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A Discrete Transmission Line Model for Discrete-time Quantum Walks

Yusuke FUKUSHIMA[1,*] and Tatsuaki WADA[2*]

[1]*Electrical and Electronic Engineering, Graduate School of Science and Engineering, Ibaraki University, 4-12-1 Nakanarusawa, Hitachi, Ibaraki 316-8511, Japan*
[2]*Department of Electrical and Electronic Engineering, Ibaraki University, 4-12-1 Nakanarusawa, Hitachi, Ibaraki 316-8511, Japan*

Based on the similarity between telegraph equation for transmission lines and Klein–Gordon equation, we have related a distributed element model in electrical engineering to a discrete-time quantum walk through Dirac equation. As a result, we have constructed a discrete transmission line model for a discrete-time quantum walk, and it enables us understanding the characteristics of quantum walks as those of the transmission line.

KEYWORDS: quantum walk, transmission line, telegraph equation, Klein–Gordon equation, Dirac equation

1. Introduction

Quantum walks are quantum analogs of classical random walks [1, 2]. In late years quantum computers come to be broadcast by the media and they have been attracted much attention. Quantum computer has a potential ability to perform some calculations significantly faster than the current computers, and it is expected as a next-generation computer. Quantum walk is a useful mathematical model for further developing the basic theory of the quantum computer. In addition quantum walks have been expected as powerful tools for the development of quantum algorithms. Quantum walk is often compared with classical random walk, and it is introduced as a quantum version of classical random walk. However, the behavior of persistent random walk is nearer to the behavior of quantum walk than that of random walk. For example, both in persistent random walk and in quantum walk walkers spread out to left and right ballistically depending on an initial state. In contrast, the behavior of random walk is not so. Persistent random walk also has a connection with the telegraph equation [3], which expresses the propagation of the electric voltage and current in transmission line. In addition there is the similarity between telegraph equation and the Klein–Gordon equation, which is related with a quantum walk [2]. Then, we can relate quantum walk to a model of transmission line.

In this contribution, we have related a quantum walk to a distributed element model for a transmission line, which is often used in electronics. In the next section we briefly explain the basics of random walks, persistent random walks, and quantum walks. Then Section 3 explains the characteristics of the transmission lines for a small loss (or high frequency) case and for a lossless case. In Section 4, we construct a discrete transmission line model, which is a complex equivalent circuit model, for discrete-time quantum walks. In Section 5, we show the characteristics of the complex equivalent circuit. Final Section 6 devoted to our conclusions.

2. Quantum Walk

One dimensional (1D) random walk is a mathematical stochastic model in which a particle (or a walker) moves to the right or to the left on 1D lattice sites. Let the probability of moving to the right as p and that of moving to the left as q, where p and q satisfy the condition of $p + q = 1$. The time evolution equation of the 1D random walk is described by

$$\rho^{t+1}(x) = p\,\rho^t(x+1) + q\,\rho^t(x-1), \tag{2.1}$$

where $\rho^t(x)$ is the probability of the walker at a position x and at a time step t.

We next explain 1D persistent random walk [3]. There are two walkers A and B. We write each probability of two walkers at a position x and at a time step t as $\rho^t_A(x)$ and $\rho^t_B(x)$, respectively. In each time step, one of the walkers A persistently goes to the right site with a probability p, and goes to the left site with a probability $q = 1 - p$. The other

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*Corresponding author. E-mail: 15nm617h@vc.ibaraki.ac.jp; tatsuaki.wada.to@vc.ibaraki.ac.jp

walker B persistently goes to the left site with a probability p, and goes to the right site with a probability $q = 1 - p$. Thus the difference equations for the probabilities $\rho_A^{t+1}(x)$ and $\rho_B^{t+1}(x)$ are:

$$\begin{cases} \rho_A^{t+1}(x) = p\rho_A^t(x-1) + q\rho_B^t(x-1), \\ \rho_B^{t+1}(x) = q\rho_A^t(x+1) + p\rho_B^t(x+1). \end{cases} \tag{2.2}$$

