Interdisciplinary Information Sciences Vol. 23, No. 1 (2017) 75–86

ISSN 1340-9050 print/1347-6157 online
DOI 10.4036/iis.2017.A.10

Periodicity of the Discrete-time Quantum Walk on a Finite Graph

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In this paper we discuss the periodicity of the evolution matrix of Szegedy walk, which is a special type of quantum walk induced by the classical simple random walk, on a finite graph. We completely characterize the periods of Szegedy walks for complete graphs, compete bipartite graphs and strongly regular graphs. In addition, we discuss the periods of Szegedy walk induced by a non-reversible random walk on a cycle.

KEYWORDS: quantum walk, periodicity, evolution matrix, cyclotomic polynomial, random walk, strongly regular graph

1. Introduction

Research on quantum walks has been developed in various areas [1–3, 8, 15, 16, 19, 22, 23]. Among a wide range of properties of them, we here focus on periodicity of the evolution matrix U_{sz} of Szegedy walk, the discrete-time quantum walks, on a special class of finite graphs. For the continuous-time quantum walks, periodicity is discussed in terms of "perfect state transfer" in [7]. As can be seen in the reviews on the development of quantum walks in the analogous field of random walks [11, 12], discrete-time quantum walks have some different features from classical random walks. For example, the quantum walk with a Hadamard coin returns exactly to its initial state after 8 and 24 steps on the cycle of length 4 and 8, respectively; this property may be called *periodicity*. In contrast, the classical isotropic random walk on any cycle of length n ($n \geq 3$) has no such behaviour. For details and more observation, refer to [11, 12] and references therein. In this paper, we restrict ourselves to considering Szegedy walks, which belong to a special class of quantum walks and relate closely to the underlying classical random walks. Our object is to identify all finite graphs to have periodic Szegedy walks; as a valid starting point, our graphs in this paper are restricted to complete graphs, complete bipartite graphs, strongly regular graphs and cycles.

We first show our setting. Graphs treated here are finite only. Let $G = (V(G), E(G))$ be a connected graph (having possibly multiple edges and self-loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges. We say two vertices u and v are *adjacent* if there exists an unoriented edge joining u and v; $uv \in E(G)$. Considering each edge in $E(G)$ to have two orientations, we can introduce the set of all oriented edges; it is denoted by $D(G)$. For an oriented edge $e \in D(G)$, the origin vertex and the terminal one of e are denoted by $o(e)$ and $t(e)$, respectively; the inverse edge of e is denoted by e^{-1}. The *degree* $\deg v = \deg_G v$ of a vertex v of G stands for the number of oriented edges whose origin is v. For a connected graph G with n vertices and m unoriented edges, we often set $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$.

Let $p : D(G) \to (0, 1]$ be a transition probability such that

$$\sum_{e:o(e)=v} p(e) = 1 \tag{1.1}$$

for every vertex $v \in V(G)$. A classical random walk on G is defined by this probability p, that is, a particle at $u = o(e)$ can be considered to move to a neighbour $v = t(e)$ along the oriented edge e with probability $p(e)$ in one unit time. For a finite graph G, we consider the transition matrix T_p such that T_p is an $n \times n$-matrix and

Received April 1, 2016; Accepted December 13, 2016
2016 Mathematics Subject Classification: Primary 05C50, Secondary 15B99, 05C81, 58J50, 39A12, 47A25, 05C75.
Dedicated to Professor Yoichiro Takahashi on his seventieth birthday.
[1]Partially supported by JSPS under the Grant-in-Aid for Scientific Research No. 25400208, 26610025 and 15H02055.
[2]Partially supported by JSPS under the Grant-in-Aid for Scientific Research No. 15K13443.
[3]Partially supported by JSPS under the Grant-in-Aid for Scientific Research No. 15K04985.
[4]Partially supported by JSPS under the Grant-in-Aid for Scientific Research No. 16K17637.
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$$(T_p)_{u,v} = \begin{cases} \sum_{o(e)=u,t(e)=v} p(e), & \text{if } uv \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

With respect to the transition probability of a classical random walk, the evolution matrix of the Szegedy walk, which is a kind of quantum walk introduced in [22], is defined as follows (cf. [17, 22]): U_{sz} is a $2m \times 2m$-matrix and

$$(U_{sz})_{e,f} = \begin{cases} 2\sqrt{p(e)p(f^{-1})} - \delta_{e^{-1},f}, & \text{if } t(f) = o(e), \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Here $\delta_{a,b}$ is the Kronecker delta function, that is,

$$\delta_{a,b} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

In the above, we consider an (e, f)-element stands for the hopping rate of wave's traveling from $f \in D(G)$ to $e \in D(G)$.

The Szegedy matrix with respect to the simple random walk is called the Grover matrix, whose original form can be seen in [23]. In fact, setting $p(e) = 1/\deg_G o(e)$ in (1.3), we can get the following form: the Grover matrix $U_{gr}(G) = (U_{e,f})_{e,f\in D(G)}$ of G is defined by

$$(U_{gr})_{e,f} = \begin{cases} 2/\deg_G o(e), & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/\deg_G o(e) - 1, & \text{if } f = e^{-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

We often use the symbol T_0 for the transition matrix for the simple random walk.

Let us introduce the notion of "periodicity", which is the main theme in this paper. Let U_{sz} be the Szegedy matrix for a finite graph G with m unoriented edges; I_{2m} be the $2m \times 2m$ identity matrix.

Definition 1.1 (Periodicity of Szegedy walk)**.** If there exists a positive integer k such that

$$U_{sz}^k = I_{2m}, \tag{1.5}$$

then we call U_{sz} *periodic* and the minimum positive integer satisfying (1.5) its *period*. If there does not exist such a positive integer k satisfying (1.5), then we call U_{sz} *aperiodic*.

The main theorems in this paper are as follows:

Theorem 1.2 (Complete graphs)**.** *Let U_{sz} be the Szegedy matrix induced by an ℓ-lazy simple random walk on a complete graph K_n for $n \geq 2$. For a rational number $\ell \in [0, 1)$, U_{sz} is periodic if and only if $(n, \ell) = (2, 0)$, $(3, 0)$, $(n, 1/n)$, $(2, 1/4)$ or $(n, (n + 1)/(2n))$, whose period is* 2, 3, 4, 6 *and* 6, *respectively.*

An ℓ-lazy version of T and more detail setting on graphs will be discussed in Section 2.2.1.

Theorem 1.3 (Complete bipartite graphs)**.** *Let U_{sz} be the Szegedy matrix induced by an ℓ-lazy simple random walk on a complete bipartite graph $K_{m,n}$ with $m, n > 0$ and $m + n \geq 3$. For a rational number $\ell \in [0, 1)$, U is periodic if and only if $\ell = 0$ or $\ell = 1/2$, whose period is* 4 *and* 12, *respectively.*

Theorem 1.4 (Strongly regular graph $SRG(n, k, \lambda, \mu)$)**.** *Let U_{gr} be the Grover matrix induced by the simple random walk on a strongly regular graph $SRG(n, k, \lambda, \mu)$. U_{gr} is periodic if and only if*

$$(n, k, \lambda, \mu) = (2k, k, 0, k), (3\lambda, 2\lambda, \lambda, 2\lambda), (5, 2, 0, 1),$$

whose period is 4, 12 and 5, respectively.

