博　士　論　文

Large Deviations for Continuous Additive Functionals of Symmetric Markov Processes

（対称マルコフ過程の連続加法汎関数に対する大偏差）

梁　承　煥

平成28年

Large Deviations for Continuous Additive Functionals of Symmetric Markov Processes

A thesis presented
by

Seunghwan YANG

to
The Mathematical Institute
for the degree of
Doctor of Science

Tohoku University
Sendai, Japan

March 2017

Acknowledgments

First of all, I would like to express my sincere gratitude to my supervisor Professor Masayoshi Takeda for his pertinent guidance, valuable advices and continuous encouragement throughout my master and doctoral programs. Without them, the thesis would never been completed.

I am deeply thankful to Shigeki Aida for his careful reading of the draft of this thesis and valuable advice. I would like to profoundly thank Professor Yuu Hariya for his important suggestions, helpful comments and constant encouragement. I wish to thank Professor Kaneharu Tsuchida (National Defense Academy) for his constructive comments.

I would like to thank to all the members of Probability Seminar in Tohoku University. I am grateful to all the members of Mathematical Institute of the Graduate School of Science in Tohoku University.

I am deeply thankful to Kazushige Shibata and all the members of Sendai rotary club for their encouragement.

Finally, I would like to express my appreciation to my parents, sister and wife, without their faith, support and encouragement, I could not have done my master and doctoral programs.

Contents

Chapter 1

Introduction

In [15], [18], a large deviation principle was proved for additive functionals of Brownian motion corresponding to Kato measures. In [21], they used the Gärtner-Ellis theorem to show a large deviation principle for additive functionals of symmetric α-stable processes. For applying the Gärtner-Ellis theorem, they proved the differentiability of logarithmic moment generating functions of the additive functional. A main objective of this paper is to extend these results in [15], [18] and [21] to more general symmetric Markov processes, especially in the case that the logarithmic moment generating function is not differentiable.

In [22], he established a sufficient condition for uniform large deviation principle. In [20], he proved the uniform large deviation principle for symmetric Markov processes under certain assumptions. Second object of this paper is to show the locally uniform lower bound of the large deviations for occupation times of symmetric Markov processes with finite life time by using the ground state transform.

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $\mathbf{M} = (P_x, X_t)$ be an irreducible, conservative, m-symmetric Markov process on E with the doubly Feller property. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(E; m)$ generated by $\mathbf{M}$. We assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and transient. Let μ be a positive Radon measure in the *Green-tight Kato class* (in notation $\mu \in \mathcal{K}_\infty$) and A_t^μ the positive continuous additive functional in the Revuz correspondence to μ.

We define

$$\gamma(\theta) := \inf \left\{ \mathcal{E}(u, u) : u \in \mathcal{D}(\mathcal{E}),\ \theta \int_E u^2 d\mu = 1 \right\}, \quad \theta \in \mathbb{R}^1. \tag{1.1}$$

Let θ_0 be a unique value such that $\gamma(\theta_0) = 1$. We define the functions $C(\theta)$ and $\widetilde{C}(\theta)$

by

$$C(\theta) = -\inf\left\{\mathcal{E}(u,u) - \theta\int_E u^2 d\mu : u \in C_0(E)\cap\mathcal{D}(\mathcal{E}),\ \int_E u^2 dm = 1\right\},$$

and

$$\widetilde{C}(\theta) = \begin{cases} C(\theta), & \theta \geq \theta_0 \\ 0, & \theta < \theta_0. \end{cases}$$

Here $C_0(E)$ is the space of continuous functions on E with compact support.
Let $I(\lambda)$ (resp. $\widetilde{I}(\lambda)$) be the Legendre transform of $C(\theta)$ (resp. $\widetilde{C}(\theta)$):

$$I(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - C(\theta)\} \quad \left(\text{resp. } \widetilde{I}(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - \widetilde{C}(\theta)\}\right), \quad \lambda \in \mathbb{R}^1.$$

Our main theorem is as follows:

Theorem 1.1. *Suppose* $\mathbf{M}$ *satisfies* **(I)**, **(DF)**, **(C)** *and* **(LU)** *below. Let* $\mu \in \mathcal{K}_\infty$. *Then*

(i) For any open set $G \subset \mathbb{R}^1$,

$$\liminf_{t\to\infty}\frac{1}{t}\log P_x\left(\frac{A_t^\mu}{t} \in G\right) \geq -\inf_{\lambda\in G} I(\lambda).$$

(ii) For any closed set $K \subset \mathbb{R}^1$,

$$\limsup_{t\to\infty}\frac{1}{t}\log P_x\left(\frac{A_t^\mu}{t} \in K\right) \leq -\inf_{\lambda\in K} \widetilde{I}(\lambda).$$

We can show that I equals $\widetilde{I}$ on $[C'(\theta_0+),\infty)$, where $C'(\theta_0+) = \lim_{\epsilon\downarrow 0} C'(\theta_0+\epsilon)$. As a corollary of Theorem 1.1, for $A \subset [C'(\theta_0+),\infty)$ with $\inf_{\lambda\in A^\circ} I(\lambda) = \inf_{\lambda\in\bar{A}} I(\lambda)$,

$$\lim_{t\to\infty}\frac{1}{t}\log P_x\left(\frac{A_t^\mu}{t} \in A\right) = -\inf_{\lambda\in A} I(\lambda).$$

In particular, if $C = \widetilde{C}$, that is, $C(\theta) = 0$ for $\theta \leq \theta_0$, then the large deviation principle for A_t^μ/t holds.

In [18], [21], they showed that C equals $\widetilde{C}$ for the Brownian motion or α-stable process. In general, C does not equal $\widetilde{C}$ when $C(0) < 0$([19, Theorem 3.1 (ii)]). Hence Theorem 1.1 turn out to be an extension of the result in [21].

In the proof of the large deviation principle for A_t^μ, we also use the Gärtner-Ellis Theorem. The function $\widetilde{C}(\theta)$ is regarded as the logarithmic moment generating function of A_t^μ. In the Gärtner-Ellis theorem, the differentiability of logarithmic moment generating functions is a sufficient condition for obtaining the lower bound.

Needless to say, it is impossible to show the differentiability for continuous additive functionals of general symmetric Markov processes. Indeed, if $\theta_0 > 0$ and $C(0) < 0$, then the right derivative of $\widetilde{C}$ at $\theta = \theta_0$ is positive because it is equal to $C'(\theta_0)$ and $\widetilde{C}(\theta)$ is convex, but the left derivative is 0. Therefore, the logarithmic moment generating function $\widetilde{C}(\theta)$ is not differentiable at θ_0.

We prove first the lower bound for the absorbing symmetric Markov process $\mathbf{M}^G$ on a relatively compact open set $G \subset E$. For $\theta \in \mathbb{R}^1$, let

$$C^G(\theta) = -\inf\left\{\mathcal{E}^{\theta\mu,G}(u,u) : u \in \mathcal{D}(\mathcal{E}^G), \int_G u^2 dm = 1\right\},$$

where $\mathcal{D}(\mathcal{E}^G) = \{u \in \mathcal{D}(\mathcal{E}) : \ u = 0 \text{ q.e. on } E \setminus G\}$. Here $\mathcal{E}^{\theta\mu,G}$ is the Schrödinger form on G defined in (3.1). Combining the local ultra-contractivity with the analytic perturbation theory, we can obtain that $C^G(\theta)$ is an analytic function in θ. Applying the Gärtner-Ellis theorem, we can show the lower bound for absorbing symmetric Markov process $\mathbf{M}^G$. Then by approximating E by G_n, where $\{G_n\}$ is an increasing sequence of relatively compact open sets with $\bigcup_{n=1}^{\infty} G_n = E$, we obtain the lower bound for the Markov process $\mathbf{M}$ on the whole space E.

On the other hand, to show the upper bound, we use two facts, L^p-independence of spectral bounds of Feynman-Kac semigroups and gaugeability for Schrödinger type operator. We show by the L^p-indepencence that for $\theta \geq \theta_0$ the logarithmic moment generating function of A^μ exists and equals $\widetilde{C}$, and by the gaugeability that for $\theta \leq \theta_0$ it equals 0. Hence, applying Gärtner-Ellis theorem, we have the upper bound. In appendix 5.1 and 5.2, we precisely treat the L^p-independence and the gaugeability, respectively.

From above results, we find different rate functions between for the upper bound and for the lower bound and see that the two rate functions coincide on a certain interval.

Finally, we treat the 1-dimensional Brownian motion (P_x^k, X_t) with a positive drift k as an example. At this time, (P_x^k, X_t) satisfies the assumptions in Theorem 1.1 . We can choose the Dirac measure δ_0 at 0 as a positive Radon measure in the Green-tight Kato class. Then the local time l_t of the Brownian motion (P_x^k, X_t) at the origin is the continuous additive functional in the Revuz correspondence to δ_0. Let $\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + k\frac{d}{dx}$ be the infinitesimal generator of (P_x^k, X_t). Then $\mathcal{L}^{\delta_0} := \mathcal{L} + \delta_0$ is a self-adjoint operator on $L^2(\mathbb{R}, e^{2kx}dx)$. Since $C(\theta)$ is equal to the bottom of spectrum of $\mathcal{L}^{\delta_0}$, $C(\theta)$ is negative on $\theta < k$. Therefore we can see that $C(\theta) \neq \widetilde{C}(\theta)$ on $\theta < k$, and hence $I(\lambda) \neq \widetilde{I}(\lambda)$ on $0 \leq \lambda < k$. In particular, for $A \subset [k, \infty)$ with

$\inf_{\lambda \in A^\circ} I(\lambda) = \inf_{\lambda \in \bar{A}} I(\lambda)$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log P_x^k \left(\frac{l_t}{t} \in A \right) = - \inf_{\lambda \in A} I(\lambda).$$

In [20], the uniform large deviation principle for a symmetric Markov processes is proved under certain assumptions. In Chapter 4, we study the conditions for satisfying the uniform large deviation principle for a symmetric Markov processes. As an application, we prove the locally uniform lower bound of the large deviations for occupation times of symmetric Markov processes with finite life time. For the proof of this fact, the ground state transform plays a crucial role. We further consider the large deviation principle for symmetric Markov processes conditioned on non-absorption up to $t > 0$.

This paper is organized as follow. After giving preliminaries in Chapter 2, we shall prove a large deviation principle for the positive continuous additive functional A_t^μ in the Revuz correspondence with μ in the Green-tight Kato class in Chapter 3. We shall give an example for our theorem to the 1-dimensional Brownian motion with a positive drift k in Section 3.2. As mentioned above, in Chapter 4, we study the uniform large deviation principle for symmetric Markov processes with finite life time. Finally, in Appendix 5.1, 5.2 and 5.3, we check the L^p-independence, the gaugeability and a property of Legendre transform.

Chapter 2

Preliminaries

2.1 The Gärtner-Ellis theorem

The large deviation principle characterizes the limiting behavior, as $\epsilon \to 0$, of a family of probability measures $\{\mu_\epsilon\}$ on $(E, \mathscr{B})$ in terms of a *rate function*. This characterization is via asymptotic upper and lower exponential bounds on the values that μ_ϵ assigns to measurable subsets of E. Throughout E is a topological space so that open and closed subsets of E are well-defined, and the simplest situation is when elements of $\mathscr{B}_E$, the Borel σ-field on E, are of interest. To reduce possible measurability questions, all probability spaces in this paper are assumed to have been completed, and, with some abuse of notations, $\mathscr{B}_E$ always denotes the thus completed Borel σ-field.

Definition 2.1. A rate function I is a lower semicontinuous mapping $I : E \to [0, \infty]$(such that for all $\alpha \in [0, \infty)$, the level set $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ is a closed subset of E). A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subsets of E. The effective domain of I, denoted $\mathcal{D}_I$, is the set of points in E of finite rate, namely, $\mathcal{D}_I := \{x : I(x) < \infty\}$. When no confusion occurs, we refer to $\mathcal{D}_I$ as the domain of I.

In our case, since E is a metric space, the lower semicontinuity property may be checked on sequences, i.e., I is lower semicontinuous if and only if $\liminf_{x_n \to x} I(x_n) \geq I(x)$ for all $x \in E$. A consequence of a rate function being good is that its infimum is achieved over closed sets.

For any set Γ, $\overline{\Gamma}$ denotes the closure of Γ, Γ^o the interior of Γ, and Γ^c the complement of Γ. The infimum of a function over an empty set is interpreted as ∞.

Definition 2.2. $\{\mu_\epsilon\}$ satisfies the large deviation principle with a rate function I if, for all $\Gamma \in \mathscr{B}$,

$$-\inf_{x\in\Gamma^o} I(x) \le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \le \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \le -\inf_{x\in\overline{\Gamma}} I(x). \tag{2.1}$$

The right- and left-hand sides of (2.1) are referred to as the upper and lower bounds, respectively.

When $\mathscr{B}_X \subset \mathscr{B}$, the large deviation principle is equivalent to the following bounds:

(i) (Upper bound) For any closed set $F \subseteq E$,

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F) \le -\inf_{x\in F} I(x). \tag{2.2}$$

(ii) (Lower bound) For any open set $G \subseteq E$,

$$\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G) \ge -\inf_{x\in G} I(x). \tag{2.3}$$

Consider a sequence of random vectors $Z_n \in \mathbb{R}^d$, where Z_n possesses the law μ_n and logarithmic moment generating function

$$\Lambda_n(\lambda) := \log E\left(e^{\langle\lambda, Z_n\rangle}\right). \tag{2.4}$$

The existence of a limit of properly scaled logarithmic moment generating functions indicates that μ_n may satisfy the large deviation principle. Specifically, the following assumption is imposed throughout this section.

Assumption 2.3. *For each $\lambda \in \mathbb{R}^d$, the logarithmic moment generating function, defined as the limit*

$$\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n}\Lambda_n(n\lambda)$$

exists as an extended real number. Further, the origin belongs to the interior of $\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < \infty\}$.

Let $\Lambda^*(\cdot)$ be the Fenchel-Legendre transform of $\Lambda(\cdot)$, that is,

$$\Lambda^*(\cdot) := \sup_{\lambda\in\mathbb{R}^d}\{\langle\lambda, x\rangle - \Lambda(\lambda)\},$$

with $\mathcal{D}_{\Lambda^*} = \{x \in \mathbb{R}^d : \Lambda^*(x) < \infty\}$. It is our goal to state conditions under which the sequence μ_n satisfies the large deviation principle with the rate function Λ^*.

Definition 2.4. $y \in \mathbb{R}^d$ is an exposed point of Λ^* if for some $\lambda \in \mathbb{R}^d$ and all $x \neq y$,

$$\langle \lambda, y \rangle - \Lambda^*(y) > \langle \lambda, x \rangle - \Lambda^*(x)$$

this λ is called an exposing hyperplane.

Definition 2.5. A convex function $\Lambda : \mathbb{R}^d \to (-\infty, \infty]$ is essentially smooth if:

(i) $\mathcal{D}_\Lambda^o$ is non-empty.

(ii) $\Lambda(\cdot)$ is differentiable throughout $\mathcal{D}_\Lambda^o$.

(iii) $\Lambda(\cdot)$ is steep, namely, $\lim_{n\to\infty} |\nabla\Lambda(\lambda_n)| = \infty$ whenever $\{\lambda\}$ is a sequence in $\mathcal{D}_\Lambda^o$ converging to a boundary point of $\mathcal{D}_\Lambda^o$.

Theorem 2.6 (The Gärtner-Ellis Theorem)**.** *Assumption 2.3 hold.*

(i) For any closed set F,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(F) \leq - \inf_{x\in F} \Lambda^*(x). \tag{2.5}$$

(ii) For any open set G,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq - \inf_{x\in G\cap\mathcal{F}} \Lambda^*(x), \tag{2.6}$$

where $\mathcal{F}$ is the set of exposed points of Λ^* whose exposing hyperplane belongs to $\mathcal{D}_\Lambda^o$.

(iii) If Λ is an essentially smooth, lower semicontinuous functions, then the large deviation holds with the good rate function $\Lambda^*(\cdot)$.

2.2 Symmetric Markov processes and Dirichlet forms

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be an m-symmetric regular irreducible Dirichlet form on $L^2(E; m)$. It is known that a regular Dirichlet form $\mathcal{E}$ has the Beurling-Deny decomposition ([10, Theorem 3.2.1]) : for $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}(u,u) = \frac{1}{2}\int_E d\mu^c_{\langle u\rangle} + \iint_{E\times E\setminus diag} (u(x)-u(y))^2 J(dxdy) + \int_E u^2 dk. \tag{2.7}$$

Here $\mu^c_{\langle u\rangle}$, J and k are the energy measure of the strongly local part, the jumping measure and the killing measure with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, respectively.

We assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient, that is, there exists a strictly positive, bounded function $g \in L^1(E; m)$ such that for $u \in \mathcal{D}(\mathcal{E})$

$$\int_E |u| g dm \leq \sqrt{\mathcal{E}(u, u)}$$

(cf. [10, p.40]).

We denote by $u \in \mathcal{D}_{loc}(\mathcal{E})$ if for any relatively compact open set D there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. We denote by $\mathcal{D}_e(\mathcal{E})$ the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $\mathbf{M} = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \{P_x\}_{x\in X}, \{X_t\}_{t\geq 0}, \zeta)$ be the m-symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the augmented filtration and ζ is the lifetime of $\mathbf{M}$. Denote by $\{p_t\}_{t\geq 0}$ and $\{G_\alpha\}_{\alpha\geq 0}$ the semigroup and resolvent of $\mathbf{M}$:

$$p_t f(x) = E_x(f(X_t)), \qquad G_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) dt.$$

Suppose that H is semibounded self-adjoint operator on $L^2(D)$ with D being a domain in $\mathbb{R}^d$ and that e^{Ht} is an irreducible positivity-preserving semigroup with integral kernel $a(t, x, y)$. We assume that the top of the spectrum λ_1 of H is an eigenvalue. In this case, λ_1 has multiplicity one and the corresponding eigenfunction ϕ_0, normalized by $\|\phi_0\|_2 = 1$, is positive almost everywhere on D. ϕ_0 is called the ground state of H.

We now define the unitary operator U from $L^2(D, \phi_0^2(x)dx)$ to $L^2(D)$ by $Uf = \phi_0 f$ and define $\widetilde{H}$ on $L^2(D, \phi_1^2(x)dx)$ by

$$\widetilde{H} = U^{-1}(H - \lambda_1)U.$$

Then $e^{\widetilde{H}t}$ is an irreducible symmetric Markov semigroup on $L^2(D, \phi_0^2(x)dx)$ whose integral kernel with respect to the measure $\phi_0^2(x)dx$ is given by

$$\frac{e^{-\lambda t} a(t, x, y)}{\phi_0(x)\phi_0(y)}.$$

Definition 2.7. H is said to be ultracontrctive if e^{Ht} is a bounded operator from $L^2(D)$ to $L^\infty(D)$ for all $t > 0$. H is said to be intrinsically ultrative if $\widetilde{H}$ is ultracontractive; that is $e^{\widetilde{H}t}$ is a bounded operator from $L^2(D, \phi_0^2(x)dx)$ to $L^\infty(D, \phi_0^2(x)dx)$ for all $t > 0$.