The probability of 1D persistent random walk $\rho_P^t(x)$ is defined as the sum of each probability:

$$\rho_P^t(x) = \rho_A^t(x) + \rho_B^t(x). \tag{2.3}$$

We next explain quantum walks. Two different kinds of quantum walks are proposed [2]: one is a continuous-time quantum walk; and the other is a discrete-time quantum walk (DTQW). For the simplicity we only consider a DTQW on the line, i.e., on 1D lattice. To describe the behavior of a DTQW, it is important to understand the probability amplitude, the rules of the time evolution and the probability. The probability amplitude $\psi(x,t)$ at a position x and a time step t is expressed as a two-component wavefunction. The time evolution of a DTQW is described by the state vector $|\psi(x,t)\rangle$. For a given initial state vector $|\psi(x,0)\rangle$, the probability amplitude evolves in one time step by applying a time-evolution operator $\hat{U}$ which consists of a quantum coin operator $\hat{C}$ and shift operator $\hat{S}$ as $\hat{U} = \hat{S} \otimes \hat{C}$. The time evolution of DTQW is expressed by

$$|\psi(x,t+1)\rangle = \hat{S} \otimes \hat{C}\,|\psi(x,t)\rangle, \tag{2.4}$$

with

$$|\psi(x,t)\rangle = \sum_{k=-\infty}^{+\infty} |k\rangle \otimes \begin{pmatrix} a_k^t \\ b_k^t \end{pmatrix}, \tag{2.5}$$

where $\hat{C}$ and $\hat{S}$ are 2×2 unitary matrices, and the position of k-th site is $x = k\Delta x$ with a suitable small distance Δx. Since the time evolution operator $\hat{U}$ is unitary, if we take an initial condition to be $|\psi(x,0)|^2 = 1$, the total probability is conserved $\|\psi(x,t)\|^2 = 1$ at all time step $t > 0$.
The quantum walker can be observed at position x at time t with a probability. The probability is defined in the square of the norm of the probability amplitude $\psi(x,t)$ at position x at time step t. There is big difference in characteristics in the time evolution of quantum walk and that of random walk. For example, the spreading width of a probability distribution for a quantum walker increases proportionally to the number of time step t, whereas that of a classical random walk is proportional to $\sqrt{t}$ [2].

3. Telegraph Equation

A distributed element (or transmission line) model represents the transmission line as the infinite series of a two-port elementary circuit, and each circuit represents an infinitesimally short segment of the transmission line [4] as shown in Fig. 1. The two-port circuit is characterized in terms of the following electrical elements: the distributed resistance $R[\Omega/\mathrm{m}]$ of the conductors (or wires) of the transmission line; the distributed inductance $L[\mathrm{H/m}]$ due to the magnetic field around the wires, self-inductance, etc.; the shunt capacitance $C[\mathrm{F/m}]$ between the two conductors; and and the conductance $G[\mathrm{S/m}]$ of the two conductors.

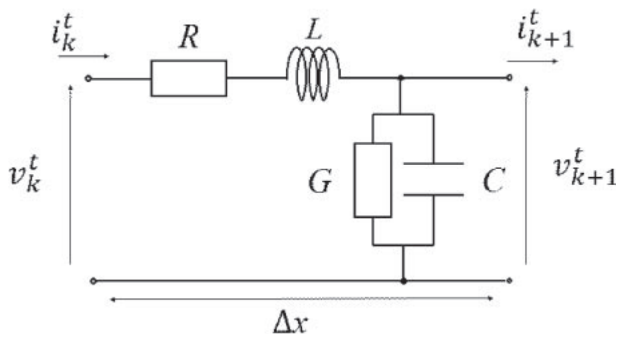

Fig. 1. A distributed element model of a transmission line.

By applying Kirchhoff's circuit laws we have

$$\begin{cases} v_k^t - v_{k+1}^t = R\,\Delta x\, i_k^t + L\,\Delta x\, \dfrac{\partial}{\partial t} i_k^t, \\ i_k^t - i_{k+1}^t = G\,\Delta x\, v_{k+1}^t + C\,\Delta x\, \dfrac{\partial}{\partial t} v_{k+1}^t. \end{cases} \tag{3.1}$$

where v_k^t is the electric voltage, i_k^t is the electric current at a time t at a position $x = k\Delta x$ with an integer k. Now introducing the forward difference operator Δ_x^+ acting a function $f(x)$ as

$$\Delta_x^+ f(x) \equiv \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{3.2}$$

and the backward difference operator Δ_x^- as

$$\Delta_x^- f(x) \equiv \frac{f(x) - f(x-\Delta x)}{\Delta x}. \tag{3.3}$$

By using these difference operators, Eq. (3.1) becomes

$$\begin{cases} -\Delta_x^+ v_k^t = Ri_k^t + L\dfrac{\partial}{\partial t} i_k^t, \\ -\Delta_x^- i_k^t = Gv_k^t + C\dfrac{\partial}{\partial t} v_k^t. \end{cases} \tag{3.4}$$

