博士論文

Hilbert尖点形式の肥田変形の重さ1での古典的な特殊化とGalois表現

小澤 友美

平成28年

Classical weight one Hilbert cusp forms in a Hida family and Galois representations

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Tomomi Ozawa

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Contents

Chapter 1

Introduction

The purpose of this thesis is to generalize a part of Dimitrov and Ghate's study [6] on classical weight one elliptic cusp forms inside a Hida family to the case of Hilbert modular forms with parallel weights. Roughly speaking, a Hida family (with respect to a prime number p) is an infinite set of modular forms parametrized by weights and powers of p in the levels so that the Fourier coefficients of the members in the family are p-adically close. One of the most important reasons that motivates us to study such families is existence of congruences between modular forms.

1.1 Background of this thesis

The study of congruences between modular forms goes back to 1910s, when Ramanujan observed in [28] that there is a congruence modulo 691 between the unique elliptic cusp form Δ (Ramanujan's delta) of weight 12 and level 1 and the Eisenstein series of the same weight and the level. Several other congruences concerning Δ had been proved, but none of them explains why such congruences exist. Later in 1960s, Deligne showed in [3] that one can associate the system of Galois representations to a normalized Hecke eigen cusp forms. This enabled Serre and Swinnerton-Dyer to open up in [44] the study of such congruences from the point of the image of the associated Galois representations, which was followed by Ribet [29], [31], [32] and Momose [24].

Hida, among with his collaborators, investigated such congruences in terms of Hecke algebras, and in the early 1980s introduced the notion of congruence modules which measures congruences between modular forms of a fixed weight. Since examples of congruences between modular forms of different weights were already known, it was natural to seek for a theory of congruence modules that could handle various weights. Eventually he was led to study the p-adic Hecke algebras which parametrize Hecke eigenvalues of p-ordinary modular forms of all weight at least two. This is the starting point of Hida theory.

The study of congruences between modular forms has made a significant influence on number theory, especially on Iwasawa theory. Among those, Ribet gave a criterion for p-divisibility of the class number of the cyclotomic field $\mathbb{Q}(\mu_p)$ in [30], and Wiles' proof of the Iwasawa main conjecture for totally real fields in [47] was achieved by the systematic use of Λ-adic Hilbert modular forms whose idea was based on aforementioned Hida's theory.

The work in this thesis has much to do with the so-called "control theorem" (Section 3.5), namely, the controllability of specializations of a Λ-adic form. Let F be a totally real field and p an odd prime. We consider a primitive ordinary Λ-adic cusp form $\mathcal{F}$ of parallel weight Hilbert cusp forms defined over F. It is proved by Hida in [13] that a specialization of $\mathcal{F}$ at any arithmetic point of weight two or more is a classical (holomorphic) Hilbert cusp form. Contrary to the higher weight specializations, understanding of weight one specializations is much less satisfactory. By studying those weight one specializations, one would expect to get a better grasp on important issues such as intersections of irreducible components of the ordinary Hecke algebra at classical weight one points and a suitable construction of a p-adic L-function attached to a classical weight one eigenform.

It is Greenberg's question (Question 6.2.1) on the local indecomposability of the p-adic Galois representation attached to an elliptic eigenform of weight at least two that directly leads to the study in this thesis. In the case of $F = \mathbb{Q}$, Ghate and Vatsal was motivated by his question and solved a similar question for Λ-adic forms in [9]. In the course of their proof, they showed (as a byproduct) that a primitive ordinary Λ-adic cusp form admits infinitely many classical weight one specializations if and only if it is a CM family (namely, constructed from a Hecke character of an imaginary quadratic field). Further the number of such forms inside a non-CM family is bounded by an explicit constant due to Dimitrov and Ghate in [6]. The former result in [9] was generalized to the case of totally real fields by Balasubramanyam, Ghate and Vatsal in [1], which contains an outline of a proof even for nearly ordinary families with not necessarily parallel weights.