Remark that there exists a complete balanced bipartite graph, $K_{k,k}$, a complete balanced tripartite graph, $K_{\lambda,\lambda,\lambda}$ and a cycle of length 5, C_5, for such parameters, respectively.

Definition and some fundamental properties on a strongly regular graph are given in Section 2.2.2.

Finally we discuss a non-reversible random walk on the cycle C_n of length n: $V(C_n) = \{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1}\}$ for $i = 1, \ldots, n$ (mod n).

Theorem 1.5 (Szegedy walk induced by a non-reversible random on C_n)**.** *Let U_{sz} be the Szegedy matrix induced by a non-reversible random walk on the cycle C_n of length n. For a rational number p_0 with $0 < p_0 < 1/2$, we set a non-reversible probability as $p(e) = p_0$ for $o(e) = v_i$ and $t(e) = v_{i+1}$. Then the following hold:*

(0) *U_{sz} is aperiodic for any $n \neq 2, 4, 8$;*

(1) *For $n = 2$, U_{sz} is periodic if and only if $p_0 = (2 - \sqrt{3})/4$, $(2 - \sqrt{2})/4$ or $1/4$, whose period is* 6, 8 *or* 12, *respectively;*

(2) *For $n = 4$, U_{sz} is periodic if and only if $p_0 = (2 - \sqrt{3})/4$, $(2 - \sqrt{2})/4$ or $1/4$, whose period is* 12, 8, *or* 12, respectively;

(3) *For $n = 8$, U_{sz} is periodic if and only if $p_0 = (2 - \sqrt{2})/4$, whose period is* 24.

Detailed setting can be seen in Section 2.2.3. It is easy to check the period of U_{sz} is n if $p = 1/2$; that of U is 4 if $p = 0$. See the end of Section 2.2.3. We should add a remark that, after this research is established, the period of the Hadamard walk on a cycle is studied in a slightly different way in [13].

This paper is organized as follows: In Section 2, we briefly explain the setting for graphs and the main tools, which include the spectrum mapping theorem in the context of the Szegedy walks, a lazy version of random walk and the cyclotomic polynomial and so on. In Section 3, we give the proofs of theorems above: those of Theorems 1.2, 1.3, 1.4, and 1.5 can be seen in Sections 3.1, 3.2, 3.3, and 3.4, respectively. We discuss also a strategy on deciding the periodicity of the Grover walk G_{gr} induced by the simple random walk T_0 on a general finite graph in Section 4.

2. Setting and Tools

2.1 Spectral Mapping Theorem

Here we explain the spectral mapping theorem, which is one of main tools. First we give an $n \times n$ matrix $S = S_p$ derived from a given transition probability p as

$$(S_p)_{u,v} = \begin{cases} \sum_{o(e)=u,t(e)=v} \sqrt{p(e)p(e^{-1})}, & \text{if } uv \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the following mapping theorem has been obtained; see [9, 17]. For more general abstract quantum walks, it is generalized [18].

Theorem 2.1. *(cf. [9, 17]) For the Szegedy matrix U_{sz} of G with n vertices and m unoriented edges, we have*

$$\det(\lambda I_{2m} - U_{sz}) = (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)I_n - 2\lambda S_p).$$

□

According to this theorem, we have all eigenvalues of U_{sz} are all of the solutions of

$$\lambda^2 - 2\mu\lambda + 1 = 0 \tag{2.2}$$

for each eigenvalue μ of S_p and, if $m - n > 0$, ± 1. Naturally, since U_{sz} is a unitary matrix, all of its eigenvalues are on the unit circle on the complex plane. Moreover, it is well known that unitary matrices can be diagonalizable. Then the following obvious but important property holds. Here the complex value λ is said to be a root of unity if there exists a positive integer n such that $\lambda^n = 1$. In addition, if $\lambda^n = 1$ and $\lambda^k \neq 1$ for any positive integer $k < n$, then λ is said to be a *primitive* n-th root of unity and have the period n.

Proposition 2.2. *The Szegedy matrix U_{sz} is periodic if and only if all of its eigenvalues are roots of unity. Moreover, if U_{sz} is periodic, then its periodicity coincides with the least common multiple (LCM) of all periods of eigenvalues.*

□

For a transition probability p, if there exists a positive valued function $m : V(G) \to (0, \infty)$ such that

$$m(o(e))p(e) = m(t(e))p(e^{-1}) \tag{2.3}$$

for every oriented edge $e \in D(G)$, p is said to be *reversible*; the function m is said to be a reversible measure for p or for the random walk, which is unique, if exists, up to a multiple constant. If p is reversible, it is easy to check that $MT_pM^{-1} = S_p$, where $(M)_{u,v} = \sqrt{m(u)} \cdot \delta_{u,v}$; hence T_p and S_p are isospectral. Thus, for any reversible random walk T_p and the Szegedy walk U_{sz} induced by p, the equation (2.2) also holds for the eigenvalue of T_p instead of μ.

As a representative examples of a reversible random walk, we may display the *simple* random walk on G, which is induced by p such that $p(e) = 1/\deg_G o(e)$ for every $e \in D(G)$. Obviously $m(u) = \deg_G u$ is a reversible measure for such p.

We should remark that each random walk treated in Sections 3.1, 3.2 and 3.3 is reversible; one only in Section 3.4 is non-reversible.

2.2 Graphs and Lazy Random Walk

Finite graphs appeared in our theorems here are only a special type of graphs, that is, a complete graph K_n on n vertices, a complete bipartite graph $K_{m,n}$ on $m + n$ vertices, a strongly regular graph $SRG(n, k, \lambda, \mu)$ and a cycle graph C_n on n vertices.

For completeness, we give the definition of the above in our context.

2.2.1 Lazy random walk on complete graph and complete bipartite graph

Definition 2.3. (Complete graph and complete bipartite graph) A complete graph K_n is a graph on n vertices where every vertex is adjacent to any other vertex and has one self-loop. A complete bipartite graph $K_{m,n}$ with $V(K_{m,n}) = V_1 \sqcup V_2$ such that $|V_1| = m$ and $|V_2| = n$ is a graph where every vertex in V_1 is adjacent to any vertex in V_2 and every vertex has one self-loop. See Figs. 1 and 2.

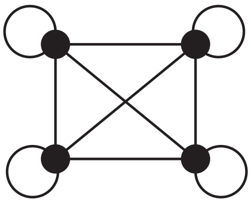

Fig. 1. Complete graph K_4 with self-loops.

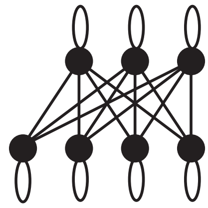

Fig. 2. Complete bipartite graph $K_{3,4}$ with self-loops.