Then the telegraph equation (TE) for the voltage is written by

$$\Delta_x^- \Delta_x^+ v_k^t = RG\, v_k^t + (RC + LG)\frac{\partial}{\partial t} v_k^t + LC\frac{\partial^2}{\partial t^2} v_k^t, \tag{3.5}$$

and that for the current is

$$\Delta_x^- \Delta_x^+ i_k^t = RG\, i_k^t + (RC + LG)\frac{\partial}{\partial t} i_k^t + LC\frac{\partial^2}{\partial t^2} i_k^t. \tag{3.6}$$

For a sinusoidal wave with frequency ω, it is often convenient to analyze in frequency domain and we can replace the time differential operator $\frac{\partial}{\partial t}$ with $j\omega$. From Eq. (3.4) we have

$$\begin{cases} -\Delta_x^+ v_k^t = (R + j\omega L)\, i_k^t, \\ -\Delta_x^- i_k^t = (G + j\omega C)\, v_k^t. \end{cases} \tag{3.7}$$

Then by eliminating v_k^t or i_k^t we obtain the following difference equations:

$$\begin{cases} \Delta_x^- \Delta_x^+ v_k^t = (R + j\omega L)(G + j\omega C)\, v_k^t, \\ \Delta_x^- \Delta_x^+ i_k^t = (R + j\omega L)(G + j\omega C)\, i_k^t. \end{cases} \tag{3.8}$$

The general solutions of Eq. (3.8) are expressed in the following form:

$$v_k^t = V_1 \exp(-\gamma\, k\Delta x) + V_2 \exp(+\gamma\, k\Delta x), \tag{3.9}$$

$$i_k^t = \frac{V_1}{Z(\omega)} \exp(-\gamma\, k\Delta x) - \frac{V_2}{Z(\omega)} \exp(+\gamma\, k\Delta x), \tag{3.10}$$

where

$$Z(\omega) = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \tag{3.11}$$

is the characteristic impedance and

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{3.12}$$

is the propagation constant. The parameter V_1 and V_2 are determined according to the given boundary conditions. The term $V_1 \exp(-\gamma\, k\Delta x)$ describes an electromagnetic wave propagating along the forward direction along the x-axis, while $V_2 \exp(+\gamma\, k\Delta x)$ describes one propagating along the backward direction. The propagation constant γ is a complex quantity in general, and can be decomposed as the real and imaginary parts: $\gamma = \alpha + j\beta$, where α is called *attenuation constant*, β *the phase constant*.

When the conditions: $R \ll \omega L$; and $G \ll \omega C$ are satisfied, i.e., in a small loss or high frequency case, α and β are approximated as

$$\alpha \simeq \frac{1}{2}\omega\sqrt{LC}\left(\frac{RC + LG}{\omega LC}\right), \tag{3.13}$$

$$\beta \simeq \omega\sqrt{LC}. \tag{3.14}$$

For the case of a lossless ($R = G = 0$) transmission line, the propagation constant γ^0 is

$$\gamma^0 = j\beta = \omega\sqrt{LC}, \tag{3.15}$$

which is no dispersion. The corresponding phase velocity v_p^0 and group velocity v_g^0 are

$$v_p^0 = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}, \quad v_g^0 = \frac{\partial \omega}{\partial \beta} = v_p^0. \tag{3.16}$$

4. A Complex Equivalent Circuit

We here relate the distributed element model of a transmission line to DTQW through Dirac equation. We firstly relate the TE to Klein–Gordon (KG) equation [5, 6]. Secondly, it relates to Dirac equation. Finally, we relate the Dirac equation to DTQW. In this way we can relate the distributed element model to DTQW.

4.1 Relating the Telegraph Equation to Klein–Gordon Equation

The continuous version of Eq. (3.5) is

$$\frac{\partial^2}{\partial x^2} v(x,t) = RG\, v(x,t) + (RC + LG) \frac{\partial}{\partial t} v(x,t) + LC \frac{\partial^2}{\partial t^2} v(x,t). \tag{4.1}$$

On the other hand, KG equation for a wavefunction $\psi(x,t)$ is described by

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(x,t) = -\hbar^2 c^2 \frac{\partial^2}{\partial x^2} \psi(x,t) + m^2 c^4 \psi(x,t). \tag{4.2}$$

If in Eq. (4.1) the coefficient of the first-order time derivative is 0, Eq. (4.1) and Eq. (4.2) become the same form. Thus we can relate the TE to KG equation by requiring the following conditions:

$$RC + LG = 0; \quad c = \frac{1}{\sqrt{LC}}; \quad \text{and} \quad RG = \left(\frac{mc}{\hbar}\right)^2. \tag{4.3}$$

Note that in order to satisfy these conditions Eq. (4.3), R and G should be pure imaginary quantities with opposite signs, i.e., $R = -j|R|, G = j|G|$.