Motivated by these works, in this doctor thesis, we pursue a generalization of the results of Dimitrov and Ghate, and give an explicit estimate on the number of classical weight one specializations obtained from a non-CM primitive ordinary Λ-adic cusp form. As it will be revealed later, we will extensively investigate the local behavior (at the prime ideals of F lying over p) of Artin representations attached to classical weight one forms. The author hopes that such investigation offers an insight to the study of (strong) Artin conjecture and relevant topics.

1.2 Main theorem of this thesis

We give a brief account of the main results of this thesis. Let G_F be the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of a totally real field F and $\mathbf{G}$ the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of F. We take a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be the ring of integers of it. The Iwasawa algebra Λ is defined to be the complete group algebra $\mathcal{O}[[\mathbf{G}]]$ of $\mathbf{G}$ over $\mathcal{O}$, which is isomorphic to the power series ring $\mathcal{O}[[X]]$ over $\mathcal{O}$ of one-variable. In this thesis we adopt Wiles' convention of Λ-adic forms in [46].

Let $\mathcal{F}$ be a primitive ordinary Λ-adic cusp form of tame level an integral ideal $\mathfrak{n}_0$ in F prime to p, $\rho_{\mathcal{F}} : G_F \to GL_2(\mathrm{Frac}(\Lambda_L))$ the Galois representation attached to $\mathcal{F}$ where Λ_L is a finite integral extension of Λ. We take a G_F-stable Λ_L-lattice $\mathcal{L}$ in

$\mathrm{Frac}(\Lambda_L)^2$ and consider the reduction $\bar{\rho}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$ of $\rho_{\mathcal{F}}$ modulo the maximal ideal of Λ_L. Here $\mathbb{F}$ is a finite field of characteristic p. The reduction $\bar{\rho}_{\mathcal{F}}$ may depend on the choice of a lattice $\mathcal{L}$ but its semi-simplification $\bar{\rho}_{\mathcal{F}}^{\mathrm{ss}} : G_F \to GL_2(\mathbb{F})$ does not (Section 5.2). On the other hand, by Ohta [26], Rogawski-Tunnell [33] and Wiles [46], any classical weight one Hilbert cuspidal eigenform f gives rise to an irreducible totally odd Artin representation $\rho_f : G_F \to GL_2(\mathbb{C})$. The image of ρ_f in $PGL_2(\mathbb{C})$ is either

- dihedral, namely isomorphic to a dihedral group D_{2n} of order $2n$ for some integer $n \geq 2$, or
- exceptional, namely isomorphic to one of the symmetric group S_4, the alternative groups A_4 or A_5.

It will be revealed in Section 5.3 that if $\mathcal{F}$ admits a classical weight one specialization f, then the image of $\bar{\rho}_{\mathcal{F}}^{\mathrm{ss}}$ in $PGL_2(\mathbb{F})$ has to be of the same type as that of ρ_f, and hence $\bar{\rho}_{\mathcal{F}} = \bar{\rho}_{\mathcal{F}}^{\mathrm{ss}}$ is irreducible and unique up to isomorphism. If $\mathcal{F}$ is a CM family, then the image of $\bar{\rho}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is dihedral. According to the above classification, we will distinguish the arguments for each case.

1.2.1 Dihedral case

Suppose first that the image of $\bar{\rho}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is a dihedral group. We will observe in Section 5.4 (Lemma 5.4.1) that there exist a quadratic extension K of F and a character $\bar{\varphi} : G_K = \mathrm{Gal}(\overline{F}/K) \to \mathbb{F}^\times$ so that $\bar{\rho}_{\mathcal{F}}$ is isomorphic to the induced representation $\mathrm{Ind}_K^F(\bar{\varphi})$ by $\bar{\varphi}$. We consider the following condition on $\mathcal{F}$:

(P) $\mathcal{F}$ has a classical weight one specialization f such that the associated representation $\rho_f : G_F \to GL_2(O)$ (O is a suitable finite integral extension of $\mathbb{Z}_p$) has the property that $\rho_f(I_{\mathfrak{q}})$ has order at least three for each prime ideal $\mathfrak{q}$ of F lying over p. Here $I_{\mathfrak{q}} \subset G_F$ is the inertia group at $\mathfrak{q}$.