Usually K_m and $K_{m,n}$ are defined without self-loops, but adding self-loops to them, we naturally derive the Szegedy quantum walks from the lazy version of random walks. For $\ell \in (0,1)$, an ℓ-lazy version of simple random walk is defined by the transition matrix $\ell I + (1-\ell)T_0$ on usual K_n and $K_{m,n}$; Equivalently, we express the probabilities p for an ℓ-lazy simple random walks on the above K_n and $K_{m,n}$ as follows: for K_n and $\ell \in (0,1)$,

$$p(e) = \begin{cases} \ell/2, & \text{if } o(e) = t(e), \\ (1-\ell)/(n-1), & \text{if } t(e) \neq o(e); \end{cases} \tag{2.4}$$

for $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ and $\ell \in (0,1)$,

$$p(e) = \begin{cases} \ell/2, & \text{if } o(e) = t(e), \\ (1-\ell)/n, & \text{if } o(e) \in V_1 \text{ and } t(e) \in V_2, \\ (1-\ell)/m, & \text{if } o(e) \in V_2 \text{ and } t(e) \in V_1. \end{cases} \tag{2.5}$$

Remark that, since there exist two oriented edges with respect to an unoriented self-loop for every vertex in the above, the transition probability from a vertex to itself, that is, that of a particle staying on a vertex in one unit time is equal to $2 \times \ell/2 = \ell$. Moreover it is trivial that, if $\ell = 0$, then the random walk with respect to the above probability coincides with the simple random walk on a usual self-loopless K_n or $K_{m,n}$. Thus, if $\ell = 0$, we assume K_n and $K_{m,n}$ are in the usual forms, that is, graphs without self-loops, and T_0 is the transition matrix of the simple random walk with respect to them. Let us summarize our complete graph and complete bipartite graph setting with respect to an ℓ lazy random walk:

(1) if $\ell = 0$, our K_n and $K_{m,n}$ are in the standard self-loop less forms;
(2) if $0 < \ell < 1$, our K_n and $K_{m,n}$ are in the form stated as in Definition 2.3.

Thanks to the definitions above of p on K_n or $K_{m,n}$ in Definition 2.3, the Szegedy matrix U_{sz}, whose (e,f)-element present for the hopping rate of travelling quantum wave, can be naturally obtained from the given lazy version of p as in (1.3); moreover we can apply Theorem 2.1.

We note that this formulation for the Szegedy walk induced by a lazy version of random walk is somewhat different from what is seen in [10].

2.2.2 Strongly regular graph

A graph is called a *regular* graph if each vertex has the same degree.

Definition 2.4. (Strongly regular graph) A strongly regular graph $SRG(n,k,\lambda,\mu)$ with parameters n,k,λ,μ is a non-complete k-regular graph with n vertices such that: 1) every two adjacent vertices have λ common neighbours; 2) every two non-adjacent vertices have μ common neighbours.

Here we use the usual definition of strongly regular graphs. See also the standard texts, for instance, Brouwer and Haemers [5], Cameron and Van Lint [6] and so on. The four parameters, n, k, λ and μ, cannot be chosen independently: for example, the following relation holds:

$$(n-k-1)\mu = k(k-\lambda-1). \tag{2.6}$$

Many strongly regular graphs are known for some kind class of four adequate parameters; On the other hand, for general class of four adequate parameters, we do not know whether such a graph exists or not, and, if it exists, whether it is unique or not. However, if a graph exists for four adequate parameters, then the spectra of T_0, the transition matrix of the simple random walk, can be determined. For details, refer to [5,6]:

Proposition 2.5. *(Spectra, cf. [5,6]) The eigenvalues of T_0 on $SRG(n,k,\lambda,\mu)$ are as follows: the value 1 with multiplicity one, and*

$$r_\pm = \frac{1}{2k}\left((\lambda-\mu) \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right), \tag{2.7}$$

whose multiplicities are

$$\frac{1}{2}\left((n-1) \mp \frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right), \tag{2.8}$$

respectively. Here $-1 \le r_- < r_+ < 1$ *and the values* $r_\pm$ *satisfy*

$$r_+ + r_- = (\lambda-\mu)/k, \text{ and } r_+ r_- = (\mu-k)/k^2. \tag{2.9}$$

We should also remark four parameters are a non-negative integers with $n \ge 2$, $1 \le k \le n-2$, $0 \le \lambda \le k-1$, and $1 \le \mu \le k$. On the other hand, noting multiplicities are always integers, we obtain further information on parameters, which is known in [5, 6] as follows:

Theorem 2.6. *(cf. [5, 6]) If* $2k+(n-1)(\lambda-\mu) \ne 0$, *then both* $r_\pm$ *are rational. Otherwise,* $r_\pm = (-1 \pm \sqrt{n})/(n-1)$. □

2.2.3 Non-reversible random walk on C_n

A graph G is called a cycle of length n if $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $D(G) = \{e_i, e_i^{-1};\ i = 0, \ldots, n-1\}$, where $o(e_i) = v_i$ and $t(e_i) = v_{i+1}$ under modulo n. This is a standard definition of cycles and denoted by C_n. Now we give a non-reversible transition probability p: For $p_0 \in (0, 1/2)$, we set a probability p as

$$p(e) = \begin{cases} p_0, & \text{if } e = e_i, \\ 1-p_0, & \text{if } e = e_i^{-1}. \end{cases} \tag{2.10}$$

Obviously this chain on C_n is non-reversible. Let us exhibit two type of simple examples of the periodicity on C_n by using Theorem 2.1 and Proposition 2.2: two cases where $p_0 = 0$ and $p_0 = 1/2$.

If $p_0 = 0$, then we can easily obtain that all eigenvalues of S_p are 0's. Thus, by Theorem 2.1, the set of eigenvalues of U_{sz} consists of $\pm\sqrt{-1}$; we have U_{sz} has the period 4 by Proposition 2.2.

On the other hand, if $p_0 = 1/2$, then p express the simple random walk, that is, $T_p = T_0$ and p is reversible. It is easy to see the set of eigenvalues of T_0, which is the same as that of S_p, consists of $\cos(2k\pi/n)$, $(k = 0, \ldots, n-1)$. Similarly, by Theorem 2.1, we know the set of eigenvalues of U_{gr} on C_n consists of $\exp(2k\pi\sqrt{-1}/n)$ $(k = 0, \ldots, n-1)$; we have U_{sz} has the period n by Proposition 2.2. Here, for the sake to discussing the periodicity, we only have to pay attention to the set of eigenvalues without their multiplicities.

2.3 Cyclotomic Polynomial

Let us explain briefly the cyclotomic polynomial $\Phi_n(x)$ for the primitive n-th roots of unity, which is well known in the area of algebra:

$$\Phi_n(x) = \prod_{\zeta}(x-\zeta), \tag{2.11}$$

where the product is taken over all of the primitive n-th roots of unity; ζ can be expressed as $\exp(2\pi\sqrt{-1}j/n)$ for $(j, n) = 1$. This polynomial is known as monic and has integer coefficients; moreover it is also known that this is irreducible over the field of the rational numbers $\mathbf{Q}$. Some properties are exhibited as follows: (1) The degree of $\Phi_n(x)$ is $\phi(n)$, which is the value at n of the Euler ϕ-function defined as the number of integers k such that $1 \le k \le n$ and $(k, n) = 1$; (2) $\sum_{d|n}\phi(d) = n$; (3) The function $\phi(n)$ is multiplicative, that is, $\phi(mn) = \phi(m)\phi(n)$ if $(m, n) = 1$ and can be expressed as

$$\phi(n) = n\prod_{q|n}(1-1/q), \tag{2.12}$$

where the product is over all prime numbers q dividing n; (4) The coefficient of $\Phi_n(x)$ can be calculated inductively: $\Phi_1(x) = x-1$ and

$$\Phi_n(x) = (x^n-1) \Big/ \prod_{d|n, d<n}\Phi_d(x), \tag{2.13}$$

where the product is over all positive integers dividing n and less than n.