4.2 Relating Klein–Gordon Equation to Dirac Equation

A general partial differential equation which is first-order partial differential in both space and time is written by

$$\frac{\partial}{\partial t} \psi(x,t) = \left(\hat{\alpha}_x \frac{\partial}{\partial x} + \hat{\alpha}_y \frac{\partial}{\partial y} + \hat{\alpha}_z \frac{\partial}{\partial z}\right)\psi, \tag{4.4}$$

where $\hat{\alpha}_x, \hat{\alpha}_y$ and $\hat{\alpha}_z$ are operators to be determined. In order to relate this equation to Eq. (4.2), changing the coefficients as

$$j\hbar \frac{\partial}{\partial t} \psi = \left\{-\hbar^2 c^2 \left(\hat{\alpha}_x \frac{\partial}{\partial x} + \hat{\alpha}_y \frac{\partial}{\partial y} + \hat{\alpha}_z \frac{\partial}{\partial z}\right) + \hat{\beta} mc^2\right\}\psi, \tag{4.5}$$

where j stands for imaginary unit. We here consider only 1D case [7]. Taking out only operator parts in formula (4.5), and squaring the both sides, it follows

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \hat{\alpha}_z^2 \frac{\partial^2}{\partial z^2} - j\hbar mc^3 \left(\hat{\alpha}_z \hat{\beta} + \hat{\beta}\hat{\alpha}_z\right) \frac{\partial}{\partial z} + \hat{\beta}^2 m^2 c^4. \tag{4.6}$$

By comparing the coefficients of Eq. (4.2) and those of Eq. (4.6), we see that $\hat{\alpha}_z^2 = 1$, $\hat{\alpha}_z\hat{\beta} + \hat{\beta}\hat{\alpha}_z = 0$, $\hat{\beta}^2 = 1$ are seen to be necessary conditions. It is known that $\hat{\alpha}_z$ and $\hat{\beta}$ in Eq. (4.4) are as follows.

$$\hat{\alpha}_z = \begin{pmatrix} 0 & \hat{\sigma}_z \\ \hat{\sigma}_z & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix}, \tag{4.7}$$

where we used Pauli's matrices [8], $\hat{I}$ is unit 2×2 matrix, and $-j\hbar \frac{\partial}{\partial z}$ is expressed by the momentum operator $\hat{p}_z$. Arranging the first and third components of ψ as $\varphi = \binom{\psi_1}{\psi_3}$. In this way we obtain Dirac equation for a free particle [9]:

$$j\hbar \frac{\partial}{\partial t} \varphi(x,t) = (c\hat{\sigma}_x \hat{p}_z + \hat{\sigma}_z mc^2)\varphi(x,t). \tag{4.8}$$

4.3 Relating Dirac Equation to a Discrete-time Quantum Walk

The Hamiltonian of Eq. (4.8) is given by

$$H = c\hat{\sigma}_x \hat{p}_z + \hat{\sigma}_z mc^2. \tag{4.9}$$

After suitable rotations around the coordinate axises, this Hamiltonian can be converted as

$$H_D = c\hat{\sigma}_z \hat{p}_z + \hat{\sigma}_x mc^2, \tag{4.10}$$

and the corresponding Dirac equation is given by

$$j\hbar \frac{\partial}{\partial t}\varphi(x,t) = (c\hat{\sigma}_z\hat{p}_z + \hat{\sigma}_x mc^2)\varphi(x,t). \tag{4.11}$$

By applying Suzuki–Trotter approximation, the infinitesimal time-evolution operator $\hat{U}(\Delta t)$ is approximated as

$$\hat{U}(\Delta t) \equiv \exp\left(-j\frac{H_D}{\hbar}\Delta t\right)\exp\left(-jc\frac{\hat{\sigma}_z\hat{p}_z}{\hbar}\Delta t - j\frac{\hat{\sigma}_x mc^2}{\hbar}\Delta t\right) \simeq \exp\left(-jc\frac{\hat{\sigma}_z\hat{p}_z}{\hbar}\Delta t\right)\exp\left(-j\frac{\hat{\sigma}_x mc^2}{\hbar}\Delta t\right). \tag{4.12}$$