It follows from (P) that ρ_f is isomorphic to the induced representation $\mathrm{Ind}_K^F(\varphi)$ for a finite order character $\varphi : G_K \to O^\times$ which lifts $\bar{\varphi}$ (Lemma 5.4.5). The condition (P) further implies that for any prime ideal $\mathfrak{q}$ of F lying over p, $\mathfrak{q}$ splits in K, say $\mathfrak{q}O_K = \mathfrak{Q}\mathfrak{Q}^\sigma$, and the character φ is ramified at exactly one of $\mathfrak{Q}$ and $\mathfrak{Q}^\sigma$ (Lemma 7.1.2). We assume that φ is ramified at $\mathfrak{Q}$ and unramified at $\mathfrak{Q}^\sigma$. We let $\mathcal{Q} = \prod_{\mathfrak{q}|p} \mathfrak{Q}$ and $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty) = \varprojlim_{r \geq 1} \mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^r)$ be the projective limit of narrow ray class groups $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^r)$ of K of modulus $\mathfrak{n}_0\mathcal{Q}^r$. For each $\mathfrak{Q}$ let $U_{K_{\mathfrak{Q}}}(\mathfrak{Q})$ denote the principal unit group of the completion $K_{\mathfrak{Q}}$ at $\mathfrak{Q}$ and $\overline{U_K}$ the closure of the image of $U_K = \left\{u \in O_K^\times \mid u \equiv 1 \bmod \mathcal{Q}\right\}$ in $\prod_{\mathfrak{q}|p} U_{K_{\mathfrak{Q}}}(\mathfrak{Q})$. By virtue of class field theory, the group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is of finite order if and only if $(\prod_{\mathfrak{q}|p} U_{K_{\mathfrak{Q}}}(\mathfrak{Q}))/\overline{U_K}$ is a finite group (Section 7.1.2). If this is the case, we put

$$M' = |\mathrm{Cl}_K| \cdot \left| \left(\prod_{\mathfrak{q}|p} U_{K_{\mathfrak{Q}}}(\mathfrak{Q}) \right) / \overline{U_K} \right| \cdot \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{split in } K}} (q_{\mathfrak{l}} - 1) \cdot \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{inert in } K}} (q_{\mathfrak{l}} + 1)$$

and $M(\mathcal{F}, K, f) = p^{\mathrm{ord}_p(M')}$, where $q_{\mathfrak{l}}$ is the cardinality of the residue field at a prime ideal $\mathfrak{l}$ of F and ord_p is the p-adic valuation normalized so that $\mathrm{ord}_p(p) = 1$. We will prove the following

Theorem 1.2.1. *Let $\mathcal{F}$ be a primitive ordinary Λ-adic cusp form of tame level $\mathfrak{n}_0$ satisfying the condition* (P)*. Then the following two statements hold:*

(1) *Assume further that Leopoldt's conjecture for F and p is true. If $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is an infinite group, then there exists a CM family $\mathcal{G}$ by K that has the same tame level and the same residual representation as those of $\mathcal{F}$. In particular, K/F is a totally imaginary extension.*

(2) *If $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is of finite order, then $\mathcal{F}$ is not a CM family and the number of classical weight one specializations of $\mathcal{F}$ is bounded by $M(\mathcal{F}, K, f)$.*

If we assume Leopoldt's conjecture for K and p and K/F is not totally imaginary, then the group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is of finite order and the second case (2) always happens. In particular in the case of $F = \mathbb{Q}$ and K is a real quadratic field, the case (2) always happens. In [6], the case where K is a real quadratic field is referred to as "residually of RM type", which is what Theorem 1.2.1 (2) generalizes. It should be noticed that if K/F is totally imaginary, then $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is always an infinite group (Remark 7.1.6 (2)).