We assume that $\mathbf{M}$ satisfies the next conditions:

Irreducibility (I). If a Borel set A is p_t-invariant, i.e., $p_t(1_A f)(x) = 1_A p_t f(x)$ m-a.e. for any $f \in L^2(E;m) \cap \mathscr{B}_b(E)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(E \setminus A) = 0$. Here $\mathscr{B}_b(E)$ is the space of bounded Borel functions on E.

Conservativeness (C). $P_x(\zeta = \infty) = 1$ for each $x \in E$.

Doubly Feller Property (DF). For each $t > 0$, $p_t(C_\infty(E)) \subset C_\infty(E)$, $\lim_{t \to 0} \|p_t f - f\|_\infty = 0$ for any $f \in C_\infty(E)$ and $p_t(\mathscr{B}_b(E)) \subset C_b(E)$, where $C_\infty(E)$ (resp. $C_b(E)$) is the space of continuous functions on E vanishing at infinity (resp. the space of bounded continuous functions on E).

Local Ultra-contractivity (LU). Let $\{p_t^G\}$ be the semigroup defined by $p_t^G f(x) = E_x(f(X_t); t < \tau_G)$ for any $f \in \mathscr{B}_b(E)$, where τ_G is the first exit time from G. Then for any relatively compact open set G, the semigroup $\{p_t^G\}$ is ultra-contractive, $\|p_t^G f\|_\infty \leq C(t)\|f\|_1$, where $C(t)$ is the operator norm $\|p^G\|_{1,\infty}$ of p_t^G from $L^1(G;m)$ to $L^\infty(G;m)$.

We remark that **(DF)** implies

Absolute Continuity Condition (AC). The transition probability of $\mathbf{M}$ is absolutely continuous with respect to m, $p(t,x,dy) = p(t,x,y)m(dy)$ for each $t > 0$ and $x \in E$.

Under **(AC)**, there exists a non-negative, jointly measurable α-resolvent kernel $G_\alpha(x,y)$ on $E \times E$:

$$G_\alpha f(x) = \int_E G_\alpha(x,y) f(y) m(dy), \ x \in E, \ f \in \mathscr{B}_b(E).$$

Moreover, $G_\alpha(x,y)$ is α-excessive in x and in y ([10, Lemma 4.2.4]). We simply write $G(x,y)$ for $G_0(x,y)$. For a measure μ, we define the α-potential of μ by

$$G_\alpha \mu(x) = \int_E G_\alpha(x,y)\mu(dy).$$

We define the *(1-)capacity* Cap associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as follows: for an open set $O \subset E$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{D}(\mathcal{E}), u \geq 1, \ m\text{-a.e. on } O\},$$

where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)_m$, for a Borel set $A \subset E$,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) : O \text{ is open}, \ O \supset A\}.$$

A statement depending on $x \in E$ is said to hold q.e. on E if there exists a set $N \subset E$ of zero capacity such that the statement is true for every $x \in E \setminus N$. The notation "q.e." is an abbreviation of "quasi-everywhere". A real valued function u defined q.e. on E is said to be *quasi-continuous* if for any $\epsilon > 0$ there exists an open set $G \subset E$ such that $\mathrm{Cap}(G) < \epsilon$ and $u|_{E \setminus G}$ is finite and continuous. Here, $u|_{E\setminus G}$ denotes the restriction of u to $E \setminus G$. It is known that each function u in $\mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$, that is, $u = \tilde{u}$ m-a.e.([10, Theorem 2.1.7]). In the sequel, we always assume that every function $u \in \mathcal{D}_e(\mathcal{E})$ is represented by its quasi-continuous version.

Let S_{00} be the set of positive Borel measures μ such that $\mu(E) < \infty$ and $G_1\mu$ is bounded. We call a Borel measure μ on E *smooth* if there exists a sequence $\{E_n\}$ of Borel sets increasing to E such that $1_{E_n} \cdot \mu \in S_{00}$ for each n and

$$P_x(\lim_{n\to\infty} \sigma_{E\setminus E_n} \geq \zeta) = 1, \quad \forall x \in E.$$

Here $\sigma_{E\setminus E_n}$ is the hitting time of $E \setminus E_n$ by $\mathbf{M}$, $\sigma_{E\setminus E_n} = \inf\{t > 0 : X_t \in E \setminus E_n\}$. We denote by S the set of positive smooth Borel measures. In [10], a measure in S is called a *smooth measure in the strict sense*. Here we omit the adjective phrase "in the strict sense" .

A stochastic process $\{A_t\}_{t\geq 0}$ is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a set $\Lambda \in \mathscr{F}_\infty = \sigma\left(\cup_{t\geq 0}\mathscr{F}_t\right)$ such that $P_x(\Lambda) = 1$, for all $x \in E$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous and has the left limit on $[0, \zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation). The set of all PCAF's is denoted by $\mathbf{A}_c^+$. The family S and $\mathbf{A}_c^+$ are in one-to-one correspondence (Revuz correspondence) as follows: for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$such that for any $f \in \mathscr{B}^+(E)$ and γ-excessive function h,

$$\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_E f(x)h(x)\mu(dx) \tag{2.8}$$

([10, Theorem 5.1.7]). Here, $E_{h\cdot m}(\,\cdot\,) = \int_X E_x(\,\cdot\,)h(x)m(dx)$. We denote by A_t^μ the PCAF in the Revuz correspondence with μ.

We define some classes of smooth measures.

Definition 2.8. Suppose that $\mu \in S$ is a positive Radon measure.

(1) A measure μ is said to be in the *Kato class* of $\mathbf{M}$ ($\mathcal{K}$ in abbreviation) if

$$\lim_{\alpha\to\infty} \|G_\alpha\mu\|_\infty = 0.$$

A measure μ is said to be in the *local Kato class* of $\mathbf{M}$ ($\mathcal{K}_{loc}$ in abbreviation) if $1_K \cdot \mu \in \mathcal{K}$ for any relatively compact open set K. Here 1_K is the indicator function of K.

(2) A measure μ is said to be in the class $\mathcal{K}_\infty$ if $\mu \in \mathcal{K}$ and for any $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$

$$\sup_{x\in E} \int_{K^c} G(x,y)\mu(dy) < \epsilon.$$

A measure μ in $\mathcal{K}_\infty$ is called *Green-tight*.

We note that every measure treated in this paper is supposed to be Radon. Thus we see from [1, Theorem 3.9] that $\mu \in \mathcal{K}$ if and only if

$$\lim_{t\downarrow 0}\sup_{x\in E} E_x(A_t^\mu) = \lim_{t\downarrow 0}\sup_{x\in E}\int_0^t\int_E p(s,x,y)\mu(dy)ds = 0. \tag{2.9}$$

Chen [2] defined the Green-tight class in slightly different way, however two definitions are equivalent under the strong Feller property ([13, Lemma 4.1]). We see from [17] that for $\alpha \geq 0$ and $\mu \in \mathcal{K}$

$$\int_E u^2 d\mu \leq \|G_\alpha\mu\|_\infty \cdot \mathcal{E}_\alpha(u,u) \quad \text{for any } u \in \mathcal{D}(\mathcal{E}). \tag{2.10}$$

Let $\mu \in \mathcal{K}$. We define the Schrödinger form by

$$\begin{cases} \mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \displaystyle\int_E u^2 d\mu \\ \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}). \end{cases} \tag{2.11}$$

We denote by $\mathcal{L}^\mu = \mathcal{L} + \mu$ the self-adjoint operator associated with the closed symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, that is, $(-\mathcal{L}^\mu u, v)_m = \mathcal{E}^\mu(u,v)$ for any $u, v \in \mathcal{D}(\mathcal{E})$.

We define the *Feynman-Kac semigroup* $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = E_x(\exp(A_t^\mu) f(X_t)), \quad x \in E,\ f \in \mathscr{B}_b(E).$$

The next two inequalities are versions of the inequality (2.10), which plays a crucial role in chapters below.

Theorem 2.9. *([17]) Let $\mu \in \mathcal{K}$. For any $\epsilon > 0$ there exists $M(\epsilon) > 0$ such that for any $u \in \mathcal{D}(\mathcal{E})$*

$$\int_E u^2 d\mu \leq \epsilon\mathcal{E}(u,u) + M(\epsilon)\int_E u^2 dm.$$

Theorem 2.9 follows from the next theorem and the definition of Kato measures.

Theorem 2.10. *([17]) Let $\mu \in \mathcal{K}_\infty$. Then for any $u \in \mathcal{D}(\mathcal{E})$*

$$\int_E u^2 d\mu \leq \|G\mu\|_\infty \cdot \mathcal{E}(u,u).$$

Proof of Theorem 2.9. The inequality in Theorem 2.9 follows from Theorem 2.10. Indeed, for $\alpha \geq 0$ and $\mu \in \mathcal{K}$,

$$\begin{aligned} \int_E u^2 d\mu &\leq \|G_\alpha\mu\|_\infty \cdot \mathcal{E}_\alpha(u,u) \\ &= \|G_\alpha\mu\|_\infty \cdot \mathcal{E}_(u,u) + \alpha\|G_\alpha\mu\|_\infty \cdot (u,u). \end{aligned}$$

From the definition of Kato measures, we can choose $\epsilon > 0$ so that $\|G_\alpha\mu\|_\infty < \epsilon$ and put $M(\epsilon) = \alpha\|G_\alpha\mu\|_\infty$. $\square$

Chapter 3

Large deviation principle (LDP)

3.1 LDP for additive functionals

Let $G \subset E$ be a relatively compact open set. We set

$$\mathcal{D}(\mathcal{E}^G) = \{u \in \mathcal{D}(\mathcal{E}) : \ u = 0 \text{ q.e. on } E \setminus G\}.$$

Here $\mathcal{E}^G$ is the part of the Dirichlet form $\mathcal{E}$ on G. $\mathcal{D}(\mathcal{E}^G)$ is a closed subspace of the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. It is known that $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$ is a regular Dirichlet form on $L^2(G; m)$. Let $\mathbf{M}^G$ be the associated Markov process of $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$, namely, the part process of $\mathbf{M}$ on G ([10, A.2]). Indeed, $\mathbf{M}^G$ is an absorbing Markov process on G with an m-symmetric transition function p_t^G on $(G, \mathcal{B}(G))$ defined by $p_t^G(x, B) = P_x(X_t \in B; t < \tau_G)$, where τ_G is the first exit time of G.

For $\theta \in \mathbb{R}^1$ define

$$\mathcal{E}^{\theta\mu, G}(u, u) = \mathcal{E}^G(u, u) - \theta \int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^G) \tag{3.1}$$

and

$$C^G(\theta) = -\inf \left\{ \mathcal{E}^{\theta\mu, G}(u, u) : u \in \mathcal{D}(\mathcal{E}^G), \int_G u^2 dm = 1 \right\}. \tag{3.2}$$

Let I^G be the Legendre transform of C^G:

$$I^G(\lambda) = \sup_{\theta \in \mathbb{R}^1} \left\{ \lambda\theta - C^G(\theta) \right\}, \quad \lambda \in \mathbb{R}^1.$$

Lemma 3.1. *For $u_1, \ u_2 \in \mathcal{D}(\mathcal{E})$ and $0 \le \alpha \le 1$, $u := \sqrt{\alpha u_1^2 + (1-\alpha) u_2^2} \in \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(u, u) \le \alpha \mathcal{E}(u_1, u_1) + (1-\alpha)\mathcal{E}(u_2, u_2).$$

Proof. First, we consider the energy measure of the strongly local part of (2.7).

It follows from Theorem 5.6.2 in [10] that for any $\Phi \in C^1(\mathbb{R}^d)$ with $\Phi(0) = 0$ and $v_1, \ldots, v_d \in \mathcal{D}(\mathcal{E})_b := \mathcal{D}(\mathcal{E}) \cap L^\infty(E; m)$, the composite function $\Phi(v) = \Phi(v_1, \ldots, v_d)$ is in $\mathcal{D}(\mathcal{E})_b$ and

$$d\mu^c_{\langle \Phi(v), w\rangle} = \sum_{i=1}^d \Phi_{x_i}(v) d\mu^c_{\langle v_i, w\rangle}, \quad \text{for any } \omega \in \mathcal{D}(\mathcal{E})_b,$$

where Φ_{x_i} is the partial derivative of Φ with respect to x_i. We call the formula above the derivation property of $\mu^c_{<u,v>}$.

By applying the formula above to $x = (x_1, x_2)$ and $\Phi(x) = \sqrt{\alpha x_1^2 + (1-\alpha) x_2^2}$, we have for $u = \Phi(u_1, u_2)$, $u_1, u_2 \in \mathcal{D}(\mathcal{E})_b$

$$d\mu^c_{\langle u\rangle} = \frac{\alpha^2 u_1^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1\rangle} + 2\frac{\alpha(1-\alpha) u_1 u_2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1, u_2\rangle} + \frac{(1-\alpha)^2 u_2^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_2\rangle}.$$

Since

$$\begin{aligned}
&\int_E \frac{\alpha(1-\alpha) u_1 u_2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1, u_2\rangle} \\
\leq &\left(\int_E \frac{\alpha(1-\alpha) u_2^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1\rangle}\right)^{1/2} \left(\int_E \frac{\alpha(1-\alpha) u_1^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_2\rangle}\right)^{1/2} \\
\leq &\int_E \frac{\alpha(1-\alpha) u_2^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1\rangle} + \int_E \frac{\alpha(1-\alpha) u_1^2}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_2\rangle},
\end{aligned}$$

by Lemma 5.6.1 in [10], we have

$$\begin{aligned}
\int_E d\mu^c_{\langle u\rangle} &\leq \int_E \frac{\alpha(\alpha u_1^2 + (1-\alpha) u_2^2)}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_1\rangle} + \int_E \frac{(1-\alpha)(\alpha u_1^2 + (1-\alpha) u_2^2)}{\alpha u_1^2 + (1-\alpha) u_2^2} d\mu^c_{\langle u_2\rangle} \\
&\leq \alpha \int_E d\mu^c_{\langle u_1\rangle} + (1-\alpha) \int_E d\mu^c_{\langle u_2\rangle}.
\end{aligned}$$

Moreover, noting

$$\begin{aligned}
u(x)u(y) =& \sqrt{\alpha u_1^2(x) + (1-\alpha) u_2^2(x)} \sqrt{\alpha u_1^2(y) + (1-\alpha) u_2^2(y)} \\
&\geq \alpha u_1(x) u_1(y) + (1-\alpha) u_2(x) u_2(y),
\end{aligned}$$

we have

$$(u(x) - u(y))^2 \leq \alpha(u_1(x) - u_1(y))^2 + (1-\alpha)(u_2(x) - u_2(y))^2$$

and thus $\mathcal{E}^j(u, u) \leq \alpha \mathcal{E}^j(u_1, u_1) + (1-\alpha) \mathcal{E}^j(u_2, u_2)$. The proof of this lemma is completed. □

Define

$$\tilde{J}^G(\lambda) := \inf\left\{\mathcal{E}^G(u,u) : u \in \mathcal{D}(\mathcal{E}^G),\ \int_G u^2 d\mu = \lambda,\ \int_G u^2 dm = 1\right\}, \quad \lambda \in \mathbb{R}^1$$

and

$$J^G(\lambda) = \lim_{\epsilon\to 0} \inf_{|\lambda'-\lambda|<\epsilon} \tilde{J}^G(\lambda').$$

J^G is the lower semi-continuous modification of $\tilde{J}^G$. From Lemma 3.1, we have

Lemma 3.2. *The function $\tilde{J}^G$ is convex: for $0 \le \alpha \le 1$ and $\lambda_1, \lambda_2 \in \mathbb{R}^1$*

$$\tilde{J}^G(\alpha\lambda_1 + (1-\alpha)\lambda_2) \le \alpha\tilde{J}^G(\lambda_1) + (1-\alpha)\tilde{J}^G(\lambda_2).$$

Proof. For any $u_1, u_2 \in \mathcal{D}(\mathcal{E}^G)$ such that

$$\int_G u_i^2 d\mu = \lambda_i,\ \int_G u_i^2 dm = 1,\ i = 1, 2,$$

let $u := \sqrt{\alpha u_1^2 + (1-\alpha)u_2^2}$, $0 \le \alpha \le 1$. Then u belongs to $\mathcal{D}(\mathcal{E}^G)$,

$$\int_G u^2 d\mu = \alpha\lambda_1 + (1-\alpha)\lambda_2 \text{ and } \int_G u^2 dm = 1.$$

We see from the definition of $\tilde{J}^G(\lambda)$ and Lemma 3.1 that for any $u_1, u_2 \in \mathcal{D}(\mathcal{E}^G)$ satisfying above conditions,

$$\begin{aligned}\tilde{J}^G(\alpha\lambda_1 + (1-\alpha)\lambda_2) &\le \mathcal{E}(u,u)\\ &\le \alpha\mathcal{E}(u_1,u_1) + (1-\alpha)\mathcal{E}(u_2,u_2).\end{aligned}$$

Therefore, we have the lemma. □

Lemma 3.3. *The function J^G is convex.*

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{R}^1$. For λ' and λ'' with $|\lambda' - \lambda_1| < \epsilon$ and $|\lambda'' - \lambda_2| < \epsilon$,

$$\begin{aligned}\inf_{|\lambda-(\alpha\lambda_1+(1-\alpha)\lambda_2)|<\epsilon} \tilde{J}^G(\lambda) &\le \tilde{J}^G(\alpha\lambda' + (1-\alpha)\lambda'')\\ &\le \alpha\tilde{J}^G(\lambda') + (1-\alpha)\tilde{J}^G(\lambda'')\end{aligned}$$

by Lemma 3.2, and thus

$$\inf_{|\lambda-(\alpha\lambda_1+(1-\alpha)\lambda_2)|<\epsilon} \tilde{J}^G(\lambda) \le \alpha \inf_{|\lambda'-\lambda_1|<\epsilon} \tilde{J}^G(\lambda') + (1-\alpha) \inf_{|\lambda''-\lambda_2|<\epsilon} \tilde{J}^G(\lambda'').$$

The proof is completed by letting $\epsilon \to 0$. □

Lemma 3.4. *The function C^G is the Legendre conjugate of J^G,*

$$C^G(\theta) = \sup_{\lambda\in\mathbb{R}^1}\{\theta\lambda - J^G(\lambda)\}.$$

Proof. Let

$$\mathcal{A} = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \int_G u^2 dm = 1\right\}$$
$$\mathcal{A}_\lambda = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \int_G u^2 d\mu = \lambda,\ \int_G u^2 dm = 1\right\}, \quad \lambda \in \mathbb{R}^1.$$

For any $\epsilon > 0$, set

$$\mathcal{A}_{\lambda,\epsilon} = \left\{u \in \mathcal{D}(\mathcal{E}^G) : \lambda - \epsilon < \int_G u^2 d\mu < \lambda + \epsilon,\ \int_G u^2 dm = 1\right\}.$$

Then

$$\inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_{u\in\mathcal{A}_{\lambda,\epsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \lim_{\epsilon\to 0}\inf_{u\in\mathcal{A}_{\lambda,\epsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_{u\in\mathcal{A}_\lambda} \mathcal{E}^{\theta\mu,G}(u,u)$$

and thus

$$\inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_\lambda \lim_{\epsilon\to 0}\inf_{u\in\mathcal{A}_{\lambda,\epsilon}} \mathcal{E}^{\theta\mu,G}(u,u) \le \inf_\lambda \inf_{u\in\mathcal{A}_\lambda} \mathcal{E}^{\theta\mu,G}(u,u) = \inf_{u\in\mathcal{A}} \mathcal{E}^{\theta\mu,G}(u,u).$$

Hence we have

$$\begin{aligned}
C^G(\theta) &= -\inf_\lambda \lim_{\epsilon\to 0}\inf_{u\in\mathcal{A}_{\lambda,\epsilon}} \mathcal{E}^{\theta\mu,G}(u,u)\\
&= -\inf_\lambda \lim_{\epsilon\to 0}\inf_{|\lambda'-\lambda|<\epsilon}\left(\inf_{u\in\mathcal{A}_{\lambda'}} \mathcal{E}^{\theta\mu,G}(u,u)\right)\\
&= -\inf_\lambda \lim_{\epsilon\to 0}\inf_{|\lambda'-\lambda|<\epsilon}\left(\tilde{J}^G(\lambda') - \theta\lambda'\right).
\end{aligned}$$

Noting

$$\lim_{\epsilon\to 0}\inf_{|\lambda'-\lambda|<\epsilon}\left(\tilde{J}^G(\lambda') - \theta\lambda'\right) = J^G(\lambda) - \theta\lambda,$$

we have

$$C^G(\theta) = -\inf_\lambda\{J^G(\lambda) - \theta\lambda\} = \sup_\lambda\{\theta\lambda - J^G(\lambda)\}.$$

□

As a result, we see that

Lemma 3.5.

$$I^G = J^G.$$

Proof. The function J^G is lower semi-continuous, convex and not identically infinite. Hence, it follows from Lemma 3.4 and Theorem 5.15 in Appendix 5.3 that $J^G = I^G$. □

We use the notations J (resp. $\widetilde{J}$) for J^G (resp. $\widetilde{J}^G$) when $G = E$.