For details, refer to some standard textbooks on algebra, for instance, Lang [14], Stillwell [21] and so on. Therein the following fundamental theorem helps us in this paper:

Theorem 2.7. *(cf. [14, 21]) Factorization of a polynomial into irreducible ones over* $\mathbf{Q}$ *is unique up to order and constant factor.* □

The following is an easy consequence from Theorem 2.7, but plays an important role in Section 3.4.

Proposition 2.8. *Let* $f(x)$ *be a monic polynomial with* $\mathbf{Q}$*-coefficients. If all solutions of* $f(x) = 0$ *are roots of unity, then* $f(x)$ *can be uniquely factorized into cyclotomic polynomials. Hence* $f(x)$ *must be a monic polynomial with*

Z-*coefficients.* □

For convenience, we exhibit some cyclotomic polynomials up to $n = 12$; it is easy to check $\phi(n) = \deg \Phi_n(x) \geq 6$ for $n \geq 13$:

$$\Phi_1(x) = x - 1,\ \Phi_2(x) = x + 1,\ \Phi_3(x) = x^2 + x + 1,\ \Phi_4(x) = x^2 + 1,\ \Phi_5(x) = \sum_{k=0}^{4} x^k,\ \Phi_6(x) = x^2 - x + 1,$$
$$\Phi_7(x) = \sum_{k=0}^{6} x^k,\ \Phi_8(x) = x^4 + 1,\ \Phi_9(x) = x^6 + x^3 + 1,\ \Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1,\ \Phi_{11}(x) = \sum_{k=0}^{10} x^k, \tag{2.14}$$
$$\Phi_{12}(x) = x^4 - x^2 + 1.$$

3. Proofs of Theorems

3.1 Szegedy Walk on Complete Graphs K_n

As is seen in Section 2.2, we assume that $T_p = T_0$ is the transition matrix for the simple random walk on a standard self-loopless K_n if $\ell = 0$ and T_p is the one for the ℓ-lazy random walk in (2.4) on our K_n with self-loops in (2.3) if $0 < \ell < 1$. Then $2m = 2|E(K_n)| = n(n-1)$ if $\ell = 0$, and $2m = n(n+1)$ if $0 < \ell < 1$. We can easily calculate the spectra of T_p, for $n \geq 2$ and $\ell \in [0, 1)$, as

$$Spec(T_p) = \{1, (n\ell - 1)/(n-1)\} \tag{3.1}$$

with multiplicity 1 and $n - 1$, respectively.

Proof of Theorem 1.2. First we should note that $(n\ell - 1)/(n-1) < 1$ since $\ell < 1$ and $(n\ell - 1)/(n-1) = -1$ if and only if $(n, \ell) = (2, 0)$. In case where $(n, \ell) = (2, 0)$, the eigenvalues of T_p are ± 1, so, by Theorem 2.1, the eigenvalues of U_{sz} are also ± 1. Thus we know U_{sz} has the period 2. Hereafter we may assume $(n, \ell) \neq (2, 0)$ and $-1 < (n\ell - 1)/(n-1) < 1$. Also, by Theorem 2.1, we obtain that the eigenvalues of U_{sz} derived from those of T_p are 1 and the solutions of $F(x) = 0$, where $F(x)$ is a monic polynomial with **Q**-coefficients and

$$F(x) = x^2 - 2(n\ell - 1)/(n-1)x + 1. \tag{3.2}$$

Recall ℓ is assumed to be a rational number in $[0, 1)$ and the equation $F(x) = 0$ cannot have the solutions ± 1. If U_{sz} is periodic, then all of the solutions of $F(x) = 0$ must be the roots of unity from Proposition 2.2. In addition, it follows from Proposition 2.8 that $F(x)$ in (3.2) must be a cyclotomic polynomial of degree 2 if U_{sz} is periodic. Referring to (2.14), we can see $F(x)$ in (3.2) becomes $\Phi_3(x)$, $\Phi_4(x)$ or $\Phi_6(x)$; equivalently,

$$-2(n\ell - 1)/(n-1) = 1, 0 \text{ or } -1. \tag{3.3}$$

If $F(x) = \Phi_3(x)$, then $\ell = (3 - n)/(2n)$; thus we have $(n, \ell) = (2, 1/4)$ or $(3, 0)$. If $F(x) = \Phi_4(x)$, then $\ell = 1/n$. If $F(x) = \Phi_6(x)$, then $\ell = (n+1)/(2n)$. Finally we should remark that $|E(K_n)| - n > 0$ if and only if $(n, \ell) \neq (3, 0)$, that is, U_{sz} has the eigenvalue -1 if and only if $(n, \ell) \neq (3, 0)$. Conversely, for such a pair of n and ℓ, it is obvious that U_{sz} has the period as stated in Theorem 1.2 in Introduction. Thus the proof is completed. □

If we consider our complete graph with self-loops for $\ell = 0$, the conclusions in Theorem 1.2 still hold except only the case where $(n, \ell) = (3, 0)$. Since U_{sz} has the eigenvalue -1 for our K_3 in Definition 2.3, the period of U_{sz} changes to 6 for $(n, \ell) = (3, 0)$.

3.2 Szegedy Walk on Complete Bipartite Graphs $K_{m,n}$

As is seen in Section 2.2, we assume that $T_p = T_0$ is the transition matrix for the simple random walk on a standard self-loopless $K_{m,n}$ if $\ell = 0$ and T_p is the one for the ℓ-lazy random walk in (2.5) on our $K_{m,n}$ with self-loops in (2.3) if $0 < \ell < 1$. Then $2|E(K_n)| = 2mn$ if $\ell = 0$, and $2|E(K_n)| = 2mn + m + n$ if $0 < \ell < 1$. We can easily calculate the spectra of T_p, for $\ell \in [0, 1)$, as

$$Spec(T_p) = \{1, \ell, 2\ell - 1\} \tag{3.4}$$

with multiplicity 1, $m + n - 2$ and 1, respectively.