We will show that if $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is infinite and Leopoldt's conjecture for F and p is true, then there are infinitely many classical weight one forms such that all of them occur as specializations of a primitive ordinary Λ-adic cusp form $\mathcal{G}$ having the same tame level and the same residual representation as those of $\mathcal{F}$ (see Sections 7.1.2 through 7.1.4 for details). In view of Theorem 6.3.1 which concerns finiteness of classical weight one specializations of a non-CM family and can be regarded as a variant of Theorem 3 in [1], we see that $\mathcal{G}$ has CM by K and thus K/F is a totally imaginary extension.

The first case (1) of Theorem 1.2.1 is referred to as "residually of CM type" in [6], which is the other dihedral case treated in the paper. In the case (1), if $\mathcal{F}$ is a non-CM primitive ordinary Λ-adic cusp form whose residual representation $\bar{\rho}_{\mathcal{F}}$ is isomorphic to the induced representation $\mathrm{Ind}_K^F(\bar{\varphi})$ for some totally imaginary quadratic extension K of F and a character $\bar{\varphi} : G_K \to \mathbb{F}^\times$, then one can also give a bound on the number of classical weight one specializations in $\mathcal{F}$. This is a generalization of Lemma 6.5 in [6]. Since $\mathcal{F}$ is not a CM family, there is at least one prime ideal $\mathfrak{l}$ in F that is inert in K, prime to $\mathfrak{n}_0 p$ and the trace of $\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{l}})$ is non-zero (otherwise we have $\rho_{\mathcal{F}} \cong \varepsilon_{K/F} \otimes \rho_{\mathcal{F}}$ and hence $\mathcal{F}$ has CM by K: see Definition 6.1.1). Here $\mathrm{Frob}_{\mathfrak{l}}$ is an arithmetic Frobenius at $\mathfrak{l}$. Let $\lambda_{\mathcal{F},\mathfrak{l}}$ be the number of height one prime ideals of Λ_L which contain $\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{l}})$ and sit above prime ideals of Λ corresponding to weight one specializations. We put $\lambda_{\mathcal{F}} = \min\{\lambda_{\mathcal{F},\mathfrak{l}} \mid \mathfrak{l} \text{ is inert in } K \text{ and prime to } \mathfrak{n}_0 p\}$. In Section 7.1.5, we will prove the following

Proposition 1.2.2. *Let $\mathcal{F}$ be a non-CM primitive ordinary Λ-adic cusp form which is residually of dihedral type and $\bar{\rho}_{\mathcal{F}}$ is isomorphic to the induced representation*

$\mathrm{Ind}_K^F(\bar{\varphi})$ *for some totally imaginary quadratic extension* K *of* F *and a character* $\bar{\varphi} : G_K \to \mathbb{F}^\times$*. Then the number of classical weight one specializations of* $\mathcal{F}$ *is bounded by* $\lambda_{\mathcal{F}}$*.*

Secondly we consider the exceptional case.

1.2.2 Exceptional case

As the image of the Artin representation ρ_f in $PGL_2(\mathbb{C})$ for any classical weight one form f in $\mathcal{F}$ has bounded order, say 24, 12 or 60, our analysis in this case is simpler than the dihedral case. Let p^r be the p-part of the class number of F and t the number of the prime ideals of F lying over p.

Theorem 1.2.3. *Let* $\mathcal{F}$ *be a non-CM primitive ordinary* Λ*-adic cusp form such that the image of the residual representation* $\bar{\rho}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$ *in* $PGL_2(\mathbb{F})$ *is exceptional. Then* $\mathcal{F}$ *has at most* b *classical weight one specializations, where*

- $b = p^r$*, except* $p = 3$ *or* 5*, in which case* $b = 2^t \cdot p^r$*.*

Further, under some assumptions on a Hida community $\{\mathcal{F}\}$ (refer to Definition 7.1.4 for the notion of Hida communities), we prove the existence of a classical weight one specialization in some member of $\{\mathcal{F}\}$.