Lemma 3.6. *Let $\{G_n\}$ be an increasing sequence of relatively compact open sets with $\bigcup_{n=1}^{\infty} G_n = E$. Then for an open set $O \subset \mathbb{R}^1$*

$$\inf_{\lambda\in O} J(\lambda) = \inf_n \inf_{\lambda\in O} J^{G_n}(\lambda).$$

Proof. By the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$,

$$\begin{aligned}
\inf_{\lambda\in O} \widetilde{J}(\lambda) &= \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_E u^2 d\mu \in O, \int_E u^2 dm = 1\right\} \\
&= \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}) \cap C_0(X), \int_E u^2 d\mu \in O, \int_E u^2 dm = 1\right\} \\
&= \inf_n \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}) \cap C_0(G_n), \int_E u^2 d\mu \in O, \int_E u^2 dm = 1\right\} \\
&= \inf_n \inf_{\lambda\in O} \widetilde{J}^{G_n}(\lambda).
\end{aligned}$$

Noting that $\inf_{\lambda\in O} \widetilde{J}^G(\lambda) = \inf_{\lambda\in O} J^G(\lambda)$ for any open set $O \subset \mathbb{R}^1$, we have the lemma. □

Let $\mu \in \mathcal{K}_{loc}$. Let G be a relatively compact open set of E. Denote by $\{G_\alpha^G\}_{\alpha\geq 0}$ the resolvent of the part process $\mathbf{M}^G$ of $\mathbf{M}$ on G. Then the part process $\mathbf{M}^G$ is *tight* in the sense that for any $\epsilon > 0$, there exists a compact set $K \subset G$ such that

$$\sup_{x\in G} G_1^G 1_{K^c}(x) \leq \epsilon.$$

Here 1_{K^c} is the indicator function of $G \setminus K$. In fact, note that for $x \in G$,

$$G_1^G 1_{K^c}(x) = \int_0^\infty e^{-t} p_t^G 1_{K^c}(x) dt = \int_0^\delta e^{-t} p_t^G 1_{K^c}(x) dt + \int_\delta^\infty e^{-t} p_t^G 1_{K^c}(x) dt.$$

We see from (**LU**) and inequality (4.20) that the right hand side is dominated by

$$\begin{aligned}
\int_0^\delta e^{-t} dt + \int_\delta^\infty e^{-t} \|p_t^G\|_{1,\infty} m(G\setminus K) dt \leq& 1 - e^{-\delta} + \int_\delta^\infty e^{-t} C(\delta) m(G\setminus K) dt \\
\leq& 1 - e^{-\delta} + e^{-\delta} C(\delta) m(G\setminus K).
\end{aligned}$$

For any $\epsilon > 0$, we choose $\delta > \log(1 - \frac{\epsilon}{2})$ and a compact set $K \subset G$ satisfying $m(G\setminus K) < \frac{e^\delta \epsilon}{2c(\delta)}$, and obtain the tightness of $\mathbf{M}^G$.

Let $\{p_t^{\mu,G}\}_{t>0}$ be the semigroup defined by

$$p_t^{\mu,G} f(x) = E_x\left(e^{A_t^\mu} f(X_t); t < \tau_G\right), \quad \text{for } f \in \mathscr{B}_b(G).$$

Define the L^p-spectral bounds of $\{p_t^{\mu,G}\}_{t>0}$ by

$$\lambda_p^G(\mu) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^{\mu,G}\|_{p,p}, \quad 1 \le p \le \infty,$$

where $\|p_t^{\mu,G}\|_{p,p}$ is the operator norm of $p_t^{\mu,G}$ from $L^p(G;m)$ to $L^p(G;m)$. We omit 'G' from $\lambda_p^G(\mu)$ when $G = E$.

The L^p-independence of the spectral bounds of $\{p_t^{\mu,G}\}_{t>0}$ means that

$$\lambda_p^G(\mu) = \lambda_2^G(\mu), \quad 1 \le p \le \infty.$$

As mentioned above, the Markov process $\mathbf{M}^G$ is tight, so $\lambda_p^G(\theta\mu)$ is independent of p by [2, Theorem 4.1]. We easily see the following inequality

$$\begin{aligned} -\lambda_2^G(\theta\mu) \le \liminf_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^\mu}; t < \tau_G\right) &\le \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in G} E_x\left(e^{\theta A_t^\mu}; t < \tau_G\right) \\ &= \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in G} p_t^{\theta\mu,G} 1(x) \\ &= \limsup_{t\to\infty} \frac{1}{t} \log \|p_t^{\theta\mu,G}\|_\infty \\ &= -\lambda_\infty^G(\theta\mu). \end{aligned}$$

By combining the L^p-independence of the spectral bounds of $\{p_t^{\theta\mu,G}\}_{t>0}$ and the variational formula for $\lambda_2^G(\theta\mu)$,

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^\mu}; t < \tau_G\right) = C^G(\theta). \tag{3.3}$$

By using (**LU**), the transition function $p_t^{\theta\mu,G}(x,y)$ of $p_t^{\theta\mu,G}$ is bounded for each $t > 0$ and $x, y \in E$, and thus $p_t^{\theta\mu,G}$ is a Hilbert-Schmidt integral operator, in particular, a compact operator. Hence, we see that $C^G(\theta)$ is an analytic function in θ because it is the principal eigenvalue of $\mathcal{L}^\mu$. Then, combining (3.3) with the Gärtner-Ellis theorem ([6, Section 2.3]), we obtain the next lower estimate: For any open set $O \subset \mathbb{R}^1$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O; t < \tau_G\right) \ge -\inf_{\lambda\in O} I^G(\lambda), \tag{3.4}$$

where I^G is the Legendre transform of C^G.

We use the notations I for I^G when $G = E$.

Theorem 3.7. *Let $\mu \in \mathcal{K}_{loc}$. Then, for any open set $O \subset \mathbb{R}^1$*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O\right) \geq -\inf_{\lambda\in O} I(\lambda).$$

Proof. Let $\{G_n\}$ be a sequence of relatively compact open sets such that $G_n \uparrow E$ and simply write I^n for I^{G_n}. Then we have from (3.4) that

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O\right) \\
&\qquad \geq \sup_n \liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in O; t < \tau_{G_n}\right) \\
&\qquad \geq -\inf_n \inf_{\lambda\in O} I^n(\lambda).
\end{aligned}$$

Combining Lemma 3.5 and Lemma 3.6, we have

$$\inf_n \inf_{\lambda\in O} I^n(\lambda) = \inf_{\lambda\in O} I(\lambda).$$

Hence we obtain the theorem. □

Define

$$\gamma(\theta) := \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}),\ \theta\int_E u^2 d\mu = 1\right\}, \quad \theta \in \mathbb{R}^1. \tag{3.5}$$

Lemma 3.8.

$$\gamma(\theta) \leq 1 \Longleftrightarrow \inf\left\{\mathcal{E}^{\theta\mu}(u,u) : \int_E u^2 dm = 1\right\} \leq 0. \tag{3.6}$$

Proof. We can prove this lemma by the same argument as in [21, Lemma 2.2]. Assume that $\gamma(\theta) \leq 1$. Then there exists a $\varphi_0 \in C_0(X)$ with $\theta\int_E \varphi_0^2 d\mu = 1$ such that $\mathcal{E}(\varphi_0,\varphi_0) \leq 1$. Hence we see

$$\mathcal{E}(\varphi_0,\varphi_0) \leq \theta\int_E \varphi_0^2 d\mu.$$

Letting

$$u_0 = \frac{\varphi_0}{\sqrt{\int_E \varphi_0^2 dm}},$$

we have

$$\mathcal{E}^{\theta\mu}(u_0,u_0) \leq 0.$$

On the other hand, we assume that $\inf\left\{\mathcal{E}^{\theta\mu}(u,u) : \int_E u^2 dm = 1\right\} \leq 0$. Then there exists a $\psi_0 \in C_0(E)$ with $\int_E \psi_0^2 dm = 1$ such that $\mathcal{E}^{\theta\mu}(\psi_0,\psi_0) \leq 0$. Letting

$$u_0 = \frac{\psi_0}{\sqrt{\theta\int_E {\psi_0}^2 d\mu}},$$

we have

$$\mathcal{E}(u_0, u_0) \leq 1.$$

$\square$

Let $\theta_0 > 0$ be a unique value such that $\gamma(\theta_0) = 1$. Suppose that $\mu \in \mathcal{K}_\infty$. Under the assumptions (**C**) and (**DF**), if $\lambda_2(\mu) \leq 0$, $\lambda_p(\mu)$ is independent of p by [19, Theorem 3.1]. By combining Lemma 3.8, we can derive the following in a similar way of (3.3): for $\theta \geq \theta_0$

$$C(\theta) = \lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^\mu}\right).$$

On the other hand, by Lemma 3.8 and [2, Theorem 5.1] on the Schrödinger type operator, we see that $\gamma(\theta) > 1$ is equivalent to

$$\sup_{x\in E} E_x\left(e^{\theta A_\infty^\mu}\right) < \infty.$$

Since A_t^μ is positive, for $\theta < \theta_0$

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^\mu}\right) \leq \lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_\infty^\mu}\right) = 0.$$

Hence we have

Theorem 3.9. *Let $\mu \in \mathcal{K}_\infty$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(e^{\theta A_t^\mu}\right) = \widetilde{C}(\theta),$$

where $\widetilde{C}(\theta)$ is the function defined by

$$\widetilde{C}(\theta) = \begin{cases} C(\theta), & \theta \geq \theta_0, \\ 0, & \theta < \theta_0. \end{cases}$$

Let $\widetilde{I}$ be the Legendre transform of $\widetilde{C}(\theta)$,

$$\widetilde{I}(\lambda) = \sup_{\theta\in\mathbb{R}^1} \{\lambda\theta - \widetilde{C}(\theta)\}.$$

We see from Theorem 3.9 that $\widetilde{C}(\theta)$ is the logarithmic moment generating function of A_t^μ. Then, combining Theorem 3.9 with the Gärtner-Ellis theorem ([6, Section 2.3]), we have the upper bound:

Theorem 3.10. *Let $\mu \in \mathcal{K}_\infty$. Then for any closed set $K \subset \mathbb{R}^1$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x\left(\frac{A_t^\mu}{t} \in K\right) \leq -\inf_{\lambda\in K} \widetilde{I}(\lambda).$$

The Legendre transform of $C(\theta)$ and $\widetilde{C}(\theta)$ are expressed as follows:

$$I(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - C(\theta)\}$$
$$= \begin{cases} \lambda(C')^{-1}(\lambda) - C((C')^{-1}(\lambda)), & \lambda \geq C'(0) \\ C(0), & 0 \leq \lambda < C'(0) \\ \infty, & \lambda < 0. \end{cases} \tag{3.7}$$

$$\widetilde{I}(\lambda) = \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - \widetilde{C}(\theta)\}$$
$$= \begin{cases} \lambda(C')^{-1}(\lambda) - C((C')^{-1}(\lambda)), & \lambda \geq C'(\theta_0+) \\ \lambda\theta_0, & 0 \leq \lambda < C'(\theta_0+) \\ \infty, & \lambda < 0. \end{cases} \tag{3.8}$$

Hence, I equals $\widetilde{I}$ on $[C'(\theta_0+), \infty)$.

3.2 An example – Brownian motion with constant drift

We give a simple example that our main theorem can be applied.

Example 3.11. Let us consider the 1-dimensional Brownian motion (P_x^k, X_t) with a positive drift k. Then the process (P_x^k, X_t) is transient and its infinitesimal generator $\mathcal{L}$ is given by $\frac{1}{2}\frac{d^2}{dx^2} + k\frac{d}{dx}$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(\mathbb{R}^1; e^{2kx}dx)$ generated by (P_x^k, X_t), that is,

$$\begin{cases} \mathcal{E}(u,v) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^1} \frac{du}{dx}\frac{dv}{dx} e^{2kx}dx, \quad u,\ v \in \mathcal{D}(\mathcal{E}) \\ \mathcal{D}(\mathcal{E}) = \text{the closure of } C_0^\infty(\mathbb{R}^1) \text{ with respect to } \mathcal{E}_1^{1/2}. \end{cases}$$

By using integration by parts,

$$\begin{aligned} \mathcal{E}(u,v) &= -\frac{1}{2}\int_{\mathbb{R}} \left(\frac{d^2u}{dx^2} + 2k\frac{du}{dx}\right) v e^{2kx}dx \\ &= (-\mathcal{L}u, v)_{e^{2kx}dx}. \end{aligned}$$

Then (P_x^k, X_t) satisfies the assumptions **(I), (DF), (C)** and **(LU)**.

Let μ be the Dirac measure at the origin. i.e., $\mu = \delta_0$. Then $\mu \in \mathcal{K}_\infty$. Let l_t be the local time at 0. Then l_t is the continuous additive functional corresponding to μ.

We define the functions $C(\theta)$ and $\widetilde{C}(\theta)$ by

$$C(\theta) = -\inf\left\{\mathcal{E}(u,u) - \theta u^2(0) : u \in C_0^\infty(\mathbb{R}^1),\ \int_{\mathbb{R}^1} u^2 e^{2kx}dx = 1\right\},$$

$$\widetilde{C}(\theta) = \begin{cases} C(\theta), & \theta \geq \theta_0 \\ 0, & \theta < \theta_0. \end{cases}$$

The function $C(\theta)$ is equal to the bottom of spectrum of the self-adjoint operator $\mathcal{L}^{\delta_0} := \mathcal{L} + \delta_0$. We first consider $C(\theta)$ for $\theta \geq 0$. For $u \in C_0^\infty(\mathbb{R}^1)$, the boundary condition

$$u'(0+) - u'(0-) = -2\theta u(0)$$

must be satisfied. Since $u \in L^2(\mathbb{R}^1, e^{2kx}dx)$, the eigenfunction corresponding to an eigenvalue λ forms

$$u(x) = \begin{cases} Ce^{-(k+\sqrt{k^2-2\lambda})x}, & x \geq 0 \\ Ce^{-(k-\sqrt{k^2-2\lambda})x}, & x < 0, \end{cases}$$

where C is a constant. From the boundary condition, we have

$$\sqrt{k^2 - 2\lambda} = \theta.$$

Hence,

$$\lambda = \frac{k^2 - \theta^2}{2}.$$

Since $C(\theta) = C(0)$ for $\theta < 0$, we have

$$C(\theta) = \begin{cases} \dfrac{\theta^2}{2} - \dfrac{k^2}{2}, & \theta \geq 0 \\ -\dfrac{k^2}{2}, & \theta < 0. \end{cases}$$

Moreover, since $\theta_0 = k$, we have

$$\widetilde{C}(\theta) = \begin{cases} \dfrac{\theta^2}{2} - \dfrac{k^2}{2}, & \theta \geq k \\ 0, & \theta < k. \end{cases}$$

Let $I(\lambda)$ (resp. $\widetilde{I}(\lambda)$) be the Legendre transform of $C(\theta)$ (resp. $\widetilde{C}(\theta)$):

$$\begin{aligned} I(\lambda) &= \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - C(\theta)\} \\ &= \begin{cases} \dfrac{\lambda^2}{2} + \dfrac{k^2}{2}, & \lambda \geq 0 \\ \infty, & \lambda < 0. \end{cases} \end{aligned}$$

$$\begin{aligned}\widetilde{I}(\lambda) &= \sup_{\theta\in\mathbb{R}^1}\{\lambda\theta - \widetilde{C}(\theta)\}\\ &= \begin{cases}\dfrac{\lambda^2}{2}+\dfrac{k^2}{2}, & \lambda\geq k\\ \lambda k, & 0\leq\lambda<k\\ \infty, & \lambda<0.\end{cases}\end{aligned}$$

Since $C'(0)$ are equal to 0, if $\lambda \geq 0$, then $\lambda\theta - C(\theta)$ have a maximum value at $\theta = \lambda$ for all $\theta \in \mathbb{R}^d$. Since $C'(k) = k$ and $\widetilde{C}(\theta) = 0$ for $\theta < k$, for all $\theta \in \mathbb{R}^d$, $\lambda\theta - \widetilde{C}(\theta)$ have a maximum value at $\theta = k$ if $0 \leq \lambda < k$, and have a maximum value at $\theta = \lambda$ if $\lambda \geq k$. Hence, I equals $\widetilde{I}$ on $[k, \infty)$.

When $x = 0$, we see by direct calculation that l_t satisfies the large deviation principle with rate function $\widetilde{I}$. The author is told by professor Hariya. This example says that the large deviation principle holds with the Legendre transform of logarithmic moment generating function (LMGF), even if LMGF does not satisfy the sufficient condition in the Gartner-Ellis theorem.

Finally, for $A \subset [k, \infty)$ with $\inf_{\lambda\in A^\circ} I(\lambda) = \inf_{\lambda\in\bar{A}} I(\lambda)$,

$$\lim_{t\to\infty}\frac{1}{t}\log P_x^k\left(\frac{l_t}{t}\in A\right) = -\inf_{\lambda\in A} I(\lambda).$$

We can think that the Brownian motion on hyperbolic space is in the same situation as the diffusion process treated in this example.