Proof of Theorem 1.3. We may assume $m \leq n$. Since $\ell \in [0, 1)$, it is obvious that $-1 \leq 2\ell - 1 < \ell < 1$ and that $2\ell - 1 = -1$ if and only if $\ell = 0$. First we set $\ell = 0$. In this case, we easily see, by Theorem 2.1, the eigenvalues of U_{sz} are $\sqrt{-1}$, $-\sqrt{-1}$, 1, -1. Then U_{sz} has the period 4. Hereafter we assume $\ell \neq 0$; then $-1 < 2\ell - 1 < \ell < 1$. Also, by Theorem 2.1, we obtain that the eigenvalues of U_{sz} derived from those of T_p are 1 and the solutions of $F_i(x) = 0$, $(i = 1, 2)$, where $F_1(x)$ and $F_2(x)$ are monic polynomials with **Q**-coefficients and

$$F_1(x) = x^2 - 2\ell x + 1 \text{ and } F_2(x) = x^2 - 2(2\ell - 1)x + 1. \tag{3.5}$$

Recall ℓ is assumed to be a rational number in $[0, 1)$ and both of the equations $F_i(x) = 0$ cannot have the solutions ± 1. By the same way as in the proof in Section 3.1, we find U_{sz} is periodic if and only if all of the solutions of $F_1(x) = 0$ and $F_2(x) = 0$ are the roots of unity from Proposition 2.2. Assume U_{sz} is periodic. Referring to (2.14), we find

$F_1(x) = \Phi_6(x)$ since $-2 < -2\ell < 0$, which implies that $\ell = 1/2$ and $F_2(x) = \Phi_4(x)$. Here let us remark that U_{sz} has the eigenvalue -1 since $|E(K_{m,n})| - |V(K_{m,n})| = mn > 0$. Conversely, it is obvious that U_{sz} has the period 12 for $\ell = 1/2$. This completes the proof of Theorem 1.3 in Introduction. □

3.3 Grover Walk on Strongly Regular Graphs $SRG(n, k, \lambda, \mu)$

Proof of Theorem 1.4. By Theorem 2.1, we obtain that the eigenvalues of U_{sz} derived from those of T_p are 1 and the solutions of $F_i(x) = 0$, $(i = 1, 2)$, where $F_1(x)$ and $F_2(x)$ are monic polynomials with $\mathbf{R}$-coefficients and

$$F_1(x) = x^2 - 2(r_-)x + 1 \text{ and } F_2(x) = x^2 - 2(r_+)x + 1. \tag{3.6}$$

Recall $r_\pm$ satisfy (2.8) and (2.9), and both of the equations $F_i(x) = 0$ cannot have the solution 1. By the same way as in the proofs in Sections 3.1 and 3.2, we find U_{sz} is periodic if and only if all of the solutions of $F_1(x) = 0$ and $F_2(x) = 0$ are the roots of unity from Proposition 2.2.

Now let us divide this proof into two cases where $r_\pm$ are rational or not.

3.3.1 For rational $r_\pm$

In this subsection, we assume $r_\pm$ are rational numbers. Then both of the monic polynomials in (3.6) are with $\mathbf{Q}$-coefficients. When $r_- = -1$, we find that $\mu = k$, $\lambda = 0$, and $r_+ = 0$ according to (2.9) and the fact $1 \le \mu \le k$; we have $n = 2k$ by (2.6). Therefore, $F_1(x) = (x+1)^2$ and $F_2(x) = x^2 + 1$. In result, we obtain U_{sz} has the period 4. Recall a strongly regular graph is not a complete one. For $k \ge 2$, we can find $K_{k,k}$ as $SRG(2k, k, 0, k)$.

Under the condition that U_{sz} is periodic, we may now assume that neither of ± 1 are the solutions of the equations $F_i(x) = 0$ for $i = 1, 2$. Then $F_1(x)$ and $F_2(x)$ are irreducible over $\mathbf{Q}$, so each of them is a monic cyclotomic polynomial of degree 2, that is, coincides with $\Phi_3(x)$, $\Phi_4(x)$ or $\Phi_6(x)$ in (2.14).

Let $F_1(x) = \Phi_3(x)$ and $F_2(x) = \Phi_4(x)$, equivalently, $r_- = -1/2$ and $r_+ = 0$. Then we find that $k = \mu = 2\lambda$ and $n = 3\lambda$ according to (2.9) and (2.6). In this case, we obtain U_{sz} has the period 12. For $k \ge 2$, we can find $K_{k,k,k}$ as $SRG(3k, 2k, k, 2k)$. Here A complete tripartite graph $K_{k,m,n}$ with $V(K_{k,m,n}) = V_1 \sqcup V_2 \sqcup V_3$ such that $|V_1| = k$, $|V_2| = m$ and $|V_3| = n$ is a graph where every vertex in V_i is adjacent to any vertex in V_j for every distinct pair (i, j).

Next let $F_1(x) = \Phi_3(x)$ and $F_2(x) = \Phi_6(x)$, equivalently, $r_- = -1/2$ and $r_+ = 1/2$. Then we find that $\mu = k - k^2/4$ by (2.9). Since $0 < \mu < k$ and $k \ge 2$, it is impossible. Finally let $F_1(x) = \Phi_4(x)$ and $F_2(x) = \Phi_6(x)$, equivalently, $r_- = 0$ and $r_+ = 1/2$. Then we find that $\mu = k$ and $\lambda = 3k/2$ by (2.9), which contradicts $\lambda < k$.

3.3.2 For irrational $r_\pm$

In this subsection, we assume $r_\pm$ are irrational number. Then we may set $n = 4\mu + 1$, $k = 2\mu$ and $\lambda = \mu - 1$ by Theorem 2.6; therefore

$$r_\pm = \frac{-1 \pm \sqrt{4\mu + 1}}{2k}. \tag{3.7}$$

Assume U_{sz} is periodic. Then all of the solutions of $F_1(x) = 0$ and $F_2(x) = 0$ are the roots of unity. Although both of $r_\pm$ are distinct irrational numbers, $G(x) = F_1(x)F_2(x)$ is a monic polynomial with $\mathbf{Q}$-coefficients:

$$\begin{aligned} G(x) &= x^4 - 2(r_+ + r_-)x^3 + (4r_+r_- + 2)x^2 - 2(r_+ + r_-)x + 1 \\ &= x^4 - \frac{2(\lambda - \mu)}{k}x^3 + \frac{4(\mu - k) + 2k^2}{k^2}x^2 - \frac{2(\lambda - \mu)}{k}x + 1 \\ &= x^4 + \frac{1}{\mu}x^3 + \frac{-1 + 2\mu}{\mu}x^2 + \frac{1}{\mu}x + 1. \end{aligned} \tag{3.8}$$

Noting that the solutions of $F_1(x) = 0$ and $F_2(x) = 0$ are mutually distinct roots of unity and not equal to ± 1, we find $G(x)$ is a monic polynomial with $\mathbf{Q}$-coefficients, which is irreducible over $\mathbf{Q}$; this must be a cyclotomic polynomial of degree 4. Thus this coincides with $\Phi_5(x)$, $\Phi_8(x)$, $\Phi_{10}(x)$ or $\Phi_{12}(x)$ in (2.14). Observing the form in (3.8), we obtain $\mu = 1$; then $G(x) = \Phi_5(x)$ and $(n, k, \lambda, \mu) = (5, 2, 0, 1)$. When G is a $SRG(5, 2, 0, 1)$, $|E(G)| = |V(G)|$, which implies that U_{sz} does not have the eigenvalue -1. Therefore U_{sz} has the period 5. In this case we can find C_5 as $SRG(5, 2, 0, 1)$.