If $c\Delta t$ is replaced with the infinitesimal step Δx in spatial direction, and introducing a parameter

$$\theta = \frac{mc^2}{\hbar}\Delta t, \tag{4.13}$$

Eq. (4.12) becomes

$$\hat{U}(\Delta t) = \exp\left(-j\frac{\hat{\sigma}_z\hat{p}_z}{\hbar}\Delta x\right)\exp(-j\hat{\sigma}_x\theta). \tag{4.14}$$

Here the second exponential factor is rewritten as

$$\exp\big(-j\hat{\sigma}_x\theta\big) = \hat{I}\cos\theta - j\hat{\sigma}_x\sin\theta = \begin{pmatrix} \cos\theta & -j\sin\theta \\ -j\sin\theta & \cos\theta \end{pmatrix}, \tag{4.15}$$

which is nothing but a coin operator $C(\theta)$ in DTQW. For the first exponential factor in Eq. (4.14), we see from Euler's formula that

$$\begin{aligned} \exp\left(-j\frac{\hat{\sigma}_z\hat{p}_z}{\hbar}\Delta x\right) &= \left[\cos\left(\frac{\hat{p}_z}{\hbar}\Delta x\right) - j\hat{\sigma}_z\sin\left(\frac{\hat{p}_z}{\hbar}\Delta x\right)\right] \\ &= \frac{1}{2}\left[(\hat{I}+\hat{\sigma}_x)\exp\left(-j\frac{\hat{p}_z}{\hbar}\Delta x\right) + (\hat{I}-\hat{\sigma}_x)\exp\left(+j\frac{\hat{p}_z}{\hbar}\Delta x\right)\right]. \end{aligned} \tag{4.16}$$

Introducing the infinitesimal translation operator,

$$T^{\pm}(\Delta x) = \exp\left(\pm j\frac{\hat{p}_z}{\hbar}\Delta x\right), \tag{4.17}$$

we see that the infinitesimal time-evolution operator of Dirac equation is expressed as that of DTQW:

$$\hat{U}(\Delta t) = \begin{pmatrix} T^- & 0 \\ 0 & T^+ \end{pmatrix}\begin{pmatrix} \cos\theta & -j\sin\theta \\ -j\sin\theta & \cos\theta \end{pmatrix} = \hat{S}(\Delta x)\,\hat{C}(\theta). \tag{4.18}$$

4.4 Relating the Distributed Element Model to Dirac Equation

The continuous version of Eq. (3.1) are written as

$$\begin{cases} -\dfrac{\partial}{\partial x}v(x,t) = \left(R + L\dfrac{\partial}{\partial t}\right)i(x,t), \\ -\dfrac{\partial}{\partial x}i(x,t) = \left(G + C\dfrac{\partial}{\partial t}\right)v(x,t), \end{cases} \tag{4.19}$$

respectively. After some algebra they are rewritten as

$$\begin{cases} \dfrac{\partial}{\partial t}\sqrt{C}v(x,t) = -\dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}\sqrt{L}i(x,t) - \dfrac{G}{C}\sqrt{C}v(x,t), \\ \dfrac{\partial}{\partial t}\sqrt{L}i(x,t) = -\dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}\sqrt{C}v(x,t) - \dfrac{R}{L}\sqrt{C}i(x,t). \end{cases} \tag{4.20}$$

By using the condition of the $RC + LG = 0$ in Eq. (4.3), we introduce a new parameter δ

$$\delta \equiv \frac{R}{L} = -\frac{G}{C}. \tag{4.21}$$

Adding and subtracting both equations in Eq. (4.20) we have

$$\begin{cases} \dfrac{\partial}{\partial t}(\sqrt{C}v(x,t) + \sqrt{L}i(x,t)) = -\dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}(\sqrt{C}v(x,t) + \sqrt{L}i(x,t)) + \delta(\sqrt{C}v(x,t) - \sqrt{L}i(x,t)), \\ \dfrac{\partial}{\partial t}(\sqrt{C}v(x,t) - \sqrt{L}i(x,t)) = \dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}(\sqrt{C}v(x,t) - \sqrt{L}i(x,t)) + \delta(\sqrt{C}v(x,t) + \sqrt{L}i(x,t)). \end{cases} \tag{4.22}$$