Chapter 4

LDP for occupation distributions

4.1 Uniform LDP

We consider the uniform Large deviation principle with respect to starting point $x \in E$. The sufficient condition for Uniform LDP is obtained in Wu [22]. He called this property *uniform hyper-exponential recurrence*. In this section we will prove that the conservative symmetric Markov processes with tightness property satisfy the property.

Tightness Property (T). For any $\epsilon > 0$, there exists a compact set K such that $\sup_{x \in E} R_1 1_{K^c}(x) \leq \epsilon$.

Definition 4.1. A positive smooth measure μ is said to be in the class $\mathcal{K}_\infty$ if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{x \in E} \int_{K^c \cup B} R_1(x, y) \mu(dy) \leq \epsilon.$$

Under the condition for $\mathbf{M}$ being transient, the class $\mathcal{K}_\infty$ is usually defined by using the Green kernel, i.e., the 0-resolvent density, and a measure μ in the class is said to be *Green-tight*. Here we use the 1-resolvent density to deal with recurrent processes. The next lemma is proven by Chen ([2, Theorem 4.2.]). We give a proof for completion.

Lemma 4.2. *If* $\mathbf{M}$ *satisfies* (**DF**) *and* (**T**), *then the measure* m *belongs to* $\mathcal{K}_\infty$.

Proof. By the definition of property (**T**), there exists a compact set K such that $\sup_{x \in E} \int_{K^c} R_1(x, y) m(dy) \leq \epsilon/2$. Suppose that for any $\delta > 0$ there exists a Borel set $B \subset K$ with $m(B) \leq \delta$ such that $\sup_{x \in E} R_1 1_B(x) > \epsilon/2$. Then there exists a sequence

$\{B_n\}_{n=1}^{\infty}$ of Borel subsets of K such that $m(B_n) \le 1/2^n$ and $\sup_{x\in K} R_1 1_{B_n}(x) > \epsilon/2$. Define $A_n = \cup_{k=n}^{\infty} B_k$. Then $m(A_n)$ is less than $1/2^{n-1}$ and decreasingly converges to zero as $n \to \infty$. Hence $R_1 1_{A_n}$ decreasingly converges to zero pointwise. Since $R_1 1_{A_n}$ is continuous by the property (**DF**), $R_1 1_{A_n}$ uniformly converges to zero on K. This is contradictory to $\sup_{x\in K} R_1 1_{A_n}(x) \ge \sup_{x\in K} R_1 1_{B_n}(x) > \epsilon/2$. □

We denote by $\mathcal{P}$ the set of probability measures on E. Define the function $I_{\mathcal{E}}$ on $\mathcal{P}$ by

$$I_{\mathcal{E}}(\nu) = \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}), & \text{if } \nu = f \cdot m,\ \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty, & \textit{otherwise}. \end{cases} \tag{4.1}$$

The space $\mathcal{P}$ is supposed to be equipped with the weak topology. Given $w \in \Omega$ with $0 < t < \zeta(w)$, let $L_t(w) \in \mathcal{P}$ be the normalized occupation distribution: for a Borel set A of E,

$$L_t(w)(A) = \frac{1}{t}\int_0^t 1_A(X_s(w))ds.$$

Takeda [20] proved the next theorem.

Theorem 4.3. *Assume that* $\mathbf{M}$ *satisfies* (**I**), (**DF**) *and* (**T**) .

(i) For each open set $G \subset \mathcal{P}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x(L_t \in G, t < \zeta) \ge -\inf_{\nu\in G} I_{\mathcal{E}}(\nu).$$

(ii) For each closed set $K \subset \mathcal{P}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x(L_t \in K, t < \zeta) \le -\inf_{\nu\in K} I_{\mathcal{E}}(\nu).$$

Note that the uniform upper bound holds. This fact follows from the symmetry of Markov processes.

We define the function space $\mathcal{D}^+$ by

$$\mathcal{D}^+ = \{R_\alpha f : \alpha > 0, f \in L^2(E;m) \cap C_b^+(E) \text{ and } f \not\equiv 0\},$$

where $C_b^+(E)$ denotes the set of non-negative bounded continuous functions. We see that any function in $\mathcal{D}^+$ is strictly positive by the irreducibility (**I**). Define the operator A on $\mathcal{D}^+$ by $AR_\alpha f = \alpha R_\alpha f - f$ and the function I on $\mathcal{P}$ by

$$I(\nu) = -\inf_{u\in\mathcal{D}^+, \epsilon>0} \int_E \frac{Au}{u+\epsilon} d\nu. \tag{4.2}$$

The function I is a version of the Donsker-Varadhan I-*function* introduced in [8]. Note that since the Markov process $\mathbf{M}$ is allowed to have a finite lifetime, the function $u = R_\alpha f \in \mathcal{D}^+$ is not always uniformly lower-bounded by a positive constant even if f is so, and consequently the function Au/u is not always bounded. By adding a positive constant ϵ, the function $Au/(u+\epsilon)$ is bounded and continuous, and consequently the I-function defined by (4.2) is lower semicontinuous on $\mathcal{P}$ with respect to the weak topology. This is a reason why we need to modify the Donsker-Varadhan I-function. In spite of this modification, we can identify the I-function with the Dirichlet form ([10, Theorem 6.4.2]).

Proposition 4.4.

$$I(\nu) = I_{\mathcal{E}}(\nu),\ \ \nu \in \mathcal{P}.$$

We define the subset $\mathcal{P}_M$ of $\mathcal{P}$ by

$$\mathcal{P}_M = \left\{ u^2 \cdot m : u \in \mathcal{D}(\mathcal{E}), \int_E u^2 dm = 1, \mathcal{E}(u,u) \leq M \right\},\ \ M > 0.$$

Lemma 4.5. *The set $\mathcal{P}_M$ is compact in $\mathcal{P}$.*

Proof. Recall the inequality in [17]: for any $\beta > 0$ and any smooth measure μ,

$$\int_E u^2(x)\mu(dx) \leq \|R_\beta \mu\|_\infty \cdot \left(\mathcal{E}(u,u) + \beta \int_E u^2 dm \right),\ \ u \in \mathcal{D}(\mathcal{E}). \tag{4.3}$$

Combining property (**T**) with this inequality, we see $\mathcal{P}_M$ is tight. Indeed, for any compact set $K \subset E$ and any $u^2 \cdot m \in \mathcal{P}_m$,

$$\int_{K^c} u^2 dm \leq \|R_1 1_{K^c}\|_\infty \cdot \left(\mathcal{E}(u,u) + \int_E u^2 dm \right) \leq (M+1)\|R_1 1_{K^c}\|_\infty. \tag{4.4}$$

Since $\mathcal{P}_M = \{\nu \in \mathcal{P} : I(\nu) \leq M\}$ is closed by the lower semicontinuity of I, we have the lemma. □

Let λ_2 be the bottom of the spectrum:

$$\lambda_2 = \inf \left\{ \mathcal{E}(f,f) : f \in \mathcal{D}(\mathcal{E}), \int_E f^2 dm = 1 \right\}. \tag{4.5}$$

A function ϕ_0 on E is called a *ground state* of the L^2-generator for $\mathcal{E}$ if $\phi_0 \in \mathcal{D}(\mathcal{E})$, $\|\phi_0\|_2 = 1$ and $\mathcal{E}(\phi_0, \phi_0) = \lambda_2$.

Lemma 4.6 ([20])**.** *Assume that* $\mathbf{M}$ *satisfies* (**I**), (**DF**) *and* (**T**). *Then there exists a ground state ϕ_0 uniquely up to a sign. ϕ_0 can be taken to be strictly positive on E.*

Proof. Let $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E})$ be a minimizing sequence, $\|u_n\|_2 = 1$, and $\lambda_2 = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$. We see from Lemma 4.5 that there exists a subsequence $\{u_{n_k}^2 \cdot m\}_{k=1}^\infty$ such that $u_{n_k}^2 \cdot m$ converges weakly to a probability $\nu = \phi_0^2 \cdot m$, $\phi_0 \in \mathcal{D}(\mathcal{E})$, $\phi_0 \geq 0$. Since the function $I_\mathcal{E}$ is lower semicontinuous by Proposition 4.4, $I_\mathcal{E}(\phi^2 m) \leq \lambda_2$. Hence the function ϕ_0 is just a ground state.

It follows from the inequality $\|\phi_0 + \epsilon g\|_\mathcal{E}^2 \geq \lambda_2 \|\phi_0 + \epsilon g\|_2^2$ holding for any $g \in \mathcal{D}(\mathcal{E})$ for any $\epsilon > 0$ that $\mathcal{E}(\phi_0, g) = \lambda_2(\phi_0, g)$. Hence $\alpha R_{\alpha-\lambda_2}\phi_0 = \phi_0$, $\alpha > \lambda_2$, which implies that ϕ_0 is strictly positive by irreducibility.

To prove the uniqueness of the ground state, we introduce a closed symmetric form $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ on $L^2(E; \phi_0^2 m)$ by

$$\begin{cases} \mathcal{E}^{\phi_0}(u, v) = \mathcal{E}(u\phi_0, v\phi_0) - \lambda_2(u\phi_0, v\phi_0) \\ \mathcal{D}(\mathcal{E}^{\phi_0}) = \{u \in L^2(E; \phi_0^2 m) : u\phi_0 \in \mathcal{D}(\mathcal{E})\}. \end{cases} \tag{4.6}$$

Since $1 \in \mathcal{D}(\mathcal{E}^{\phi_0})$, $\mathcal{E}^{\phi_0}(1,1) = 0$ and the associated resolvent $R_\alpha^{\phi_0}$ satisfies $R_\alpha^{\phi_0} f = \phi_0^{-1} R_{\alpha-\lambda_2}(f\phi_0)$, $\alpha > \lambda_2$, we see from the strict positivity of ϕ_0 that $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ is an irreducible recurrent Dirichlet form so that f is constant whenever $f \in \mathcal{D}(\mathcal{E}^{\phi_0})$, $\mathcal{E}^{\phi_0}(f,f) = 0$. Let ψ_0 be another ground state. Then $\psi_0 = f\phi_0$ with $f = \psi_0/\phi_0 \in \mathcal{D}(\mathcal{E}^{\phi_0})$, $\mathcal{E}^{\phi_0}(f,f) = \mathcal{E}(\psi_0, \psi_0) - \lambda_2 = 0$, which yields that f is constant and $\psi_0 = \pm\phi_0$. □

Lemma 4.7. *Assume* $\mathbf{M}$ *satisfies* $(\mathbf{I})$, $(\mathbf{DF})$ *and* $(\mathbf{T})$ *and is, in addition, conservative, then it is positively recurrent.*

Proof. If $\mathbf{M}$ is conservative, then the tightness property $(\mathbf{T})$ implies that for any $\epsilon > 0$, there exists a compact set K such that $\inf_{x\in E} R_1 1_K(x) \geq 1 - \epsilon$. Since the function $R_1 1_K$ is in $L^1(E; m)$, m is finite, and thus $1 \in \mathcal{D}(\mathcal{E})$, $\mathcal{E}(1,1) = 0$. Hence $\mathbf{M}$ is positively recurrent ([10, Theorem 1.6.3]). □

Lemma 4.8. *Assume* $\mathbf{M}$ *satisfies* $(\mathbf{AC})$. *Then*

$$\sup_{x\in E} p_t 1(x) = \mathrm{esssup}_{x\in X} p_t 1(x).$$

Proof. Let $M = \sup_{x\in E} p_t 1(x)$, $\widetilde{M} = \mathrm{esssup}_{x\in E}\, p_t 1(x)$. Suppose $M > \widetilde{M}$ and take r so that $M > r > \widetilde{M}$. Since the function $p_t 1$ is excessive, the set $O = \{x \in E : p_t 1(x) > r\}$ is finely open and $m(O) = 0$ by the definition of $\widetilde{M}$. Hence by the Lemma 4.1.4 and Theorem 4.1.2 in [10], the set O is polar and thus empty by the argument in the proof of Lemma 3.1 in [20]. Therefore $p_t 1(x) \leq r$, which is contradictory to $M > r$. □

Let us denote by $\|p_t\|_{p,p}$ the operator norm of p_t from $L^p(E;m)$ to $L^p(E;m)$ and put

$$-\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \|p_t\|_{p,p}, \ 1 \le p \le \infty.$$

$-\lambda_p$ is the long time exponential growth bound of the semigroup $\{p_t\}_{t\ge 0}$. The next theorem gives us a probabilistic interpretation of λ_∞ (cf. [16]).

Theorem 4.9. *Assume* $\mathbf{M}$ *satisfies* $(\mathbf{AC})$. *Then*

$$\lambda_\infty = \sup\left\{\lambda \ge 0; \sup_{x\in E} E_x(e^{\lambda\zeta}) < \infty\right\}. \tag{4.7}$$

Proof. Let γ be the right hand side of (4.7). Since for $\lambda < \gamma$,

$$\|p_t\|_{\infty,\infty} = \sup_{x\in E} P_x(t < \zeta) \le e^{-\lambda t} \sup_{x\in E} E_x(e^{\lambda\zeta}),$$

$\gamma \le \lambda_\infty$. In particular, if $\lambda_\infty = 0$, then $\gamma = 0$.

For $0 < \lambda < \lambda_\infty$, let $p_t^\lambda = e^{\lambda t} p_t$. Then since

$$\lim_{t\to\infty} \frac{1}{t} \log \|p_t^\lambda\|_{\infty,\infty} = \lambda - \lambda_\infty < 0,$$

$$\int_0^\infty \|p_t^\lambda\|_{\infty,\infty} dt = \int_0^\infty \sup_{x\in E} E_x(e^{\lambda t}; t < \zeta) dt < \infty.$$

Hence

$$\sup_{x\in E} \int_0^\infty E_x(e^{\lambda\zeta}; t < \zeta) dt = \sup_{x\in E} \left(\frac{E_x(e^{\lambda\zeta}) - 1}{\lambda}\right) < \infty, \tag{4.8}$$

and so $\gamma \ge \lambda_\infty$. $\square$

Let us extend the resolvent operator; for $\lambda \ge 0$,

$$R_{-\lambda} f(x) = E_x\left(\int_0^\infty e^{\lambda t} f(X_t) dt\right).$$

We then see from (4.8) that for $\lambda > 0$,

$$\|R_{-\lambda}\|_{\infty,\infty} < \infty \iff \sup_{x\in E} E_x(e^{\lambda\zeta}) < \infty. \tag{4.9}$$

It holds that if $\lambda_\infty > 0$, then $\sup_{x\in E} E_x(e^{\lambda_\infty\zeta}) = \infty$. Indeed, we see from (4.9) that if $\sup_{x\in E} E_x(e^{\lambda_\infty\zeta}) < \infty$, then $\|R_{-\lambda_\infty}\|_{\infty,\infty} < \infty$. Noting that

$$R_{-\lambda_\infty-\epsilon} = R_{-\lambda_\infty} + \epsilon R_{-\lambda_\infty}^2 + \epsilon^2 R_{-\lambda_\infty}^3 + \cdots$$

([12, III, §6]), we see that if $0 < \epsilon < 1/\|R_{-\lambda_\infty}\|_{\infty,\infty}$, then $\|R_{-\lambda_\infty-\epsilon}\|_{\infty,\infty} < \infty$. Using (4.9) again, we have $\sup_{x\in E} E_x(e^{(\lambda_\infty+\epsilon)\zeta}) < \infty$, which is contradictory to Theorem 4.9. Therefore, we have the next corollary.

Corollary 4.10. *Suppose* $\lambda_\infty > 0$. *Then*

$$\sup_{x\in E} E_x(\exp(\lambda\zeta)) < \infty \iff \lambda < \lambda_\infty.$$

Chen [2, Theorem 4.1] proved:

Theorem 4.11. *Suppose* $\mathbf{M}$ *is irreducible and satisfies* (**AC**). *If the measure* m *belongs to* $\mathcal{K}_\infty$, *then* λ_p *is independent of* p.

Combining Theorem 4.11 with Corollary 4.10, we have

Corollary 4.12. *Suppose* $\mathbf{M}$ *is irreducible and satisfies* (**AC**). *If* $m \in \mathcal{K}_\infty$ *and* $\lambda_2 > 0$, *then*

$$\sup_{x\in E} E_x(\exp(\lambda\zeta)) < \infty \iff \lambda < \lambda_2.$$

Let $K \subset E$ be a compact set and $D := K^c$, the complement of K. Let $\mathbf{M}^D = (P_x, X_t^D)$ be the part process on D:

$$X_t^D = \begin{cases} X_t, & t < \tau_D, \\ \Delta, & t > \tau_D,\ \tau_D = \inf\{t \geq 0 : X_t \notin D\}. \end{cases} \tag{4.10}$$

Define the (quasi-regular) Dirichlet form $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$ on $L^2(E;m)$ by

$$\begin{cases} \mathcal{E}^D = \mathcal{E}, \\ \mathcal{D}(\mathcal{E}^D) = \{u \in \mathcal{D}(\mathcal{E}) : u = 0 \text{ q.e. on } K\}. \end{cases} \tag{4.11}$$

Then $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$ is the Dirichlet space generated by X^D [10, Theorem 4.4.3].

Let λ^D be the principal eigenvalue of the spectrum of $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$:

$$\lambda^D = \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}^D), \int_D u^2 dm = 1\right\}. \tag{4.12}$$

Lemma 4.13. *Suppose that* $\mathbf{M}$ *satisfies* (**I**), (**DF**) *and* (**T**) *and is conservative. For any compact set* K *with non-empty interior* K^o, *the principal eigenvalue* λ^D, $D = K^c$, *is positive.*

Proof. Let $\{\phi_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E}^D) \cap C_0(D)$ be an approximating sequence in (4.12) such that $\mathcal{E}(\phi_n, \phi_n) \to \lambda^D$. Let $\{\phi_{n_k}^2 \cdot m\}_{k=1}^\infty$ be weakly converging to $\phi_0^2 \cdot m$, $\phi_0 \in \mathcal{D}(\mathcal{E})$.Then

$$1 = \limsup_{k\to\infty} \int_{E\setminus K^o} \phi_{n_k}^2 dm \leq \int_{E\setminus K^o} \phi_0^2 dm,$$

and thus ϕ_0 equals 0, m-a.e. on K^o. In particular, the function ϕ_0 is not constant on E, because $m(K^o) > 0$ by the assumption on m. Hence we have $\mathcal{E}(\phi_0, \phi_0) > 0$.

In fact, if $\mathcal{E}(\phi_0, \phi_0) = 0$, then ϕ_0 must be a constant by the irreducible recurrence of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ [11, Theorem 1.3]. We now conclude that

$$\lambda^D = \liminf_{k\to\infty} \mathcal{E}(\phi_{n_k}, \phi_{n_k}) \geq \mathcal{E}(\phi_0, \phi_0) > 0.$$

□

We write $\mathcal{K}_\infty(R_1)$ for $\mathcal{K}_\infty$ to express the dependence of the 1-resolvent. Let R_1^D be the 1-resolvent of $\mathbf{M}^D$. Denote by m^D the restriction of m to D, $m^D(\cdot) = m(D \cap \cdot)$.

