 5$), we see that there exists a polynomial $G_N(\lambda)$ such that

$$\det(\lambda I_{2^{n+1}} - U_{2^n}^{(s)}) = \det(\lambda I_{2^5} - U_{2^4}^{(s)}) \times G_N(\lambda). \quad (4.3)$$

In view of (4.2), we conclude that $T_N = \infty$ for any $N = 2^n$ ($n \geq 5$).

5. Summary

In this paper, we proved that the period $T_N = \infty$ except with $N = 2, 4, 8$ for the Hadamard walk on C_N . On the other hand, $T_2 = 2, T_4 = 8, T_8 = 24$ was previously shown by Dukes [10]. Our method is based on a path counting and cyclotomic polynomials which is different from his approach based on the property of eigenvalues for $U_N^{(s)}$. An implementation of a Hadamard-like QW on C_N was proposed by Moqadam *et al.* [16] by using optomechanical systems. We hope that our result is helpful in building new quantum algorithms. Chou and Ho [17] investigated numerically the asymptotic behaviour of space-inhomogeneous QWs on \mathbb{Z} , where \mathbb{Z} is the set of integers. Their model is defined by a periodic quantum coin $U_x (x \in \mathbb{Z})$ given by H or I_2 , where I_2 is the 2×2 identity matrix, e.g., $U_x = H$ for $x = 0 \pmod{N}$, $U_x = I_2$ for $x \neq 0 \pmod{N}$ with $N \geq 2$. They discussed localization of the QWs, so one of the interesting future problems is to consider the periodicity of space-inhomogeneous QWs on C_N .

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