All of the proof of Theorem 1.4 in Introduction is completed. □

3.4 Szegedy Walk Induced by a Non-reversible Random on C_n

Proof of Theorem 1.5. For the setting given in 2.2.3, we can easily calculate the spectra of S_p, for $0 < p_0 < 1/2$, as

$$Spec(S_p) = \left\{ 2\sqrt{p_0(1 - p_0)} \cos\frac{2k\pi}{n};\ k = 0, 1, \ldots n - 1 \right\}. \tag{3.9}$$

By Theorem 2.1, we obtain that the eigenvalues of U_{sz} are the solutions of $F_k(x) = 0$, $(k = 0, \ldots, n-1)$, where

$$F_k(x) = x^2 - 2\lambda_k x + 1, \tag{3.10}$$

where $\lambda_k = 2\sqrt{p_0(1-p_0)}\cos\frac{2k\pi}{n}$. Remark that U_{sz} does not have the eigenvalues ± 1, since $\pm 1 \notin Spec(S_p)$ and $|E(C_n)| - |V(C_n)| = 0$.

Moreover, we should note that the form of spectra of S_p in (3.9) tells us that, for a fixed p_0, the set of eigenvalues $Spec(S_p)$ on C_m is a subset of that on C_n if $m|n$; thus every eigenvalue of U_{sz} on C_m is that of U_{sz} on C_n if $m|n$ from Theorem 2.1. Hence, if U_{sz} on C_m is aperiodic, then so is that on C_n for $m|n$.

Recall we assume p_0 is rational, and that U_{sz} is periodic if and only if all the solutions of $F_k(x) = 0$, $(k = 0, \dots, n-1)$, are roots of unity by Proposition 2.2. The plan in this section is as follows:
(0) We characterize p_0 such that the roots of $F_0(x) = 0$ are roots of unity in Section 3.4.1. In fact, we find U_{sz} is aperiodic unless $p_0 = (2-\sqrt{3})/4$, $(2-\sqrt{2})/4$ or $1/4$;
(1) When $n = 2$, we characterize p_0 such that U_{sz} is periodic in Section 3.4.2;
(2) When $n = 2^2$, we characterize p_0 such that U_{sz} is periodic in Section 3.4.3;
(3) When $n = 2^3$, we characterize p_0 such that U_{sz} is periodic in Section 3.4.4;
(4) When $n = 2^4$, we show U_{sz} is aperiodic for any p_0 in Section 3.4.5;
(5) When n is odd prime, we show U_{sz} is aperiodic for any p_0 in Section 3.4.6.
Summarizing the above, if n has an odd prime factor, then U_{sz} is aperiodic from **(5)**; if n has a factor 2^k for $k \geq 4$, then U_{sz} is aperiodic from **(4)**; thus, if U_{sz} is periodic, then $n = 2$, or 4 or 8.

3.4.1 The roots of $F_0(x) = 0$

For $k = 0$, that is, $\lambda_0 = 2\sqrt{p_0(1-p_0)}$, we consider the solutions of $F_0(x) = 0$. Assume U_{sz} is periodic. Then those solutions of $F_0(x) = 0$ must be roots of unity.

First let $\sqrt{p_0(1-p_0)}$ be rational. Then $F_0(x)$ is a monic polynomial with $\mathbf{Q}$-coefficients of degree 2 and irreducible over $\mathbf{Q}$; in result, $F_0(x)$ is a cyclotomic polynomial of degree 2. Here, since $\lambda_0 > 0$, we find $F_0(x) = \Phi_6(x)$. So we get $p_0(1-p_0) = 1/16$, equivalently, $p_0 = (2-\sqrt{3})/4$.

Next let $\sqrt{p_0(1-p_0)}$ be irrational. It is easy to see that, for any solution λ of $F_0(x) = 0$, $-\lambda$ is also a root of unity, that is, all of the solutions of $F_0(-x)$ are roots of unity. Consider $F_0(x)F_0(-x)$:

$$F_0(x)F_0(-x) = x^4 + 2(1 - 8p_0(1-p_0))x^2 + 1.$$

Here all of the solutions of $F_0(x)F_0(-x) = 0$ are root of unity. This polynomial is also a monic polynomial with $\mathbf{Q}$-coefficients of degree 4 and irreducible over $\mathbf{Q}$; this is a cyclotomic polynomial of degree 4. Referring (2.14), we find $F_0(x)F_0(-x) = \Phi_8(x)$ or $\Phi_{12}(x)$. So we get $p_0(1-p_0) = 1/8$ or $3/16$, equivalently, $p_0 = (2-\sqrt{2})/4$, or $1/4$.

3.4.2 For $n = 2$

From the observation in Section 3.4.1, we only have to discuss when $p_0(1-p_0) = 1/8$, $3/16$ or $1/16$. Recall $Spec(S_p) = \{\pm 2\sqrt{p_0(1-p_0)}\}$.

Let us first assume that $p_0(1-p_0) = 1/8$, equivalently, $p_0 = (2-\sqrt{2})/4$. Then we have $Spec(S_p) = \{\pm 1/\sqrt{2}\}$, thus

$$Spec(U_{sz}) = \{\exp(2k\pi\sqrt{-1}/8);\ k = 1, 3, 5, 7\},$$

therefore we find U_{sz} has the period 8.

Let us secondly assume that $p_0(1-p_0) = 3/16$, equivalently, $p_0 = 1/4$. Then we have $Spec(S_p) = \{\pm\sqrt{3}/2\}$, thus

$$Spec(U_{sz}) = \{\exp(2k\pi\sqrt{-1}/12);\ k = 1, 5, 7, 11\},$$

therefore we find U_{sz} has the period 12.

Let us finally assume that $p_0(1-p_0) = 1/16$, equivalently, $p_0 = (2-\sqrt{3})/4$. Then we have $Spec(S_p) = \{\pm 1/2\}$, thus

$$Spec(U_{sz}) = \{\exp(2k\pi\sqrt{-1}/6);\ k = 1, 2, 4, 5\},$$

therefore we find U_{sz} has the period 6.

3.4.3 For $n = 2^2$

Also in this section, we only have to discuss when $p_0(1-p_0) = 1/8$, $3/16$ or $1/16$. Recall $Spec(S_p) = \{0, \pm 2\sqrt{p_0(1-p_0)}\}$. Then we have

$$Spec(U_{sz}) = \{\pm\sqrt{-1}\} \cup \{\exp(2k\pi\sqrt{-1}/8);\ k = 1, 3, 5, 7\}$$

if $p_0 = (2-\sqrt{2})/4$,

$$Spec(U_{sz}) = \{\pm\sqrt{-1}\} \cup \{\exp(2k\pi\sqrt{-1}/12);\ k = 1, 5, 7, 11\}$$

if $p_0 = 1/4$, and

$$Spec(U_{sz}) = \{\pm\sqrt{-1}\} \cup \{\exp(2k\pi\sqrt{-1}/6);\ k = 1, 2, 4, 5\}$$

if $p_0 = (2-\sqrt{3})/4$. Therefore we find U_{sz} has the period 8, 12 and 12, respectively.