Lemma 4.14. *Let K be a compact set. Then $m^D \in \mathcal{K}_\infty(R_1^D)$, $D = K^c$.*

Proof. Let $\widetilde{K}$ and δ be a compact set and a positive constant in Definition 4.1. We can suppose that the interior of $\widetilde{K}$ contains K. Let G be a relatively compact open set such that $K \subset G \subset \overline{G} \subset \widetilde{K}$ and $m(G \setminus K) < \delta$. Then $\widetilde{K} \cap G^c$ is a compact subset of D and

$$R_1^D 1_{(\widetilde{K}\cap G^c)^c} = R_1^D 1_{\widetilde{K}^c \cup (G\setminus K)} \leq R_1 1_{\widetilde{K}^c} + R_1 1_{G\setminus K} \leq \epsilon.$$

Moreover, $R_1^D 1_B \leq R_1 1_B$ for any Borel set $B \subset \widetilde{K} \cap G^c$. □

It follows from (4.4) that

$$\int_D u^2 dm = \int_E u^2 1_D dm \leq \|R_1 1_D\|_\infty \cdot \left(\mathcal{E}(u,u) + \int_E u^2 dm\right), \ u \in \mathcal{D}(\mathcal{E}),$$

and thus

$$1 \leq \|R_1 1_D\|_\infty \cdot (\lambda^D + 1). \tag{4.13}$$

The tightness property implies that there exists a sequence $\{K_n\}_{n=1}^\infty$ of compact sets such that $\cup_{n=1}^\infty K_n = E$ and $\|R_1 1_{K_n^c}\|_\infty \to 0$ as $n \to \infty$. Hence we see from (4.13) that for $D_n = K_n^c$,

$$\lambda^{D_n} \uparrow \infty \ \text{ as } \ n \to \infty. \tag{4.14}$$

Note that if $\mathbf{M}$ is conservative, then the lifetime of $\mathbf{M}^D$ equals the hitting time of K. Combining Lemma 4.14 with Corollary 4.12, we know that if $\mathbf{M}^{D_n}$ is irreducible, then

$$\sup_{x\in D_n} E_x(\exp(\gamma\sigma_{K_n})) < \infty \iff \gamma < \lambda^{D_n}. \tag{4.15}$$

Note that

$$\sup_{x\in D} E_x(\exp(\sigma_K)) = \sup_{x\in E} E_x(\exp(\sigma_K)). \tag{4.16}$$

Indeed, let $x_0 \in K \setminus K^r$, where K^r is the regular set of K, i.e., $K^r = \{x \in E : P_x(\sigma_K = 0) = 1\}$. Then since

$$\begin{aligned} E_{x_0}(\exp(\sigma_K)) &= E_{x_0}(\exp(\sigma_K); X_t \in K) + E_{x_0}(\exp(\sigma_k); X_t \in D) \\ &\le e^t P_{x_0}(X_t \in K) + E_{x_0}(\exp(t + \sigma_K(\theta_t)); \sigma_K < t) \\ &\le e^t P_{x_0}(X_t \in K) + E_{X_t}(\exp(\sigma_K); X_t \in D) \\ &\le e^t P_{x_0}(X_t \in K) + \sup_{x\in D} E_x(\exp(\sigma_K)) \end{aligned}$$

and

$$P_{x_0}(X_t \in K) \le P_{x_0}(\sigma_K \le t) \to 0 \quad \text{as} \quad t \downarrow 0,$$

we have (4.16) and thus

$$\sup_{x\in E} E_x(\exp(\gamma\sigma_{K_n})) < \infty \iff \gamma < \lambda^{D_n}. \tag{4.17}$$

Hence we have from (4.14) and (4.16) the following:

Lemma 4.15. *Suppose that* $\mathbf{M}$ *satisfies* (**I**), (**DF**) *and* (**T**) *and is conservative. If there exists an increasing sequence* $\{K_n\}_{n=1}^{\infty}$ *of compact sets such that* $\cup_{n=1}^{\infty} K_n = E$ *and* $\mathbf{M}^{D_n}$, $D_n = K_n^c$, *are irreducible, then* $\mathbf{M}$ *has the following property:*

For any $\gamma > 0$ *there exists a compact set* K *such that* $\sup_{x\in E} E_x(\exp(\gamma\sigma_K)) < \infty$. (4.18)

Property (4.18) is said to be a *uniform hyper-exponential recurrence* ([22]). We will give sufficient conditions for the part process $\mathbf{M}^D$ being irreducible.

Noting that

$$p_t(x, U) = 0,\ \forall t > 0 \iff P_x(\sigma_U < \infty) = 0,$$

we see that if $\mathbf{M}$ is irreducible, the semigroup $\{p_t\}_{t\ge 0}$ is *topological transitive*; that is, for all non-empty open sets U and $x \in E$, there exists $t > 0$ such that $p_t(x, U) > 0$. Therefore, Theorem 1.2 in Wu [22] leads us to:

Theorem 4.16. *Suppose* $\mathbf{M}$ *satisfies* (**I**), (**DF**) *and* (**T**) *and is conservative. If there exists an increasing sequence* $\{K_n\}_{n=1}^{\infty}$ *of compact sets such that* $\cup_{n=1}^{\infty} K_n = E$ *and* $\mathbf{M}^{D_n}$, $D_n = K_n^c$, *are irreducible, then the uniform large deviation principle holds: for each open set* G *of* $\mathcal{P}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in E} P_x(L_t \in G) \ge -\inf_{\mu\in G} I_{\mathcal{E}}(\mu).$$

4.2 Locally uniform lower bound

In this section, as an application of uniform LDP, we consider the locally uniform lower bound of the large deviations for occupation times of symmetric Markov processes with finite life time. We further consider the large deviation principle for symmetric Markov processes conditioned on non-absorption up to $t > 0$.

Let $\mathbf{M} = (X_t, P_x)$ be the Markov process on E with the semigroup $\{p_t\}_{t\geq 0}$. We assume $m(E) < \infty$. We also assume that the semigroup $\{p_t\}_{t\geq 0}$ is ultra-contractive (**UC**), that is, $\|p_t\|_{1,\infty} = C_t < \infty$. Here $\|\cdot\|_{1,\infty}$ means the operator norm from $L^1(E;m)$ to $L^\infty(E;m)$.

Note that

$$\begin{aligned}
R_1 1_{K^c}(x) &= \int_0^\infty e^{-t} p_t 1_{K^c}(x)dt \\
&= \int_0^\delta e^{-t} p_t 1_{K^c}(x)dt + \int_\delta^\infty e^{-t} p_t 1_{K^c}(x)dt \\
&\leq \int_0^\delta e^{-t}dt + \int_\delta^\infty e^{-t}\|p_t\|_{1,\infty} m(K^c)dt.
\end{aligned}$$

Indeed, we have last inequality from the following :

$$\|p_t 1_{K^c}\|_\infty \leq \|p_t\|_{1,\infty}\|1_{K^c}\|_1 \quad \text{and} \quad \|1_{K^c}\|_1 = \int_X 1_{K^c}(x)dm = m(K^c).$$

Since there exists δ such that $\int_0^\delta e^{-t}dt \leq \epsilon/2$ for all $\epsilon > 0$, we have

$$\int_0^\delta e^{-t}dt = 1 - e^{-\delta}. \tag{4.19}$$

If $t > s$, we have

$$\begin{aligned}
\|p_t f\|_\infty &= \|p_s \cdot p_{t-s} f\|_\infty \\
&\leq \|p_s\|_{1,\infty}\|p_{t-s} f\|_1 \\
&\leq \|p_s\|_{1,\infty}\|p_{t-s}\|_{1,1}\|f\|_1.
\end{aligned}$$

Since $\|p_{t-s}\|_{1,1} \leq 1$, we have

$$\|p_t\|_{1,\infty} \leq \|p_s\|_{1,\infty} \quad \text{for } t > s, \tag{4.20}$$

that is, C_t is monotone decrease. Hence,

$$\begin{aligned}
\int_\delta^\infty e^{-t}\|p_t\|_{1,\infty} m(K^c)dt &\leq \int_\delta^\infty e^{-t} \cdot C_\delta \cdot m(K^c)dt \\
&= C_\delta e^{-\delta} m(K^c) < \epsilon/2,
\end{aligned} \tag{4.21}$$

for a sufficiently large compact set K. Combining (4.19) and (4.21), we then have the tightness of $\mathbf{M}$.

Lemma 4.17. *If $m(E) < \infty$ and $\|p_t\|_{1,\infty} = C_t < \infty$, then $\mathbf{M}$ is tight.*

Applying the uniform large deviation principle, Theorem 4.16, we show the locally uniform large deviation principle and the conditional large deviation principle for the part process $\mathbf{M}^D$ on D in (4.10). By Lemma 4.6, we can find the bottom of the spectrum λ_2 and a ground state ϕ_0 that is strictly positive on E. We define the semigroup $\{p_t^{\phi_0}\}_{t\geq 0}$ by

$$p_t^{\phi_0} f = e^{\lambda_0 t}\frac{1}{\phi_0} p_t^D(\phi_0 f).$$

Let $\mathbf{M}^{\phi_0} = (X_t, P_x^{\phi_0})$ be the Markov process on D with the semigroup $\{p_t^{\phi_0}\}_{t\geq 0}$. Then, $\mathbf{M}^{\phi_0}$ is $\phi_0^2 m$-symmetric, $p_t^{\phi_0} 1 = 1$ and $\phi_0^2 m(D) = \int_D \phi_0^2 dm < \infty$. We assume that the semigroup $\{p_t^D\}_{t\geq 0}$ is intrinsically ultra-contractive (**IUC**), that is, the semigroup $\{p_t^{\phi_0}\}_{t\geq 0}$ is ultra-contractive.

Let us denote by $\mathcal{P}(D)$ the set of probability measures on D. Note that for an open set $G \subset \mathcal{P}(D)$

$$P_x(L_t \in G, t < \tau_D) = e^{-\lambda_2 t}\phi_0(x) E_x^{\phi_0}\left(\frac{1}{\phi_0(X_t)}; L_t \in G\right)$$

because

$$P_x^{\phi_0}(X_t \in G) = e^{-\lambda_2 t}\frac{1}{\phi_0} P_x^D\left(\phi_0(X_t); X_t \in G\right).$$

Since

$$\begin{aligned}
\frac{1}{t}\left(f - p_t^{\phi_0} f, f\right)_{\phi_0^2 m} &= \frac{1}{t}\left(\phi_0 f - e^{\lambda_2 t} p_t^D(\phi_0 f), \phi_0 f\right)_m \\
&= \frac{1}{t}\left(\phi_0 f - p_t^D(\phi_0 f), \phi_0 f\right)_m + \frac{1}{t}\left((1 - e^{\lambda_2 t}) p_t^D(\phi_0 f), \phi_0 f\right)_m \\
&\to \mathcal{E}^D(\phi_0 f, \phi_0 f) - \lambda_2 \int (\phi_0 f)^2 dm \text{ as } t \to \infty,
\end{aligned}$$

by definition of $\mathcal{E}^D$, we have

$$\mathcal{E}^{\phi_0}(f, f) = \mathcal{E}^D(\phi_0 f, \phi_0 f) - \lambda_2 \int (\phi_0 f)^2 dm.$$

For $K \subset D$ being compact,

$$\inf_{x\in K} P_x(L_t \in G, t < \tau_D) \geq e^{-\lambda_2 t}\left(\inf_{x\in K}\phi_0(x)\right)\frac{1}{\|\phi_0\|_\infty}\inf_{x\in K} P_x^{\phi_0}(L_t \in G).$$

Then, by Theorem 4.16, we have

$$\begin{aligned}\liminf_{t \to\infty} \frac{1}{t} \log \inf_{x\in K} P_x(L_t \in G, t < \tau_D) &\geq -\lambda_2 - \inf_{\phi^2\phi_0^2 m\in G} \mathcal{E}^{\phi_0}(\phi,\phi)\\ &= -\inf_{\phi^2 m\in G} \mathcal{E}^{D}(\phi,\phi).\end{aligned}$$

Hence we have

Theorem 4.18 (locally uniform lower bound)**.** *For any open set $G \in \mathscr{P}(D)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} P_x(L_t \in G; t < \tau_D) \geq -\inf_{\mu\in G} I_{\mathcal{E}^D}(\mu). \tag{4.22}$$

Here K is a compact set of D.

We now consider the locally uniform lower bound for symmetric Markov processes conditioned on non-absorption up to t. Since

$$P_x(L_t \in G|t < \tau_D) = \frac{P_x(L_t \in G; t < \tau_D)}{P_x(t < \tau_D)},$$

note that

$$\log P_x(L_t \in G|t < \tau_D) = \log P_x(L_t \in G; t < \tau_D) - \log P_x(t < \tau_D).$$

and

$$\begin{aligned}\inf_{x\in K} P_x(L_t \in G|t < \tau_D) &= \inf_{x\in K} \left(\frac{P_x(L_t \in G; t < \tau_D)}{P_x(t < \tau_D)}\right)\\ &\geq \frac{\inf_{x\in K} P_x(L_t \in G; t < \tau_D)}{\sup_{x\in D} P_x(t < \tau_D)}.\end{aligned}$$

By Theorems 4.3 and 4.18, we have

$$\begin{aligned}\frac{1}{t} \log \inf_{x\in K} P_x(L_t \in G|t < \tau_D) &\geq \frac{1}{t} \log \inf_{x\in K} P_x(L_t \in G; t < \tau_D) - \frac{1}{t} \log \sup_{x\in D} P_x(t < \tau_D)\\ &\geq -\inf_{u^2 m\in G} \mathcal{E}^D(u,u) - \lambda_2.\end{aligned}$$

Let $I_\tau = I_{\mathcal{E}^D} + \lambda_2$. Hence we have the following conditional lower bound.

Theorem 4.19. *For any open set $G \in \mathscr{P}(D)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} P_x(L_t \in G|t < \tau_D) \geq -\inf_{\mu\in G} I_\tau(\mu). \tag{4.23}$$

4.3 An example – killed Brownian motion

In this section, applying the results obtained in the previous chapter to killed Brownian motions, we give another proof of the main theorem in [3]. Let (P_x, X_t) be a standard d-dimensional Brownian motion ($d \geq 1$) on $\{\Omega, \mathscr{F}_t\}$, where $\Omega = C([0,\infty), \mathbb{R}^d)$ is the family of all continuous maps from $\mathbb{R}_+$ to $\mathbb{R}^d$ and $\mathscr{F}_t = \sigma\{X_s; 0 \leq s \leq t\}$ is the σ-algebra generated by $\{X_s; 0 \leq s \leq t\}$. Denote by $\{P_x; x \in \mathbb{R}^d\}$ the corresponding Markov family. Let D be an open bounded connected set in $\mathbb{R}^d$ and we set

$$X_t^D = \begin{cases} X_t, & \text{if } \tau_D \geq t, \\ \partial, & \text{if } \tau_D \leq t, \end{cases}$$

where ∂ is an extra point and τ_D is the first exit time of the domain D. In this section, we simply write τ for τ_D. We call X_t^D the Brownian motion killed outside D. Note that $P_x(\tau > t) > 0$ for any $x \in D$. The state space of X_t^D is $D \cup \partial$ and the transition function is

$$p_t^D(x, B) = P_x(X_t \in B; \tau > t),\ t > 0,\ x \in D,\ B \in \mathscr{B}(D), \tag{4.24}$$

where $\mathscr{B}(D)$ is the Borel σ-algebra on D. The transition function has a density with respect to the Lebesgue measure.

Proposition 4.20 (See [5, p.33])**.**

$$P_x(X_t \in B; \tau > t) = \int_B p^D(t; x, y)dy,\ x \in D,\ t > 0,\ B \in \mathscr{B}(D) \tag{4.25}$$

The density function $p_t^D(\cdot,\cdot)$ is symmetric continuous, and strictly positive on $D \times D$. Furthermore, it satisfies that

$$p_t^D(x, y) = \int_D p_l^D(x, z)p_{t-l}^D(z, y)dz,\ x, y \in D,\ t > l > 0. \tag{4.26}$$

Let $C_0^\infty(D) = \{f; f \in C^\infty(\mathbb{R}^d)$ and the support of f is in $D\}$. We define

$$\nabla f = \left(\frac{\partial}{\partial x_1}f, \cdots, \frac{\partial}{\partial x_d}f\right) \quad \text{and} \quad \Delta f = \sum_{i=1}^d \frac{\partial^2}{\partial x_d^2}f(x), \quad \text{for } f \in C_0^\infty(D).$$

Moreover, Let $H_0^1(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm

$$\|f\| = \left(\int_D f^2(x)dx + \frac{1}{2}\int_D \nabla f \cdot \nabla f dx\right)^{1/2}.$$

Let $L^2(D) = L^2(D; dx)$ be the real Hilbert space with inner product $\langle f, g\rangle = \int_D fg dx$, $f, g \in L^2(D)$. We can define a family of operators $\{p_t^D\}_{t\geq 0}$ on $L^2(D)$ associated with X_t^D as follows,

$$p_t^D f(x) = \int_D p_t^D(x,y) f(y) dy = E_x(f(X_t); \tau > t),\ t > 0,\ x \in D,\ f \in L^2(D). \quad (4.27)$$

$\{p_t^D\}_{t\geq 0}$ also has the strong Feller property, i.e.,

$$p_t^D f \in C_b(D),\ f \in L^\infty(D; dx),\ t > 0,$$

where $C_b(D) = \{f : f \text{ is a real valued bounded continuous function on } D\}$. We state further properties of the semigroup.

Proposition 4.21 (See [5, p.33 and p.56]). *$\{p_t^D\}_{t\geq 0}$ is a strong continuous, symmetric, compact, and contraction semigroup on $L^2(D)$. The infinitesimal generator is $\frac{\Delta}{2}$, $\mathcal{D} = \{f \in H_0^1(D) : \nabla f$ exists weakly and $\Delta f \in L^2(D)\}$. The corresponding Dirichlet form $\mathcal{E}(\cdot,\cdot)$ is*

$$\mathcal{E}(f,g) = \frac{1}{2}\int_D \nabla f \cdot \nabla g dx,\ f, g \in \mathcal{D}(\mathcal{E}) = H_0^1(D).$$

The followings are the eigenfunction expansion for the density p_t^D of the killed Brownian motion, and some estimates which are based on this expansion.

Proposition 4.22. *(i) (See, [14, p.123]) The density p_t^D has the following expansion :*

$$p_t^D(x,y) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y),$$

where $\{\lambda_n\}$ are the (nondecreasing) Dirichlet eigenvalues of $\frac{\Delta}{2}$ counting multiplicity, and ϕ_n are the corresponding eigenfunctions which form a complete orthonormal system of $L^2(D)$ and satisfy

$$\phi_n^2(x) \leq \exp(\lambda_n \epsilon)\left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}}.$$

Furthermore, for $0 < \epsilon < t$,

$$\sum_{n=1}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y) \leq \left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}} \sum_{n=1}^{\infty} \exp(-\lambda_n(t-\epsilon)) < +\infty. \quad (4.28)$$

Thus as $n_0 \to \infty$, $\sum_{n=n_0}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y)$ converges to 0 absolutely and uniformly on $D \times D$.