Proposition 1.2.4. *Let* $p \geq 7$ *be a prime number that splits completely in* F *and* $\{\mathcal{F}\}$ *a Hida community of exceptional type which is* p*-distinguished* (*cf. Definition* 6.2.2) *and the residual representation* $\bar{\rho}_{\mathcal{F}}$ *is absolutely irreducible when restricted to* $\mathrm{Gal}(\overline{F}/F(\zeta_p))$*, where* ζ_p *is a primitive* p*-th root of unity in* $\overline{F}$*. Assume further that the tame level* $\mathfrak{n}_0$ *of* $\{\mathcal{F}\}$ *is the same as the Artin conductor of* $\bar{\rho}_{\mathcal{F}}$*. Then* $\{\mathcal{F}\}$ *has at least one classical weight one specialization* f*. Moreover, any other classical weight one specialization of* $\{\mathcal{F}\}$ *can be written as* $f \otimes \eta$*, where* $\eta : G_F \to \mathbb{Q}_p^\times$ *is a* p*-power order character of conductor dividing* $\mathfrak{n}_0$*.*

As mentioned in the last paragraph of Section 1.1, Theorems 1.2.1 and 1.2.3 extend Theorems 6.4 and 5.1 of [6], respectively, to the case of Hilbert modular forms with parallel weights. The proof is basically the same as [6], but some additional argument are necessary in the dihedral case for the following reason. When $F = \mathbb{Q}$ and K is a real quadratic field, the finiteness of $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is a consequence of Leopoldt's conjecture. However, for a general number field K, Leopoldt's conjecture is still open and we do not know whether or not the group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is finite. Therefore, a new analysis around the group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is necessary to get around assuming Leopoldt's conjecture for K and p.

1.3 Organization

This doctor thesis is organized as follows:

The first two chapters are devoted to introducing modular forms and p-adic families of them. To be specific, in Chapter 2 we recall basics of Hilbert modular forms with parallel weights and the Hecke operators acting on the space of such forms. In Chapter 3, we introduce Hilbert modular forms defined over arithmetic rings, recall the notion of Λ-adic forms which was invented by Wiles in [46], and give an account on the control theorem concerning the space of ordinary Λ-adic forms.

In the next two chapters, we focus on Galois representations attached to forms. Precisely, in Chapter 4 we recall main properties of the Galois representations attached to Hilbert cusp forms of weight at least two and of weight one, separately. We also discuss the image of those representations in the projective linear groups. In Chapter 5, we briefly review the construction of the Galois representation arising from a Λ-adic cusp form, define the notion of reduction of such a representation, and compare the image in the projective linear group of the Galois representations attached to a Λ-adic form and its specializations including in weight one. This chapter includes some facts on dihedral and induced representations as well.

In Chapter 6, we will first recall the notion and an explicit construction of CM families, and discuss specializations (including in weight one) of CM and non-CM families. Then we introduce the works in [9] and in [1] which characterize CM families in terms of local indecomposability of the associated Galois representations. Then we prove the finiteness result for non-CM families (Theorem 6.3.1).

The last chapter is dedicated to giving an upper bound on the number of classical weight one specializations of a non-CM family. In Section 7.1 we discuss the dihedral case and prove Theorem 1.2.1 (Theorem 7.1.5) and Proposition 1.2.2 (Proposition 7.1.14). In Section 7.2, we deal with the exceptional case and prove Theorem 1.2.3 (Theorem 7.2.1) and Proposition 1.2.4 (Proposition 7.2.3).

1.4 Notation and Terminology

Throughout this thesis, we adopt the following notation and terminology:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ indicate respectively the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers. For a complex number $z \in \mathbb{C}$, the real part and the imaginary part of z are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. The symbol $\mathfrak{H}$ indicates the upper-half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We denote by $\sqrt{-1}$ the square root of -1 that lies in $\mathfrak{H}$ and by $\infty = \lim_{t \to +\infty} t\sqrt{-1}$ the point at infinity. For a rational prime number p, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers.