3.4.4 For $n = 2^3$

Also in this section, we only have to discuss when $p_0(1-p_0) = 1/8$, $3/16$ or $1/16$. Recall

$$Spec(S_p) = \{\pm 2\sqrt{p_0(1-p_0)}/\sqrt{2}, 0, \pm 2\sqrt{p_0(1-p_0)}\}.$$

If $p_0 = (2-\sqrt{2})/4$, equivalently $p_0(1-p_0) = 1/8$, then

$$Spec(U_{sz}) = \{\exp(2k\pi\sqrt{-1}/6);\ k = 1,2,4,5\} \cup \{\pm\sqrt{-1}\} \cup \{\exp(2k\pi\sqrt{-1}/8);\ k = 1,3,5,7\};$$

thus we find U_{sz} has the period 24.

Let us consider the case where $p_0(1-p_0) = 3/16$ and $1/16$. From (3.10), all of the solutions of

$$F_1(x) = F_7(x) = x^2 - 2\sqrt{2p_0(1-p_0)}x + 1 \quad \text{and} \quad F_3(x) = F_5(x) = x^2 + 2\sqrt{2p_0(1-p_0)}x + 1 \tag{3.11}$$

must be roots of unity if U_{sz} is periodic. In these p_0, neither $F_1(x)$ nor $F_3(x)$ are with $\mathbf{Q}$-coefficients, but $F_1(x)F_3(x)$ becomes a monic polynomial with $\mathbf{Q}$-coefficients, which is irreducible over $\mathbf{Q}$:

$$F_1(x)F_3(x) = x^4 + 2(1 - 4p_0(1-p_0))x^2 + 1. \tag{3.12}$$

However, for $p_0(1-p_0) = 3/16$ or $1/16$, the polynomial in (3.12) is a monic polynomial having not $\mathbf{Z}$-coefficients, which contradicts Proposition 2.8.

3.4.5 For $n = 2^4$

From the observation in Section 3.4.4, we only have to discuss when $p_0(1-p_0) = 1/8$. Recall

$$Spec(S_p) = \left\{2\sqrt{p_0(1-p_0)}\cos\frac{k\pi}{8};\ k = 0,\dots,15\right\}.$$

Here we pay attention to the eigenvalues $2\sqrt{p_0(1-p_0)}\cos\frac{(2k-1)\pi}{8}$ for $k = 1,2,3,4$, and the equations

$$x^2 - 4\sqrt{p_0(1-p_0)}\cos\frac{(2k-1)\pi}{8}\cdot x + 1 = 0 \tag{3.13}$$

by (3.10). Let us first remark any solution of

$$x^2 - 2\cos\frac{(2k-1)\pi}{8}x + 1 = 0$$

is a primitive 16-th root of unity for each $k = 1,2,3,4$. Then

$$\begin{aligned}
\Phi_{16}(x) &= \prod_{k=1}^{4}\left(x - \exp\left(\frac{(2k-1)\pi\sqrt{-1}}{8}\right)\right)\left(x - \exp\left(\frac{-(2k-1)\pi\sqrt{-1}}{8}\right)\right) \\
&= \prod_{k=1}^{4}\left(x^2 - 2\cos\frac{(2k-1)\pi}{8}x + 1\right) \\
&= x^4\prod_{k=1}^{4}\left(\left(x + \frac{1}{x}\right) - 2\cos\frac{(2k-1)\pi}{8}\right).
\end{aligned} \tag{3.14}$$

On the other hand, we know

$$\Phi_{16}(x) = x^8 + 1 = x^4((x+1/x)^4 - 4(x+1/x)^2 + 2). \tag{3.15}$$

Combining (3.14) with (3.15), we obtain

$$\prod_{k=1}^{4}\left(X - 2\cos\frac{(2k-1)\pi}{8}\right) = X^4 - 4X^2 + 2. \tag{3.16}$$

Let us assume that, for $k = 1,2,3,4$, all of the solutions of equations in (3.13) are roots of unity. By using (3.16), we have

$$\begin{aligned}
&\prod_{k=1}^{4}\left(x^2 - 4\sqrt{p_0(1-p_0)}\cos\frac{(2k-1)\pi}{8}\cdot x + 1\right) \\
&\quad = (2\sqrt{p_0(1-p_0)}x)^4\prod_{k=1}^{4}\left(\left(\frac{1}{2\sqrt{p_0(1-p_0)}}\right)(x + 1/x) - 2\cos\frac{(2k-1)\pi}{8}\right)
\end{aligned}$$

$$= (2\sqrt{p_0(1-p_0)}x)^4\left(\left(\frac{1}{2\sqrt{p_0(1-p_0)}}(x+1/x)\right)^4 - 4\left(\frac{1}{2\sqrt{p_0(1-p_0)}}(x+1/x)\right)^2+2\right)$$

$$= x^8+4(1-4p_0(1-p_0))x^6+2(1-4p_0(1-p_0))(3-4p_0(1-p_0))x^4+4(1-4p_0(1-p_0))x^2+1. \qquad (3.17)$$

Putting $p_0(1-p_0)=1/8$ into (3.17), we have $x^8+2x^6+(5/2)x^4+2x^2+1$, which is a monic polynomial having not $\mathbf{Z}$-coefficients; this contradicts Proposition 2.8.

3.4.6 For odd prime integer *n*

In this section, we assume n is an odd prime integer. As is seen in Section 3.4.1, we only have to discuss when $p_0(1-p_0)=1/8$, $3/16$ or $1/16$. Recall

$$Spec(S_p)=\left\{2\sqrt{p_0(1-p_0)}\cos\frac{2k\pi}{n};\ k=0,\dots,n-1\right\}.$$

Moreover, by (3.10), assuming all of the solutions of the equations

$$x^2-4\sqrt{p_0(1-p_0)}\cos\frac{2k\pi}{n}\cdot x+1=0 \qquad (3.18)$$

are root of unity, we shall derive a contradiction.

First we set $\lambda_0=\exp(2\pi\sqrt{-1}/n)$. Since n is an odd prime integer, λ_0^k for each $k=1,\dots,n-1$ is a primitive n-th root of unity. In the same way in Section 3.4.5, let us see $\Phi_n(x)$: we know

$$\begin{aligned}\Phi_n(x)&=\sum_{j=0}^{n-1}x^j\\&=x^m\left((x+1/x)^m+(x+1/x)^{m-1}+\frac{3-n}{2}(x+1/x)^{m-2}+\sum_{j=0}^{m-3}a_j(x+1/x)^j\right),\end{aligned} \qquad (3.19)$$

where $m=(n-1)/2$ and a_j's are rational. On the other hand, we have

$$\begin{aligned}\Phi_n(x)&=\prod_{j=1}^{n-1}(x-\lambda_0^j)=\prod_{j=1}^{m}(x-\lambda_0^j)(x-\lambda_0^{-j})\\&=\prod_{j=1}^{m}\left(x^2-2\cos\frac{2j\pi}{n}x+1\right)\\&=x^m\prod_{j=1}^{m}\left(\left(x+\frac{1}{x}\right)-2\cos\frac{2j\pi}{n}\right).\end{aligned} \qquad (3.20)$$

Combining (3.19) with (3.20), we obtain

$$\prod_{j=1}^{m}\left(X-2\cos\frac{2j\pi}{n}\right)=X^m+X^{m-1}+\frac{3-n}{2}X^{m-2}+\sum_{j=0}^{m-3}a_jX^j. \qquad (3.21)$$