(ii) *(See, [9, p.336]) λ_1 is simple, so $\lambda_1 < \lambda_n$, for $n > 1$. and $\phi_1 \in C^\infty(D)$ with $\varphi > 0$.*

From now, we study the large deviation principle for the killed Brownian motion. Firstly, we define the Donsker-Varadhan I-function of the killed Brownian motion on the domain D by

$$I(\mu) = \begin{cases} \mathcal{E}(f, f), & \text{if } f = \left(\frac{d\mu}{dx}\right)^{1/2} \in H_0^1(D),, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.29}$$

The following is the known large deviation principle. An important point is that the rate function I attains the unique minimum at μ_0

Theorem 4.23. (i) *(See, [10, p.367]) I is a good rate function, i.e., for $r \in [0, \infty)$, the level set $\Psi_I(r) = \{\mu \in \mathcal{P}(D); I(\mu) \leq r\}$ is compact in $\mathcal{P}(D)$.*

(ii) *(See, [10, p.367] and [9, p.336]) I attains its unique minimum at μ_0 which is just the mean ratio qusai-stationary distribution, and $\lambda_1 = I(\mu_0) = \inf_{\mu \in \mathscr{P}_1(D)} I(\mu)$ is the first Dirichlet eigenvalue of $-\frac{\Delta}{2}$.*

(iii) *(See, [10, p.349]) (Lower bound) For any open set $G \in \mathcal{P}(D)$ and $\nu \in \mathcal{P}(D)$,*

$$\liminf_{t \to \infty} \frac{1}{t} \log P_\nu(L_t \in G, \tau > t) \geq - \inf_{\mu \in G} I(\mu). \tag{4.30}$$

(iv) *(See, [10, p.349]) (Uniform upper bound) For any set $C \in \mathcal{P}(D)$,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in D} P_x(L_t \in C, \tau > t) \leq - \inf_{\mu \in C} I(\mu). \tag{4.31}$$

We give the following slight generalization of the lower bound.

Theorem 4.24 (Local uniform lower bound)**.** *For any open set $G \in \mathcal{P}(D)$ and compact set $K \in D$,*

$$\liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in K} P_x(L_t \in G, \tau > t) \geq - \inf_{\mu \in G} I(\mu). \tag{4.32}$$

Proof. For any $\mu \in G$, there exist $\epsilon > 0$ and $f_i \in C_b(D)$ with $|f_i| \leq 1$, $i = 1, 2, \ldots, n$, such that

$$U_\epsilon = \left\{ \nu \in \mathcal{P}(D) : \left| \int f_i d(\nu - \mu) \right| < \epsilon, i = 1, 2, \cdots, n \right\} \subset G.$$

If $t > \frac{4}{\epsilon}$, we have for $f \in C_b(D)$ with $|f| \leq 1$

$$\begin{aligned}
&\left|\int f(x)L_{t-1}(\theta_1 w)(dx) - \int f(x)L_t(x)(dx)\right| \\
&\quad = \left|\frac{1}{t-1}\int_1^t f(w_s)ds - \frac{1}{t}\int_0^t f(w_s)ds\right| \\
&\quad = \left|\frac{1}{t}\int_0^1 f(w_s)ds - \frac{1}{t(t-1)}\int_1^t f(w_s)ds\right| \le \frac{2}{t} < \frac{\epsilon}{2}.
\end{aligned}$$

Let $a = \inf_{x,y\in K} p_1^D(x,y) > 0$. Combining the above with Theorem 4.23 and the Markov property, we see that

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K} P_x(L_t \in G, \tau > t) \\
&\quad \ge \liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K} P_x(L_{t-1}(\theta_1 w) \in U_{\frac{\epsilon}{2}}, \tau(\theta_1 w) > (t-1), \tau > 1) \\
&\quad = \liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K}\int P_{X_1}(L_{t-1} \in U_{\frac{\epsilon}{2}}, \tau > (t-1))1_{\{\tau>1\}}dP_x \\
&\quad = \liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K}\int P_y(L_{t-1} \in U_{\frac{\epsilon}{2}}, \tau > (t-1))p_1^D(x,y)dy \\
&\quad \ge \liminf_{t\to\infty}\frac{1}{t}\log\int_K a\cdot P_y(L_{t-1} \in U_{\frac{\epsilon}{2}}, \tau > (t-1))dy \\
&\quad \ge -I(\mu).
\end{aligned}$$

We have the theorem. □

The following large deviation principle for the conditional process is a direct consequence of Theorems 4.23 and 4.24. Let $I_\tau = I - \lambda_1$.

Theorem 4.25 (Conditional large deviation principle)**.**

(i) (Lower bound) For any open set $G \in \mathscr{P}(D)$ and $\nu \in \mathcal{P}(D)$,

$$\liminf_{t\to\infty}\frac{1}{t}\log P_\nu(L_t \in G|\tau > t) \ge -\inf_{\mu\in G} I_\tau(\mu). \tag{4.33}$$

(ii) (Local uniform lower bound) For any open set $G \in \mathcal{P}(D)$ and compact set $K \subset D$,

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K} P_x(L_t \in C|\tau > t) \ge -\inf_{\mu\in G} I_\tau(\mu). \tag{4.34}$$

(iii) (Upper bound) For any closed set $C \in \mathcal{P}(D)$ and $\nu \in \mathcal{P}(D)$,

$$\limsup_{t\to\infty}\frac{1}{t}\log P_\nu(L_t \in C|\tau > t) \le -\inf_{\mu\in C} I_\tau(\mu). \tag{4.35}$$

(iv) *(Local uniform upper bound) For any closed set* $C \in \mathcal{P}(D)$ *and compact set* $K \subset D$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} P_x(L_t \in C|\tau > t) \leq - \inf_{\mu\in C} I_\tau(\mu). \tag{4.36}$$

(v) I_τ *is good and* $I_\tau(\mu) = 0$ *if and only if* $\mu = \mu_0$.

Proof. Note that

(i) For any initial distribution ν,

$$\log P_\nu(L_t \in A|\tau > t) = \log P_\nu(L_t \in A, \tau > t) - \log P_\nu(\tau > t),$$

and by Theorem 4.23, $\lim_{t\to\infty} \frac{1}{t} \log P_\nu(\tau > t) = -\lambda_1$.

(ii) For compact set $K \subset D$,

$$\log \inf_{x\in K} P_x(L_t \in C|\tau > t) \geq \log \inf_{x\in K} P_x(L_t \in C, \tau > t) - \log \sup_{x\in D} P_x(\tau > t),$$

and

$$\log \sup_{x\in K} P_x(L_t \in C|\tau > t) \leq \log \sup_{x\in D} P_x(L_t \in C, \tau > t) - \log \inf_{x\in K} P_{x}.(\tau > t)$$

Then by using above results, we can easily show the desired assertions.

□

For $0 \leq s < t$, let

$$Q(t-s; x, y) := \exp(\lambda_1(t-s)) \frac{\phi_1(y)}{\phi_1(x)} p^D_{t-s}(x, y). \tag{4.37}$$

$Q(t; x, y)$ is the density of a probability transition function with respect to the Lebesgue measure. Then we can construct a Markov process $\{Y_s : 0 \leq x < \infty\}$ on $C([0,\infty), \mathbb{R}^d)$. Let $\{Q_x, x \in \mathbb{R}^d\}$ be the associated Markov family on $C([0,\infty), \mathbb{R}^d)$.

Finally, we prove the large deviation principle for the limiting process. The following lemma is used to compare the limiting process with the conditional process, which is important in deriving the large deviation principle.

Lemma 4.26. *For* $A \in \mathscr{F}_t$,

$$Q_x(A) = \frac{\exp(\lambda_1 t)}{\phi_1(x)} E_x(\phi_1(x_t); A, \tau > t). \tag{4.38}$$

Proof. Let $A = A_{t_0} \times A_1 \times \cdots \times A_k$ for $A_i \in \mathscr{B}(D)$ and $0 = t_0 < t_1 < \cdots < t_k = t$. From (4.37), we have

$$\begin{aligned}
&Q_x((Y_{t_0}, Y_{t_1}, \cdots, Y_{t_k}) \in A) \\
&= \int_A \prod_{i=1}^k Q(t_i - t_{i-1}; y_{i-1}, y_i)\delta_x(dy_0)dy_1 \cdots dy_k \\
&= \int_A \prod_{i=1}^k \exp(\lambda_1(t_i - t_{i-1}))\frac{\phi_1(y_i)}{\phi_1(y_{i-1})}p^D_{t_i - t_{i-1}}(y_{i-1}, y_i)\delta_x(dy_0)dy_1 \cdots dy_k \\
&= \frac{\exp(\lambda_1 t)}{\phi_1(x)} \int_A \phi_1(y_k) \prod_{i=1}^k p^D_{t_i - t_{i-1}}(y_{i-1}, y_i)\delta_x(dy_0)dy_1 \cdots dy_k \\
&= \frac{\exp(\lambda_1 t)}{\phi_1(x)} E_x(\phi_1(x_t); A, \tau > t).
\end{aligned}$$

The proof is completed by extending the above equality for any $A \in \mathscr{F}_t$. $\square$

Lemma 4.27. *For any open set $O \subset D$, if V is an open set in $\mathcal{P}(D)$, then $V \cap \mathcal{P}(O)$ is open set in $\mathcal{P}(O)$.*

Proof. If $V \cap \mathcal{P}(O) = \emptyset$, it is open. Otherwise $\forall \mu \in V \cap \mathcal{P}(O)$, there exists a open set U in $\mathcal{P}(D)$ as follows:

$$U = \left\{ \nu \in \mathcal{P}(D) : \Big| \int_D f_i \nu(dx) - \int_D f_i \mu(dx) \Big| < \epsilon_i, i = 1, 2, \cdots, k \right\} \subset V,$$

where $f_i \in C_b(D)$. Since the function f_i is restricted to O are also bounded and continuous in O, we see that

$$U \cap \mathcal{P}(O) = \left\{ \nu \in \mathcal{P}(O) : \Big| \int_O f_i \nu(dx) - \int_O f_i \mu(dx) \Big| < \epsilon_i, i = 1, 2, \cdots, k \right\}$$

is a open set in $\mathcal{P}(O)$. $\square$

The following is an approximation result.

Lemma 4.28. *Given $\mu \in \mathcal{P}(D)$. If $I(\mu) < \infty$, then for any $\epsilon > 0$ there is an open subset O of D with $\overline{O} \subset D$, and a $\nu(dx) = g^2 dx \in \mathcal{P}(O)$ with $g \in C_0^\infty(D)$, such that $\|\mu - \nu\|_{Var} < \epsilon$ and $|I(\mu) - I(\nu)| < \epsilon$.*

Proof. By definition of I and the assumption, $I(\mu) = \frac{1}{2}\int_D \nabla f \cdot \nabla f dx$ with $f = (\frac{d\mu}{dx})^{1/2} \in H_0^1(D)$. Thus by definition of $H_0^1(D)$, we can find $f_n \in C_0^\infty(D)$, $n = 1, 2, \ldots$, such that $\int_D f_n^2 dx = 1$ and

$$\lim_{n\to\infty} \left(\int_D (f - f_n)^2(x)dx + \frac{1}{2}\int_D \nabla(f - f_n) \cdot \nabla(f - f_n)dx \right)^{1/2} = 0.$$

Since

$$\|\mu - 1_D f_n^2 dx\|_{\mathrm{Var}} = \int_D |f^2 - f_n^2| dx,$$

we have the lemma. □

Now we can state the large deviation result for the limiting process.

Theorem 4.29 (Large deviation principle for the limiting process)**.**

(i) (Local uniform upper bound) For any compact set $K \in D$ and closed set $C \subset \mathcal{P}(D)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} Q_x(L_t \in C) \leq -\inf_{\mu\in C} I_\tau(\mu). \tag{4.39}$$

(ii) (Local uniform lower bound) For any compact set $K \in D$ and open set $G \subset \mathcal{P}(D)$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} Q_x(L_t \in G) \geq -\inf_{\mu\in G} I_\tau(\mu). \tag{4.40}$$

Proof. (i) Since ϕ is continuous, strictly positive and bounded above on D,

$$M = \frac{\sup_{x\in D} \phi_1(x)}{\inf_{x\in K} \phi_1(x)}$$

is also strictly positive and finite. Thus by Theorem 4.23 and lemma 4.26, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} Q_x(L_t \in C) \\
&\quad = \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} \left\{ \frac{\exp(\lambda_1 t)}{\phi_1(x)} E_x(\phi_1(x_t), L_t \in C, \tau > t) \right\} \\
&\quad \leq \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} \{M \exp(\lambda_1 t) P_x(L_t \in C, \tau > t)\} \\
&\quad \leq -\inf_{\mu\in C} I_\tau(\mu).
\end{aligned}$$

(ii) To prove the theorem, it is enough to show that for any $\mu \in G$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} Q_x(L_t \in G) \geq I(\mu) + \lambda_1.$$

If $I(\mu) = \infty$, it is trivial. Otherwise by Lemma 4.28, $\forall \epsilon > 0$, there exists an open subset O_1 of G with $\overline{O}_1 \in G$, such that $\mu_1(dx) = f_1^2 dx$ with $f_1 \in C_0^\infty(O_1)$

and $|I(\mu_1) - I(\mu)| < \epsilon$. We can choose another open subset O of G with $\overline{O} \subset G$ and $\overline{O} \cup K \subset O$. Thus,

$$m = \frac{\inf_{x \in O} \phi_1(x)}{\sup_{x \in K} \phi_1(x)}$$

is strictly positive and finite. By combining the above with Theorem 4.24, Lemma 4.26 and Lemma 4.27, we have that

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} Q_x(L_t \in G) \\
&\quad = \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} \left\{ \frac{\exp(\lambda_1 t)}{\phi_1(x)} E_x(\phi_1(x_t), L_t \in G, \tau > t) \right\} \\
&\quad = \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} \left\{ \frac{\exp(\lambda_1 t)}{\phi_1(x)} E_x(\phi_1(x_t), L_t \in G \cap \mathcal{P}(O), \tau_O > t) \right\} \\
&\quad \leq \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} \{ m \exp(\lambda_1 t) P_x(L_t \in G \cap \mathcal{P}(O), \tau_O > t) \} \\
&\quad \geq -I(\mu_1) + \lambda_1 \\
&\quad \geq -(I(\mu) + \epsilon) + \lambda_1,
\end{aligned}$$

where $\tau_O = \inf\{t > 0; X_t(w) \in O^c\}$. Since ϵ is arbitrary, we have the theorem.

$\square$

Chapter 5

Appendix

5.1 L^p-independence of spectral bounds

In this chapter, let $C_u(X)$ be the set of continuous functions on E that have the limit as $x \to \infty$. For $f \in C_u(X)$, put $f(\infty) = \lim_{x\to\infty} f(x)$. Under the assumptions **(I)**, **(DF)** and **(C)**, we obtain the following results.

Theorem 5.1. *Let* $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$

(i) There exist constants C *and* $\kappa(\mu)$ *such that*

$$\|p_t^\mu\|_{p,p} \leq Ce^{\kappa(\mu)t},\ 1 \leq \forall p \leq \infty,\ t > 0.$$

Here $\|\cdot\|_{p,p}$ *means the operator norm on* $L^p(E;m)$.

(ii) p_t^μ *is a strongly continuous symmetric semigroup on* $L^p(E;m)$ *and the closed form corresponding to* p_t^μ *is identical to* $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$.

(iii) For each $f \in \mathscr{B}_b(E)$, $p_t^\mu f \in C_b(E)$.

(iv) $p_t^\mu(C_u(E)) \subset C_u(E)$ *and* $\lim_{x\to\infty} p_t^\mu f(x) = f(\infty)$ *for* $f \in C_u(E)$.

Proof. The statements (i) and (ii) follow from results in [1]. From [4, Theorem 3], the semigroup p_t^μ possesses the strong Feller property (iii).

(iv) By (i),

$$|E_x(\exp(-A_t^\mu)f(X_t))| \leq |E_x(\exp(-A_t^{2\mu})|^{1/2}|E_x(f^2(X_t))|^{1/2},\ for\ f \in C_\infty(E),$$

and $\sup_{x\in E} E_x(\exp(-A_t^{2\mu})) < \infty$. Hence $\lim_{x\to\infty} p_t^\mu f(x) = 0$ from the assumption **(DF)**. Since $f(x) - f(\infty) \in C_\infty(E)$ and $p_t^\mu f(x) = p_t^\mu(f - f(\infty)) + f(\infty)p_t^\mu 1(x)$, it is enough to show that

$$\lim_{x\to\infty} p_t^\mu 1(x) = \lim_{x\to\infty} E_x(\exp(-A_t^\mu)) = 1.$$

Let $\mu \in \mathcal{K}_\infty$ and $K \subset E$ a compact set.

$$\begin{aligned} E_x(\exp(A_t^{\mu_K})) =& E_x(\exp(A_t^{\mu_K}); \sigma_K > t) + E_x(\exp(A_t^{\mu_K}); \sigma_K \leq t) \\ =& P_x(\sigma_K > t) + E_x(\exp(A_t^{\mu_K}); \sigma_K \leq t), \end{aligned}$$

where $\sigma_K = \inf\{t > 0 : X_t \in E\}$. We have that $P_x(\sigma_K > t)$ converges to 1 as $x \to \infty$ from the assumption (**DF**). Indeed, let f be a strictly positive function in $C_\infty(E)$. Then

$$P_X(\sigma_K \leq t) \leq \frac{e^{\lambda t}}{c} E_x(e^{-\lambda \sigma_K} G_\lambda f(X_{\sigma_K})) \leq \frac{e^{\lambda t}}{c} G_\lambda f(x),$$

where $c = \inf_{x \in K} G_\lambda f(x) > 0$. Since

$$e^{\lambda(\sigma_K - t)} \geq 1 \text{ and } \frac{G_\lambda f(X_{\sigma_K})}{\inf_{x \to \infty} G_\lambda f(x)} \geq 1 \text{ for } \sigma_K \leq t,$$

we can easily see the first inequality. Moreover, since

$$\begin{aligned} E_x(e^{-\lambda \sigma_K} G_\lambda f(X_{\sigma_K})) =& E_x(e^{-\lambda \sigma_K} \int_0^\infty e^{-\lambda t} p_t f(X_{\sigma_K}) dt) \\ =& \int_{\sigma_K}^\infty e^{-\lambda t} p_t f(x) dt \\ \leq& G_\lambda f(x), \end{aligned}$$

we have the second inequality.
In addition, since

$$E_x(\exp(A_t^{\mu_K}); \sigma_K \leq t) \leq E_x(\exp(A_t^{2\mu_K}))^{1/2} P_x(\sigma_K \leq t)^{1/2},$$

the left-hand side above converges to 0 as $x \to \infty$. Therefore, we have

$$\lim_{x \to \infty} E_x(\exp(A_t^{\mu_K})) = 1.$$

By the definition of $\mathcal{K}_\infty$,

$$\lim_{K \uparrow E} \sup_{x \in E} E_x(A_t^{\mu_{K^c}}) = \lim_{K \uparrow E} \sup_{x \in E} \int_{K^c} G(x, y) d\mu(y) = 0. \tag{5.1}$$