For a commutative ring A, we denote by $A^\times$ the group consisting of invertible elements of A, and by $GL_2(A)$ the group of all 2×2 matrices whose entries are in A and determinants are in $A^\times$. The identity element of $GL_2(A)$ is denoted by I_2. We put $SL_2(A) = \{X \in GL_2(A) \mid \det(A) = 1\}$. If M is an A-module, $\mathrm{End}_A(M)$ denotes the ring of all A-linear endomorphisms of M.

Throughout the thesis, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. By an algebraic number field, we mean a finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. For an algebraic number field F, we denote by O_F the ring of integers in F, by $P(F)$ the set of places of F, and by $P_\infty(F)$ (resp. by $P_{\mathrm{f}}(F)$) the subset of $P(F)$ consisting of infinite places (resp. finite places) of F. Each real place $v \in P_\infty(F)$ uniquely determines an embedding σ_v of

F into $\mathbb{R}$. Each complex place $v \in P_\infty(F)$ corresponds to a unique pair $(\sigma_v, \bar{\sigma}_v)$ of embeddings of F into $\mathbb{C}$ so that $\bar{\sigma}_v$ is the complex conjugate of σ_v.

For each $v \in P(F)$, let F_v be the completion of F at v. We define the standard absolute value $|\cdot|_{F_v} : F_v^\times \to \mathbb{R}_{\geq 0}$ as follows: for each real place $v \in P_\infty(F)$, this is the usual absolute value on $F_v = \mathbb{R}$; for each complex place v, we have $|x|_{F_v} = x\bar{x}$ for $x \in F_v^\times = \mathbb{C}^\times$, where F_v is identified with $\mathbb{C}$ by σ_v and $\bar{x}$ is the complex conjugate of x. For $v \in P_{\mathrm{f}}(F)$, the ring of integers, the maximal ideal and the residue field at v are denoted by O_{F_v}, $\mathfrak{p}_{F_v}$ and $\mathbb{F}_v$, respectively. The standard absolute value at v is normalized so that $|\varpi_v|_{F_v} = (\#\mathbb{F}_v)^{-1}$ for any uniformizer ϖ_v in F_v.

For an algebraic number field F, let $\mathbb{A}_F$ be the ring of adeles of F, $\mathbb{A}_{F,\mathrm{f}}$ (resp. $\mathbb{A}_{F,\infty}$) the subring of $\mathbb{A}_F$ consisting of elements $a = (a_v)_{v \in P(F)} \in \mathbb{A}_F$ with $a_v = 0$ for any $v \in P_\infty(F)$ (resp. for any $v \in P_{\mathrm{f}}(F)$) so that $\mathbb{A}_F = \mathbb{A}_{F,\mathrm{f}} \times \mathbb{A}_{F,\infty}$. We denote by $\mathbb{A}_{F,\infty}^+$ the connected component of $\mathbb{A}_{F,\infty}^\times$ containing the identity element. For an integral ideal $\mathfrak{m}$ in O_F and each finite $v \in P_{\mathrm{f}}(F)$, let $U_{F_v}(\mathfrak{m})$ be the group consisting of units u_v in $O_{F_v}^\times$ such that $u_v - 1 \in \mathfrak{m}O_{F_v}$. Note that if $v \nmid \mathfrak{m}$, then $U_{F_v}(\mathfrak{m}) = O_{F_v}^\times$. We put $U_F(\mathfrak{m}) = \prod_{v \in P_{\mathrm{f}}(F)} U_{F_v}(\mathfrak{m})$. This is an open compact subgroup of $\mathbb{A}_{F,\mathrm{f}}^\times$.