These equations are rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix} = -\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix}$$
$$= \left(-\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\hat{\sigma}_z + \delta\hat{\sigma}_x\right)\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix}, \tag{4.23}$$

where we introduced the variables

$$a(x,t) = \sqrt{C}v(x,t) + \sqrt{L}i(x,t), \quad b(x,t) = \sqrt{C}v(x,t) - \sqrt{L}i(x,t). \tag{4.24}$$

By invoking to the same technique that is used in Sect. 4.2, the approximated solution of Eq. (4.23) can be cast into the following form.

$$\begin{pmatrix} a(x,t+\Delta t) \\ b(x,t+\Delta t) \end{pmatrix} = \begin{pmatrix} T^- & 0 \\ 0 & T^+ \end{pmatrix}\begin{pmatrix} \cos\theta & -j\sin\theta \\ -j\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix}, \tag{4.25}$$

where $T^{\pm}$ is the infinitesimal translation operator with $x \pm \Delta x$, $\Delta x = c\Delta t = \frac{\Delta t}{\sqrt{LC}}$, and $\theta = j\delta\Delta t = |\delta|\Delta t = \frac{|R|\Delta t}{L}$.

Equation (4.25) expresses the discrete transmission line model for discrete-time quantum walks. We call it the complex equivalent circuit (see Fig. 1.), because the value of the resistance R and conductance G are purely imaginary quantities.

5. The Characteristics of the Complex Equivalent Circuit

In the previous section, in order to relate the TE to DTQW, we have to choose the value of the resistance R and that of the conductance G in the distributed element model as a pure imaginary one. Since this is not the standard cases, we here show the characteristics, as a transmission line, of the complex equivalent circuit in which the resistance and conductance are pure imaginary. Recall that we imposed the conditions in Eq. (4.3), the propagation constant γ is expressed as

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = j\sqrt{LC}\sqrt{\left(\omega - j\frac{R}{L}\right)\left(\omega - j\frac{G}{C}\right)} = \frac{j}{v_p^0}\sqrt{(\omega - j\delta)(\omega + j\delta)} = \frac{j}{v_p^0}\sqrt{\omega^2+\delta^2}$$
$$= \frac{j}{v_p^0}\sqrt{\omega^2 - |\delta|^2}, \tag{5.1}$$

where we used the phase velocity v_p^0 in Eq. (3.16) of a lossless transmission line and the parameter δ defined in Eq. (4.21). Note that the parameter δ is pure imaginary. Since the time-evolution of a quantum walk is unitary, the total probability $\|\psi(t)\|^2$ is conserved at any time t. This means the attenuation constant α should be zero, i.e. $\gamma = j\beta$. This leads to the condition $\omega^2 > |\delta|^2$ and the corresponding phase constant β is

$$\beta(\omega) = \frac{1}{v_p^0}\sqrt{\omega^2 - |\delta|^2}. \tag{5.2}$$

Similarly the characteristic impedance Z is

$$Z(\omega) = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{R/L+j\omega}{G/C+j\omega}} = Z^0\sqrt{\frac{\omega+|\delta|}{\omega-|\delta|}}. \tag{5.3}$$

The phase velocity v_p and group velocity v_g are

$$v_p(\omega) = \frac{\omega}{\beta} = v_p^0\frac{\omega}{\sqrt{\omega^2-|\delta|^2}}, \quad v_g(\omega) = \frac{\partial\omega}{\partial\beta} = \frac{\sqrt{\omega^2-|\delta|^2}}{\omega}. \tag{5.4}$$

We see that these characteristic parameters of the complex equivalent circuit are frequency dependent. Note that in the standard situation ($R = G = 0$) of the lossless transmission line, the characteristic impedance Z^0 has no frequency dependency, i.e., $Z^0 = \sqrt{L/C}$. Therefore we consider some peculiar behaviors of quantum walks compared to those of classical random walks are due to these frequency dependent characteristics in the view from the complex equivalent circuit.

6. Conclusions

Based on the similarity between TE and KG equation, we have related DTQW to the distributed element model of transmission lines. The time-evolution equation of DTQW are related to those of the complex equivalent circuit of the transmission line. We have constructed the discrete transmission line model in Eq. (4.25). Consequently, the characteristics of the quantum walk can be understood as those of the equivalent circuit. Different from a typical transmission line which is often considered in basic electronics, the characteristics of the complex equivalent circuit have frequency dependency as shown in Section 5. In particular the characteristic impedance $Z(\omega)$ of the complex equivalent circuit depends on the frequency of the propagating waves. We thus expect that this property will explain some characteristic features of quantum walks. Further study is needed to confirm this.

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