By Khasminskii's lemma,

$$\sup_{x \in E} E_x(\exp(A_t^{\mu_{K^c}})) \leq \frac{1}{1 - \sup_{x \in E} E_x(A_t^{\mu_{K^c}})}. \tag{5.2}$$

From (5.1) and (5.2), we obtain

$$\lim_{K \uparrow E} \sup_{x \in E} E_x(\exp(A_t^{\mu_{K^c}})) \leq 1.$$

Since

$$\begin{aligned}\limsup_{x\to\infty} E_x(\exp(A_t^\mu)) &= \limsup_{x\to\infty} E_x(\exp(A_t^{\mu_K})\exp(A_t^{\mu_{K^c}})) \\ &\le \limsup_{x\to\infty} (E_x(\exp(A_t^{2\mu_K}))^{1/2} E_x(\exp(A_t^{2\mu_{K^c}}))^{1/2}) \\ &\le \sup_{x\in E} E_x(\exp(A_t^{2\mu_{K^c}}))^{1/2}),\end{aligned}$$

we have

$$\liminf_{x\to\infty} E_x(\exp(-A_t^\mu)) \ge \frac{1}{\limsup_{x\to\infty} E_x(\exp(A_t^\mu))} \ge 1.$$

Note that for $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$

$$E_x(\exp(-A_t^{\mu^+})) \le E_x(\exp(-A_t^\mu)) \le E_x(\exp(A_t^{\mu^-})),$$

we have

$$\lim_{x\to\infty} E_x(\exp(-A_t^\mu)) = 1.$$

$\square$

Let $\mathcal{P}(E)$ be the set of probability measures on E with the weak topology. We define a function $I_{\mathcal{E}^\mu}$ on $\mathcal{P}(E)$ by

$$I_{\mathcal{E}^\mu}(v) = \begin{cases} \mathcal{E}^\mu(\sqrt{f}, \sqrt{f}), & \text{if } v = f\cdot dx,\ \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty, & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{D}_{++}(\mathcal{H}^\mu) = \{\phi = R_\alpha^\mu g; \alpha > \kappa(\mu), g \in C_u(E) \text{ with } g \ge \exists\epsilon > 0\},$$

where $R_\alpha^\mu f(x) = \int_0^\infty e^{-\alpha t} p_t^\mu f(x) dt$. Here $\kappa(\mu)$ is the constant in Theorem 5.1(i). For $\phi = R_\alpha^\mu g \in \mathcal{D}_{++}(\mathcal{H}^\mu)$, let

$$\mathcal{H}^\mu \phi = \alpha\phi - g.$$

We define the *I-function* as follows:

$$I_\mu(v) = -\inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^\mu)} \int_X \frac{\mathcal{H}_\mu \phi}{\phi} dv,\ v \in \mathcal{P}(E).$$

It follows that

$$I_{\mathcal{E}^\mu}(v) = I_\mu(v),\ v \in \mathcal{P}(E).$$

We define a transition density $\bar{p}_t(x, dy)$ on E_∞ by

$$\bar{p}_t(x, D) = p_t(x, D \setminus \{\infty\}),\ x \in D,$$

and

$$\bar{p}_t(\infty, D) = \delta_\infty(D) := \begin{cases} 1 & \infty \in D, \\ 0, & \infty \notin E. \end{cases}$$

Let $\bar{\mathbf{M}} = (\bar{P}_x, X_t)$ be the Markov process on E_∞ with transition probability $\bar{p}_t(x, dy)$. $\bar{\mathbf{M}}$ is an extension of $\mathbf{M}$ and ∞ is to be a trap. For $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$, we define $\bar{p}^\mu_t$ and $\bar{R}^\mu_\alpha$ by

$$\bar{p}^\mu_t f(x) = \bar{E}_x(\exp(-A^\mu_t)f(X_t)),\ \bar{R}^\mu_\alpha f(x) = \int_0^\infty e^{-\alpha t}\bar{p}^\mu_t f(x)dt,\ f \in \mathcal{B}(E_\infty).$$

Then, $\bar{R}^\mu_\alpha f(x) = R^\mu_\alpha f(x)$ on $x \in E$ and $\bar{R}^\mu_\alpha f(\infty) = f(\infty)$. Let

$$\mathcal{D}_{++}(\bar{\mathcal{H}}^\mu) = \{\phi = \bar{R}^\mu_\alpha g; \alpha > \kappa(\mu), g \in C(E_\infty) \text{ with } g > 0\}.$$

By Theorem 5.1 (iv), for $\phi = R^\mu_\alpha g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$

$$\lim_{x\to\infty} \phi(x) = \frac{g(\infty)}{\alpha}. \tag{5.3}$$

We define a function $\bar{I}_\mu$ on $\mathcal{P}(E_\infty)$, the set of probability measures on E_∞, by

$$\bar{I}_\mu(v) = -\inf_{\phi\in\mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)} \int_{X_\infty} \frac{\bar{\mathcal{H}}_\mu \phi}{\phi} dv,\ v \in \mathcal{P}(E_\infty),$$

where $\bar{\mathcal{H}}^\mu\phi = \alpha\bar{R}^\mu_\alpha g - g$ for $\phi = \bar{R}^\mu_\alpha \phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$.

For $\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$, we define the multiplicative functional N^ϕ_t by

$$N^\phi_t = e^{-A^\mu_t}\left(\frac{\phi(X_t)}{\phi(X_0)}\right)\exp\left(-\int_0^t \frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_s)ds\right).$$

Let us define the sequence of sets $\{K_n\}_{n=1}^\infty$ by $K_n = \{x \in E; \phi(x) \geq \frac{1}{n}\}$ and denote by K^o_n the fine interior of K_n. Let τ_n be the first exit time from K^o_n: $\tau_n = \inf\{t > 0; X_t \notin K^o_n\}$.

Lemma 5.2. *For each n*

$$N^\phi_{t\wedge\tau_n} - 1 = \int_0^{t\wedge\tau_n} \frac{1}{\phi(X_0)}\exp\left(-\int_0^s \frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_u)du\right)dM^{\mu,\phi}_s;\ P_x\text{-a.e.}, \tag{5.4}$$

where $M^{\mu,\phi}_t = e^{-A^\mu_t}\phi(X_t) - \phi(X_0) - \int_0^t e^{-A^\mu_s}\bar{\mathcal{H}}^\mu\phi(X_s)ds$.

Proof. The right-hand side of (5.4) is equal to

$$\frac{1}{\phi(X_0)}\int_0^{t\wedge\tau_n} \exp\left(-\int_0^s \frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_u)du\right)\left(d(e^{-A^\mu_s}\phi(X_s)) - e^{-A^\mu_s}\bar{\mathcal{H}}^\mu\phi(X_s)ds\right).$$

Since

$$
\begin{aligned}
d&\left(e^{-A^\mu_s}\phi(X_s))\exp\left(-\int_0^s \frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_u)du\right)\right) \\
&= \exp\left(-\int_0^s \frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_u)du\right)\left(d(e^{-A^\mu_s}\phi(X_s)) - e^{-A^\mu_s}\bar{\mathcal{H}}^\mu\phi(X_s)ds\right),
\end{aligned}
$$

we have the lemma. □

Since $E_x(M_t^{\mu,\phi}) = 0$ and $M_t^{\mu,\phi} = M_{s+t}^{\mu,\phi} + e^{-A^\mu_s} M_t^{\mu,\phi} \circ \theta_s$, $M_t^{\mu,\phi}$ is a martingale with respect to P_x. Here θ_t, $t \geq 0$, is the shift operator satisfying $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geq 0$.

Indeed, for $\phi = \bar{R}^\mu_\alpha \phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$, $E_x(M_t^{\mu,\phi})$ is equal to

$$
E_x\left(e^{-A^\mu_t}\bar{R}^\mu_\alpha g(X_t) - \bar{R}^\mu_\alpha g(X_0) - \int_0^t e^{-A^\mu_s}(\alpha\bar{R}^\mu_\alpha g - g)(X_s)ds\right). \tag{5.5}
$$

By using definition of $\bar{R}^\mu_\alpha$ and the semigroup property of $\{p_t^\mu\}_{t>0}$,

$$
\begin{aligned}
E_x\left(e^{-A^\mu_t}\bar{R}^\mu_\alpha g(X_t)\right) =& E_x\left(e^{-A^\mu_t}E_{X_t}\left(\int_0^\infty e^{-(\alpha s + A^\mu_s)}g(X_s)ds\right)\right) \\
=& E_x\left(e^{-A^\mu_t}\int_0^\infty e^{-\alpha s}p_s^\mu g(X_t)ds\right) \\
=& \int_0^\infty e^{-\alpha s}p_{t+s}^\mu g(x)ds,
\end{aligned}
$$

$$
\begin{aligned}
E_x\left(\bar{R}^\mu_\alpha g(X_0)\right) =& E_x\left(E_{X_0}\left(\int_0^\infty e^{-(\alpha s + A^\mu_s)}g(X_s)ds\right)\right) \\
=& E_x\left(\int_0^\infty e^{-\alpha s}p_s^\mu g(X_0)ds\right) \\
=& \int_0^\infty e^{-\alpha s}p_s^\mu g(x)ds,
\end{aligned}
$$

and

$$
E_x\left(\int_0^t e^{-A^\mu_s}g(X_s)ds\right) = \int_0^t p_s^\mu g(x)ds.
$$

Finally, by using integral by parts,

$$
\begin{aligned}
E_x\left(\int_0^t e^{-A_s^\mu}\alpha\bar{R}_\alpha^\mu g(X_s)ds\right) &= E_x\left(\int_0^t \alpha e^{-A_s^\mu}\int_0^\infty e^{-\alpha k}p_k^\mu g(X_s)dkds\right)\\
&= \int_0^t\int_0^\infty \alpha e^{-\alpha k}e^{-\alpha k}p_{k+s}^\mu g(x)dkds\\
&= \int_0^t \alpha e^{-\alpha s}\int_s^\infty e^{-\alpha k}p_k^\mu g(x)dkds\\
&= e^{\alpha t}\int_t^\infty e^{-\alpha k}p_k^\mu g(x)dk - \int_0^\infty e^{-\alpha k}p_k^\mu g(x)dk + \int_0^t p_s^\mu g(x)ds\\
&= \int_0^\infty e^{-\alpha k}p_{t+k}^\mu g(x)dk - \int_0^\infty e^{-\alpha k}p_k^\mu g(x)dk + \int_0^t p_s^\mu g(x)ds.
\end{aligned}
$$

Hence, by combining above results, we see $E_x(M_t^{\mu,\phi}) = 0$.

Therefore N_t^ϕ is a local martingale with N_0^ϕ from Lemma 5.2. Then we have

$$E_x\left(e^{-A_t^\mu}\left(\frac{\phi(X_t)}{\phi(X_0)}\right)\exp\left(-\int_0^t\frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_s)ds\right)\right) \le 1.$$

So, we see

$$\sup_{x\in E} E_x\left(\exp\left(-A_t^\mu - \int_0^t\frac{\bar{\mathcal{H}}^\mu\phi}{\phi}(X_s)ds\right)\right) \le \frac{\sup_{x\in E}\phi(x)}{\inf_{x\in E}\phi(x)}.$$

Hence, for any Borel set C of $\mathcal{P}(E_\infty)$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in E}E_x(\exp(-A_t^\mu); L_t\in C) \le \inf_{\phi\in\mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)}\sup_{\mu\in C}\int_{E_\infty}\frac{\bar{\mathcal{H}}^\mu\phi}{\phi}dv. \tag{5.6}$$

Note that $\bar{\mathcal{H}}^\mu\phi/\phi \in C(E_\infty)$ and that $\mathcal{P}(E_\infty)$ is compact with respect to the weak topology. We can obtain the following proposition from (5.6) in exactly the same way as one in [8].

Proposition 5.3. *Let $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Then*

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in E}E_x(\exp(-A_t^\mu)) \le -\inf_{v\in\mathcal{P}(E_\infty)}\bar{I}_\mu(v),\ \ x\in E. \tag{5.7}$$

Lemma 5.4. *For $v \in \mathcal{P}(E_\infty)\setminus\{\delta_\infty\}$, let $\hat{v}(\cdot) = v(\cdot)/v(E) \in \mathcal{P}(E)$. Then*

$$\bar{I}_\mu(v) = I_\mu(v) = v(E)I_{\mathcal{E}^\mu}(\hat{v}).$$

Proof. By combining (5.3) and $\bar{\mathcal{H}}^\mu\phi(x) = \mathcal{H}^\mu\phi(x)$ on $x\in E$, for $\phi = \bar{R}_\alpha^\mu\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)$,

$$\bar{\mathcal{H}}^\mu\phi(x) = \alpha\phi(x) - g(x) \to 0,\ \ x\to\infty.$$

Therefore for $v \in \mathcal{P}(E_\infty)$

$$\begin{aligned}\bar{I}_\mu(v) &= -\inf_{\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^\mu)} \int_{X_\infty} \frac{\bar{\mathcal{H}}^\mu \phi}{\phi} dv = -\inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^\mu)} \int_X \frac{\mathcal{H}^\mu \phi}{\phi} dv \\ &= -\inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^\mu)} v(X) \int_X \frac{\mathcal{H}^\mu \phi}{\phi} d\hat{v} = v(E) \cdot I_{\mathcal{E}^\mu}(\hat{v}).\end{aligned}$$

□

Since there exists a one-to-one correspondence between $\mathcal{P}(E_\infty) \setminus \{\delta_\infty\}$ and $(0,1] \times \mathcal{P}(E)$ as follows:

$$v \in \mathcal{P}(E_\infty) \setminus \{\delta_\infty\} \longrightarrow \left(v(E), \hat{v}(\cdot) = \frac{v(\cdot)}{v(E)} \right) \in (0,1] \times \mathcal{P}(E),$$

and $\bar{I}_\mu(\delta_\infty) = 0$, we see that

$$\inf_{v \in \mathcal{P}(E_\infty)} \bar{I}_\mu(v) = \inf_{0 \le \eta \le 1, v \in \mathcal{P}(E)} (\eta \cdot I_{\mathcal{E}^\mu}(v)) = \inf_{0 \le \eta \le 1} \left(\eta \cdot \inf_{v \in \mathcal{P}(E)} I_{\mathcal{E}^\mu}(v) \right). \tag{5.8}$$

We define the L^p-spectral bounds of $\{p_t^\mu\}_{t>0}$ by

$$\lambda_p(\mu) = -\lim_{t \to \infty} \frac{1}{t} \log \|p_t^\mu\|_{p,p}, \quad 1 \le p \le \infty,$$

where $\|p_t^\mu\|_{p,p}$ is the operator norm of p_t^μ from $L^p(E;m)$ to $L^p(E;m)$.
We then have

Corollary 5.5. *For $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$,*

$$\lambda_\infty(\mu) \ge \inf_{0 \le \eta \le 1} \left(\eta \cdot \inf_{v \in \mathcal{P}(E)} I_{\mathcal{E}^\mu}(v) \right) = \inf_{0 \le \eta \le 1} (\eta \cdot \lambda_2(\mu)). \tag{5.9}$$

Proof. Since $\sup_{x \in X} E_x(\exp(-A_t^\mu))$ equals $\|p_t^\mu\|_{\infty,\infty}$, the left-hand side of (5.7) equals $-\lambda_\infty(\mu)$. By combining Proposition 5.3 and (5.8), we have the first inequality.

By spectral theorem, $\lambda_2(\mu)$ is identical to the principal eigenvalue of the self-adjoint operator $\mathcal{H}^\mu$. By also the variational formula for the principal eigenvalue, we have

$$\lambda_2(\mu) = \inf_{v \in \mathcal{P}(E)} I_{\mathcal{E}^\mu}(v). \tag{5.10}$$

Hence we have the second equality. □

If $\lambda_2(\mu) \le 0$, then $\inf_{0 \le \eta \le 1} (\eta \cdot \lambda_2(\mu)) = \lambda_2(\mu)$. So we have

Corollary 5.6. *If $\lambda_2(\mu) \le 0$, then*

$$\lambda_\infty(\mu) \ge \lambda_2(\mu)$$

By using the symmetry and the positivity of p_t^μ, we have

$$\|p_t^\mu\|_{2,2} \leq \|p_t^\mu\|_{p,p} \leq \|p_t^\mu\|_{\infty,\infty},\ 1 \leq p \leq \infty.$$

Therefore the following inequality holds generally:

$$\lambda_\infty(\mu) \leq \lambda_2(\mu)$$

Hence we have

Theorem 5.7. *Assume* **(I)**, **(DF)** *and* **(C)**. *Let* $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$. *If* $\lambda_2(\mu) \leq 0$, *then*

$$\lambda_p(\mu) = \lambda_2(\mu),\ 1 \leq p \leq \infty.$$

Corollary 5.8. *Assume* **(I)**, **(DF)** *and* **(C)**. *Then for* $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$ *with* $\lambda_2(\mu) \leq 0$

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(\exp(-A_t^\mu)\right) = -\inf\left\{\mathcal{E}^\mu(u,u); u \in \mathcal{D}(\mathcal{E}), \int_E u^2 dm = 1\right\},\ x \in E.$$

Proof. In [19], they showed that for a symmetric Markov process with the assumptions **(I)**, **(DF)** and **(C)**

$$\liminf_{t\to\infty} \frac{1}{t} \log E_x\left(\exp\left(-A_t^\mu\right)\right) \geq -\lambda_2(\mu),\ x \in E.$$

On the other hand, we see from Theorem 5.7 that

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left(\exp(-A_t^\mu)\right) = -\lambda_2(\mu).$$

□

5.2 Gaugeability

In this section, we assume that $\mathbf{M} = (P_x, X_t)$ is an irreducible, transient, m-symmetric Markov process on E. In [2], Chen defined the Green-tight class in slightly different way as follows:

Definition 5.9. Suppose that μ is a signed smooth measure whose associated continuous additive functional is A. Let A^+ and A^- be the PCAFs with Revuz measures μ^+ and μ^-. Let $|A| = A^+ + A^-$ and $|\mu| = \mu^+ + \mu^-$. A measure μ is said to be in the class $\mathcal{K}_\infty$ if for any $\epsilon > 0$, there are a Borel set $K = K(\epsilon)$ of finite $|\mu|$-measure and a constant $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{x\in E} \int_{K^c} G(x,y)|\mu|(dy) < \epsilon \text{ and } \sup_{x\in E} \int_B G(x,y)|\mu|(dy) < \epsilon$$

for all measurable sets $B \subset K$ with $|\mu|(B) < \delta$.

However two definitions are equivalent under the strong Feller property ([13, Lemma 4.1]).