Let us put $G_n^+(x)=\prod_{j=1}^m(x^2-4\sqrt{p_0(1-p_0)}\cos\frac{2j\pi}{n}\cdot x+1)$. Then we have,

$$\begin{aligned}G_n^+(x)&=(2\sqrt{p_0(1-p_0)}x)^m\prod_{j=1}^{m}\left(\frac{1}{2\sqrt{p_0(1-p_0)}}\left(x+\frac{1}{x}\right)-2\cos\frac{2j\pi}{n}\right)\\&=(2\sqrt{p_0(1-p_0)}x)^m\left(X^m+X^{m-1}+\frac{3-n}{2}X^{m-2}+\sum_{j=0}^{m-3}a_jX^j\right),\end{aligned} \qquad (3.22)$$

where $X=\frac{1}{2\sqrt{p_0(1-p_0)}}(x+1/x)$. Noting that $(x+1/x)^k=\sum_{\ell=0}^k\binom{k}{\ell}x^{k-2\ell}$, we obtain the following expression:

$$\begin{aligned}G_n^+(x)=&\left(x^{n-1}+\left(\frac{n-1}{2}+2p_0(1-p_0)(3-n)\right)x^{n-3}+\sum_{j=2}^{(n-1)/2}b_jx^{n-1-2j}\right)\\&+2\sqrt{p_0(1-p_0)}\left(x^{n-2}+\sum_{j=2}^{(n-1)/2}d_jx^{n-2j}\right),\end{aligned} \qquad (3.23)$$

where b_j and d_j are rational. Since it is assumed every solutions μ of $G_n^+(x)$ is a root of unity, $-\mu$ becomes also a root of unity. In other words, if we set $G_n^-(x)=G_n^+(-x)$, all of the solutions of $G_n^-(x)=0$ are roots of unity. Let us now consider $G_n(x)=G_n^+(x)G_n^-(x)$; it is obvious that every solution of $G_n(x)=0$ is a root of unity. Then we have, from (3.23),

$$
\begin{aligned}
G_n(x) &= G_n^+(x)G_n^-(x) \\
&= \left(x^{n-1} + \left(\frac{n-1}{2} + 2p_0(1-p_0)(3-n)\right)x^{n-3} + \sum_{j=2}^{(n-1)/2} b_j x^{n-1-2j}\right)^2 \\
&\quad - 4p_0(1-p_0)\left(x^{n-2} + \sum_{j=2}^{(n-1)/2} d_j x^{n-2j}\right)^2 .
\end{aligned}
\tag{3.24}
$$

Consequently we have $G_n(x)$ is a monic polynomial with $\mathbf{Q}$-coefficients:

$$
G_n(x) = x^{2n-2} + (n-1+4(2-n)p_0(1-p_0))x^{2n-4} + \sum_{j=0}^{n-3} \tilde{a}_j x^{2j}, \tag{3.25}
$$

where $\tilde{a}_j$ is rational. Putting $p_0(1-p_0) = 1/8$, $3/16$ or $1/16$ into (3.25), we have $G_n(x)$, which is a monic polynomial having not $\mathbf{Z}$-coefficients; this contradicts Proposition 2.8. □

4. Further Remarks

In this paper, we give some characterization on the periodicity of the Szegedy walks on a special class of finite graphs. However one of the things that we want to do is to determine the periodicity of the Grover walks completely for general finite graphs. We discuss some strategy on such characterization. For a finite graph G with n vertices and m unoriented edges, let μ_i's be eigenvalues of T_p with counting each multiplicity, for $i = 1, \ldots, n$. In other word, μ_i's are the roots of $\det(xI_n - T_p) = 0$. If T_p is reversible, then all of the eigenvalues of U_{sz} inherited from T_p are the solutions of

$$
x^2 - 2\mu_i x + 1 = 0 \tag{4.1}
$$

by Theorem 2.1 and (2.2). Here let us consider the following monic polynomial $F(x) = \det(xI_{2m} - U_{sz})$, which can be expressed as

$$
F(x) = \prod_{i=1}^{n} (x^2 - 2\mu_i x + 1). \tag{4.2}
$$

Then we have

$$
F(x)/(2x)^n = \prod_{i=1}^{n} \left(\frac{1}{2}\left(x + \frac{1}{x}\right) - \mu_i\right), \tag{4.3}
$$

as is essentially seen in Sections 3.3 and 3.4. It is obvious that the polynomial of X,

$$
\prod_{i=1}^{n} (X - \mu_i), \tag{4.4}
$$

is the characteristic polynomial of T_p. For the Grover matrix U_{gr} induced by a simple random walk T_0, $F(x)$ is obviously a monic polynomial with $\mathbf{Q}$-coefficients by its definition (1.4); simultaneously, $F(x)$ is also characterized as above (4.3) with (4.4).

Let us recall Proposition 2.2, that is, all of the solutions of $F(x) = 0$ are roots of unity if and only if U_{sz} is periodic. In addition, it follows from the observation above and Proposition 2.8 that the above $F(x)$ has $\mathbf{Z}$-coefficients and can be factorized by cyclotomic polynomials if and only if the Grover matrix U_{gr} is periodic.

The following is an obvious conclusion:

Proposition 4.1. *Assume the Grover matrix U_{gr} is periodic. Then U_{gr} has a primitive n-th root of unity as an eigenvalue if and only if it has all of the primitive n-th roots of unity as its eigenvalues.* □

The following is one of simple necessary conditions for U_{gr} to be periodic. Now let λ_1 be the second largest eigenvalue of T_0, that is, there exists no eigenvalue in $(\lambda_1, 1)$. The value $1 - \lambda_1$ is often called the spectral gap. Many results on it are derived in terms of random walks and so on. For instance, refer to [4].

Proposition 4.2. *If the Grover matrix U_{gr} is periodic, then there exists a positive integer k such that $\lambda_1 = \cos(2\pi/k)$.*

Proof. Under the condition that U_{gr} is periodic, all of the solutions of $F(x) = 0$ are roots of unity. Hence $F(x)$ is a monic polynomial with $\mathbf{Z}$-coefficients and can be factorized by cyclotomic polynomials. Now the solutions of

$$
x^2 - 2\lambda_1 x + 1 = 0
$$

are roots of unity; they can be expressed by

$$
\exp(\pm 2\pi j\sqrt{-1}/m),
$$

where j, m are positive integers with $(j, m) = 1$. Equivalently, $\exp(\pm 2\pi j\sqrt{-1}/m)$ is the primitive m-th root of unity for some positive integer m and $\lambda_1 = \cos(2\pi j/m)$. Let us assume $j \neq 1$ or $m-1$ modulo m. By Proposition 4.1, we find U_{gr} has the eigenvalues $\exp(\pm 2\pi\sqrt{-1}/m)$, which implies T_0 has the eigenvalue $\cos(2\pi/m)$. This contradicts λ_1 is the second largest eigenvalue of T_0. □

We believe the periodicity of U_{gr} for more wider class of finite graphs can be characterized.

Problem 4.3 By using the above $F(x)$ with some fruitful results on the simple random walk T_0, determine the periodicity of U_{gr} for more wider class of finite graphs. In particular, determine all graphs for U_{gr} to be periodic.

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