Suppose that μ is a signed smooth measure such that $\mu^+ \in \mathcal{K}_\infty$. Let A^+ and A^- be the PCAFs corresponding to μ^+ and μ^- respectively. Then $A := A^+ - A^-$ is the continuous additive functional with Revuz measure μ. We see that A^+_ζ is P_x-integrable. So the gauge function $g_\mu(x) := E_x(\exp(A_\zeta))$ is well-defined on E.

Theorem 5.10. *For smooth measure μ with $\mu^+ \in \mathcal{K}_\infty$, the gauge function g_μ is either bounded or identically infinite on E.*

Definition 5.11. Let μ be a signed smooth measure such that μ^+ is in $\mathcal{K}_\infty$. We say that μ is gaugeable if the gauge function $x \mapsto E_x(\exp(A_\zeta))$ is bounded on E.

By applying results in [16] to the 1-subprocess of $\mathbf{M}$, we obtain

$$\sup_{x\in X} E_x(e^{\lambda\zeta}) < \infty \text{ if and only if } \lambda < \lambda_\infty. \tag{5.11}$$

Theorem 5.12. *Suppose that the constant function 1 is in $\mathcal{K}_\infty$. Then $\lambda_2 = \lambda_\infty$. That is, the spectral radius λ_p is independent of $p \in [2,\infty]$*

Proof. Note that $\lambda_\infty \le \lambda_2$ and hence it is enough to show that if $\lambda < \lambda_2$, then $\lambda < \lambda_\infty$. For this, without loss of generality, we may assume $0 < \lambda < \lambda_2$. Let $\epsilon > 0$ and $q > 1$ be such that $\lambda + \epsilon < \lambda_2$ and $\frac{\lambda}{\lambda+\epsilon} + \frac{1}{q} = 1$. By definition of λ_2, we have $\|G_{-(\lambda+\epsilon)}\|_{2,2} < \infty$, where $G_{-a} = \int_0^\infty e^{as} p_s ds$. Since $1 \in \mathcal{K}_\infty$, by Definition 5.9, there exists an open set K of finite m-measure such that $\sup_{x\in E} G1_{K^c} \le (2\lambda q)^{-1}$. Since $1_K \in L^2(E;m)$, the function

$$G_{-(\lambda+\epsilon)}1_K(x) = E_x\left(\int_0^\infty e^{(\lambda+\epsilon)s} 1_K(X_s)ds\right)$$

is L^2-integrable. Using the elementary inequality

$$e^{(\lambda+\epsilon)a} - e^{(\lambda+\epsilon)b} \ge e^{(\lambda+\epsilon)(a-c)} - e^{(\lambda+\epsilon)(b-c)} \text{ for } a > b > c \ge 0,$$

we have

$$1 + (\lambda+\epsilon)G_{-(\lambda+\epsilon)}1_K(x) \ge E_x\left(\exp\left((\lambda+\epsilon)\int_0^\infty 1_K(X_s)ds\right)\right).$$

Now by Hölder's inequality,

$$\begin{aligned} E_x(e^{\lambda\zeta}) &= E_x\left(\exp\left(\lambda\int_0^\infty 1_{K^c}(X_s)ds\right)\exp\left(\lambda\int_0^\infty 1_K(X_s)ds\right)\right) \\ &\le \left(E_x\left(\exp\left(q\lambda\int_0^\infty 1_{K^c}(X_s)ds\right)\right)\right)^{1/q} \\ &\quad \times \left(E_x\left(\exp\left((\lambda+\epsilon)\int_0^\infty 1_K(X_s)ds\right)\right)\right)^{\lambda/(\lambda+\epsilon)} \\ &\le 2^{1/q}(1 + (\lambda+\epsilon)G_{-(\lambda+\epsilon)}1_K(x))^{\lambda/(\lambda+\epsilon)}. \end{aligned} \tag{5.12}$$

In the last inequality, we used Khasminskii's inequality. Thus $E_x(e^{\lambda\zeta}) < \infty$ m-a.e. on E and therefore by Theorem 5.10, $\sup_{x\in E} E_x(e^{\lambda\zeta}) < \infty$. This implies $\lambda < \lambda_\infty$ and so $\lambda_2 = \lambda_\infty$. □

Theorem 5.13. *Suppose that* $\mathbf{M}$ *satisfies* (**DF**) *and that for every* $\epsilon > 0$*, there is a compact set* K *such that* $\sup_{x\in E} G_1 1_{K^c}(x) \le \epsilon$*. Then* $1 \in \mathcal{K}_\infty(\mathbf{M}_1)$*, where* $\mathbf{M}_1 = (Y_t, P_x)$ *and* Y_t *is the 1-subprocess of* X_t *with semigroup* $\{e^{-t}p_t\}_{t>0}$*. In particular, this implies that* $\lambda_2 = \lambda_\infty$*.*

Proof. First note that the strong Feller property implies that the resolvent kernel $G_1(x, dy)$ is absolutely continuous with respect to m. Let G^Y be the Green function for Y. Then clearly $G^Y = G_1$. For any $\epsilon > 0$, let K be the compact set such that $\sup_{x\in E} G_1 1_{K^c}(x) \le \epsilon$. We claim that there is a constant $\delta > 0$ such that for any Borel measurable subset $B \subset K$ with $m(B) \le \delta$. Suppose that this is not true. Then there is a decreasing sequence of Borel measurable subsets B_k of K with $m(B_k) < 1/k$ such that $\sup_{x\in E} G_1 1_{B_k}(x) \ge \epsilon$ for each $k \ge 1$. By the strong Markov property,

$$\sup_{x\in E} G_1 1_{B_k}(x) = \sup_{x\in K} G_1 1_{B_k}(x).$$

Since $G_1 1_{B_k}$ is a bounded continuous function and K is compact, there is $x_k \in K$ so that

$$G_1 1_{B_k}(x_k) = \sup_{x\in E} G_1 1_{B_k}(x) \ge \epsilon. \tag{5.13}$$

Taking a subsequence if necessary, we may assume that $x_k \to x_0 \in K$. Since $G_1 1_{B_k}(x_0)$ decreases to 0 as $k \uparrow 0$, there is k_0 so that $G_1 1_{B_{k_0}}(x_0) < \epsilon/3$. By the continuity of $x \mapsto G_1 1_{B_{k_0}}(x)$, there is a neighborhood U of x_0 such that $\sup_{x\in U} G_1 1_{B_{k_0}}(x) < \epsilon/2$. As $x_k \to x_0$, $x_k \in U$ when $k > k_0$ is sufficiently large and so

$$G_1 1_{B_k}(x_k) \le G_1 1_{B_0}(x_k) < \epsilon/2,$$

which contradicts (5.13). This proves the claim and therefore $1 \in \mathcal{K}_\infty(\mathbf{M}_1)$. Now by Theorem 5.12, the spectral radius $\lambda_p(\mathbf{M}_1)$ of $\mathbf{M}_1$ is independent of $p \in [2, \infty]$. Since Y_t is the 1-subprocess of X_t, $\lambda_p(\mathbf{M}_1) = \lambda_p + 1$. Thus the spectral radius λ_p of $\mathbf{M}$ is independent of $p \in [2, \infty]$. □

We will give analytic characterizations of gaugeability in terms of the associated bilinear forms by using the result of L^p-independence of the spectral radius λ_p from Theorem 5.12.

Theorem 5.14. *Assume* (**I**) *and* (**DF**)*. Let* μ *be a positive measure in* $\mathcal{K}_\infty$*. Then* μ *is gaugeable if and only if*

$$\inf\left\{\mathcal{E}(u,u); u \in \mathcal{F} \textit{ with } \int_E u(x)^2 \mu(dx) = 1\right\} > 1.$$

Proof. Let τ_t be the right continuous inverse of A_t^μ; that is,

$$\tau_t = \inf\{s : A_s^\mu > t\}$$

with the convention that $\inf \emptyset = \infty$. Let $\widetilde{S} = \{x \in X : P_x(\tau_0 = 0) = 1\}$ be the fine support of μ and let S be the topological support of μ. The time-changed process Y_t^μ of X_t by A^μ is defined by $Y_t^\mu = X_{\tau_t}$, whose state space is $\widetilde{S}$. However, since $\widetilde{S} \subset S$ modulo a set having zero capacity, the semigroup of Y^μ is μ-symmetric and determines a strongly continuous semigroup on $L^2(S;\mu)$ ([10, Theorem 6.2.1]). So this time-changed process Y^μ is a μ-symmetric right process. Set $H_S u(x) := E_X(u(X_{\sigma_S}))$, where $\sigma_S = \inf\{t > 0 : X_t \in S\}$. Then the Dirichlet form $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ of Y^μ on $L^2(S;\mu)$ is given by

$$\begin{cases} \widehat{\mathcal{F}} = \{\varphi \in L^2(S;\mu) : \varphi = u \ \mu\text{-a.e. on } S \text{ for some } u \in \mathcal{F}_e\}, \\ \widehat{\mathcal{E}}(\varphi,\varphi) = \mathcal{E}(H_S u, H_S u), \varphi \in \widehat{\mathcal{F}} \text{ and } u \in \mathcal{F}_e \text{ such that } \varphi = u \ \mu\text{-a.e. on } S. \end{cases} \tag{5.14}$$

Here $\mathcal{F}_e$ stands for the extended Dirichlet space of $(\mathcal{E},\mathcal{F})$. Note that for every Borel $f \geq 0$,

$$\begin{aligned} E_x\left(\int_0^\infty f(Y_t^\mu)dt\right) =& E_x\left(\int_0^\infty f(X_{\tau_t})dt\right) = E_x\left(\int_0^\infty f(X_t)dA_t^\mu\right) \\ =& \int_S G(x,y)f(y)\mu(dy). \end{aligned}$$

So the Green function of Y^μ with respect to μ is $G(x,y)$. Hence the constant function $1 \in \mathcal{K}(Y^\mu)$. Since A_ζ^μ is the lifetime of the time-changed process Y^μ, by Theorem 5.12,

$$\sup_{x\in E} E_x(e^{A_\tau^\mu}) < \infty \text{ if and only if } \lambda_2(Y^\mu) > 1.$$

Note that

$$\lambda_2(Y^\mu) = \inf\left\{\widehat{\mathcal{E}}(\varphi,\varphi) : u \in \mathcal{F} \text{ with } \int_S \varphi(x)^2\mu(dx) = 1\right\},$$

which is equal to

$$= \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F} \text{ with } \int_E u(x)^2\mu(dx) = 1\right\}.$$

The theorem is now proved. $\square$

5.3 A property of Legendre transform

In this chapter, we consider one of the basic properties of the Legendre transform. The following theorem provides that there is a tangent line that never goes above the graph at each point on the graph of a convex function. Let X be the locally convex, Hausdorff topological (real) vector space.

Theorem 5.15 ([7]). *Let $f : X \longrightarrow (-\infty, \infty]$ be a lower semi-continuous, convex function and define $g : X^* \longrightarrow (-\infty, \infty]$ by*

$$g(\lambda) = \sup \left\{ {}_{X^*}\langle \lambda, x \rangle_X - f(x) : x \in X \right\}.$$

If f is not identically equal to ∞, then g is never equal to $-\infty$, and

$$f(x) = \sup \left\{ {}_{X^*}\langle \lambda, x \rangle_X - g(\lambda) : \lambda \in X^* \right\}, \ x \in X. \tag{5.15}$$

Proof. The first step in the proof is to develop the geometric picture alluded to above. To this end, we define

$$\mathcal{E}(f) = \{(x, \alpha) \in X \times \mathbb{R} : \alpha \leq f(x)\}$$

and

$$\mathcal{E}^*(f) = \left\{ (\lambda, \beta \in X^* \times \mathbb{R} : f(x)) \leq {}_{X^*}\langle \lambda, x \rangle_X - \beta \ \forall x \in X \right\}.$$

It is then an easy matter to check from our assumption that $\mathcal{E}(f)$ is a non-empty, closed, convex subset of $X \times \mathbb{R}$. Indeed, the closedness and convexity of $\mathcal{E}(f)$ come from the lower semi-continuity and convexity of f; and it is clear that $(x_0, f(x_0)) \in \mathcal{E}(f)$, where x_0 is any element of X for which $f(x_0) < \infty$. On the other hand, although $\mathcal{E}^*(f)$ is obviously closed and convex, it is less obvious that it is non-empty. To see that $\mathcal{E}^*(f) \neq \emptyset$, choose $x_0 \in X$ as above and apply the Hahn-Banach Theorem to find a $(\mu, \rho, \gamma) \in X^* \times \mathbb{R} \times \mathbb{R}$ with the properties that the closed affine half space

$$H(\mu, \rho, \gamma) := \{(x, \xi) \in X \times \mathbb{R} : {}_{X^*}\langle \mu, x \rangle_X - \rho\xi \leq \gamma\} \tag{5.16}$$

contains the set $\mathcal{E}(f)$ but not the point $(x_0, f(x_0) - 1)$. Then, since

$${}_{X^*}\langle \mu, x_0 \rangle_X - \rho\xi \leq \gamma \text{ for } \xi \geq f(x_0)$$

while

$${}_{X^*}\langle \mu, x_0 \rangle_X - \rho(f(x_0) - 1) > \gamma,$$

we see that $\rho > 0$ and therefore that

$$(\lambda_0, \beta_0) := \left(\frac{\mu}{\rho}, \frac{\gamma}{\rho} \right) \in \mathcal{E}^*(f). \tag{5.17}$$

Next, noting that $\beta \geq g(f)$ for every $(\lambda, \beta) \in \mathcal{E}^*(f)$ and

$$(\lambda, g(\lambda)) \in \mathcal{E}^*(f),\ \forall \lambda \in X^* \text{ with } g(\lambda) < \infty,$$

one sees that

$$g(\lambda) = \inf \{\beta : (\lambda, \beta) \in \mathcal{E}^*(f)\},$$

and therefore that is equivalent to

$$f(x) = \sup \{{}_{X^*}\langle \lambda, x\rangle_X - \beta : (\lambda, \beta) \in \mathcal{E}^*(f)\}, x \in X. \tag{5.18}$$

Since it is clear that $f(x) \geq {}_{X^*}\langle \lambda, x\rangle_X - \beta$ for any $x \in X$ and $(\lambda, \beta) \in \mathcal{E}^*(f)$, we will have proved (5.18) as soon as we show that, for each $(x, \alpha) \notin \mathcal{E}(f)$, there is a $(\lambda, \beta) \in \mathcal{E}^*(f)$ such that

$${}_{X^*}\langle \lambda, x\rangle_X - \beta > \alpha. \tag{5.19}$$

Since $(x, \alpha) \notin \mathcal{E}(f)$, the Hahn-Banach Theorem again provides the existence of $(\mu, \rho, \gamma) \in X^* \times \mathbb{R} \times \mathbb{R}$ so that the $H(\mu, \rho, \gamma)$ in (5.16)contains $\mathcal{E}(f)$ and $(x, \alpha) \notin (\mu, \rho, \gamma)$. In particular, since ${}_{X^*}\langle \mu, x_0\rangle_X - \rho\xi \leq \gamma$ for $\xi \geq f(x_0)$, we know that $\rho \geq 0$. Hence, for every $\delta > 0$,

$$(\lambda_\delta, \beta_\delta) := \left(\frac{\mu + \delta\lambda_0}{\rho + \delta}, \frac{\gamma + \delta\beta_0}{\rho + \delta}\right) \in \mathcal{E}^*(f),$$

where $(\lambda_\delta, \beta_\delta)$ is the element of $\mathcal{E}^*(f)$ described in (5.17). (The introduction of $\delta > 0$ here is to take care of the case when the tangent hyperplane is vertical and therefore $\rho = 0$.) At the same time, for sufficiently small $\delta > 0$ one has that

$${}_{X^*}\langle \lambda_\delta, x\rangle_X - \alpha = \frac{1}{\rho + \delta}\left({}_{X^*}\langle \mu + \delta\lambda_0, x\rangle_X - (\rho + \delta)\alpha\right) > \frac{\gamma + \delta\beta_0}{\rho + \delta} = \beta_\delta.$$

Hence, (5.18) holds with $(\lambda, \beta) = (\lambda_\delta, \beta_\delta)$ for any sufficiently small $\delta > 0$. $\square$

Bibliography

[1] S. Albeverio, P. Blanchard and Z.-M. Ma, Feynman-Kac semigroups in terms of signed smooth measures, Internat. Ser. Numer. Math. 102 (1991), 1–31

[2] Z.-Q. Chen, Gaugeability and conditional gaugeability, Trans. Amer. Math. Soc. 354 (2002), 4639–4679.

[3] J. Chen and S. Jian, S, Some limit theorems of killed Brownian motion, Sci. China Math. 56 (2013), 479–514.

[4] K.L. Chung, Doubly-Feller process with multiplicative functional, in: Seminar on Stochastic Processes, Birkhäuser (1985), 63–78.

[5] K.L. Chung and Z.X. Zhao, From Brownian Motion to Schrodinger's Equation, Springer, Berlin, 1995.

[6] A. Dembo and O. Zeitouni, Large deviation techniques and applications, Second edition, Applications of Mathematics 38, Springer-Verlag, New York, 1998.

[7] J.-D. Deuschel and D.W. Stroock, Large Deviations, Academic Press, Boston, 1989.

[8] M.D. Donsker and S.R.S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, I, Comm. Pure Appl. Math. 29 (1975), 1–47.

[9] L.C. Evans, Partial Diifferential Equations, American Mathematical Society, Procidence, 1998.

[10] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes 2nd ed., De Gruyter Stud. Math. 19, Walter de Gruyter & Co., Berlin, 2011.

[11] N. Kajino, Equivalence of recurrence and Liouville property for symmetric Dirichlet forms, preprint (2010).

[12] T. Kato, Perturbation theory for linear operators, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[13] D. Kim and K. Kuwae, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals, to appear in Trans. Amer. Math. Soc.

[14] S.C. Port and C.J. Stone, Brownian Motion and Classical Potential Theory, Academic Press, New York, 1978.

[15] B. Remillard, Large deviations estimates for occupation time integrals of Brownian motion, (English summary) Stochastic models (Ottawa, ON, 1998), 375–398, CMS Conf. Proc. 26, Amer. Math. Soc., Providence, RI, 2000.

[16] S. Sato, An inequality for the spectral radius of Markov processes, Kodai Math. J. 8 (1985), 5–13.

[17] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures, Potential Anal. 5 (1996), 109–138.

[18] M. Takeda, Large deviation principle for additive functionals of Brownian motion corresponding to Kato measures, Potential Anal. 19 (2003), 51–67.

[19] M. Takeda, L^p-independence of spectral bounds of Schrodinger type semigroups, J. Funct. Anal. 252 (2007), 550–565.

[20] M. Takeda, A tightness property of a symmetric Markov process and the uniform large deviation principle, Proc. Amer. Math. Soc. 141 (2013), 4371–4383.

[21] M. Takeda and K. Tsuchida, Differentiability of spectral functions for symmetric α-stable processes, Trans. Amer. Math. Soc. 359 (2007), 4031–4054.

[22] L. Wu, Some notes on large deviations of Markov processes, Acta Math. Sin. (Engl. Ser.) 16 (2000), 369–394