Chapter 2

Hilbert modular forms

In this chapter, we recall the definitions of Hilbert modular forms with parallel weights and their Fourier expansions, and give a brief account of the Hecke operators acting on the complex vector space of such forms. We will basically follow the setting in [40]. Throughout this chapter, we use the following notation:

- F: a totally real number field of degree g over $\mathbb{Q}$;
- I: the set of embeddings of F into $\mathbb{R}$;
- F_+: the set of totally positive elements in F;
- $GL_2^+(F) = \{\gamma \in GL_2(F) \mid \det(\gamma) \in F_+\}$;
- $\mathfrak{d}$: the different ideal of $F/\mathbb{Q}$;
- $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$: the norm of $F/\mathbb{Q}$;
- For $a \in F$ and $\sigma \in I$, a^σ is the image of a in $\mathbb{R}$ under σ;
- For $a \in F$ and a vector $r = (r_\sigma)_{\sigma \in I} \in (\mathbb{Z}/2\mathbb{Z})^g$, $\mathrm{sgn}(a)^r = \prod_{\sigma \in I} \mathrm{sgn}(a^\sigma)^{r_\sigma}$.

2.1 Basic definitions

We begin by recalling the definition of narrow ray class characters of F. Let $\mathfrak{m}$ be a non-zero integral ideal of F. We put

$$\begin{aligned} I(\mathfrak{m}) &= \left\{ \frac{\mathfrak{n}}{\mathfrak{l}} \,\middle|\, \mathfrak{n} \text{ and } \mathfrak{l} \text{ are integral ideals and prime to } \mathfrak{m} \right\}, \\ P_+ &= \left\{ aO_F \mid a \in F_+ \right\}, \text{ and} \\ P_+(\mathfrak{m}) &= P_+ \cap \left\{ aO_F \mid a \equiv 1 \bmod {}^\times\mathfrak{m} \right\}. \end{aligned}$$

Here $a \equiv 1 \bmod {}^\times\mathfrak{m}$ if and only if $aO_F \in I(\mathfrak{m})$ and there exists an element $b \in F_+$ such that $bO_F \in I(\mathfrak{m})$, $b \in O_F$, $ab \in O_F$, and $ab \equiv b \bmod \mathfrak{m}$. We call the quotient group $\mathrm{Cl}_F(\mathfrak{m}) = I(\mathfrak{m})/P_+(\mathfrak{m})$ the narrow ray class group modulo $\mathfrak{m}$. This group is known to be finite for any non-zero $\mathfrak{m}$. An inclusion $\mathfrak{m} \subset \mathfrak{m}'$ of integral ideals induces a canonical homomorphism $\mathrm{Cl}_F(\mathfrak{m}) \to \mathrm{Cl}_F(\mathfrak{m}')$.

Definition 2.1.1. A narrow ray class character modulo an integral ideal $\mathfrak{m}$ is a group homomorphism $\psi : \mathrm{Cl}_F(\mathfrak{m}) \to \mathbb{C}^\times$.

The conductor of a narrow ray class character ψ modulo $\mathfrak{m}$ is a unique integral ideal $\mathfrak{f}$ which has the following properties:

1. $\mathfrak{m} \subset \mathfrak{f}$;
2. the canonical homomorphism $\pi : \mathrm{Cl}_F(\mathfrak{m}) \to \mathrm{Cl}_F(\mathfrak{f})$ factors ψ, i.e., there exists a homomorphism $\psi_0 : \mathrm{Cl}_F(\mathfrak{f}) \to \mathbb{C}^\times$ such that $\psi = \psi_0 \circ \pi$;
3. the canonical homomorphism $\pi : \mathrm{Cl}_F(\mathfrak{f}) \to \mathrm{Cl}_F(\mathfrak{f}')$ does not factor ψ for any integral ideal $\mathfrak{f}'$ with $\mathfrak{f} \subsetneq \mathfrak{f}'$.

We write $\mathrm{cond}(\psi) = \mathfrak{f}$. If $\mathfrak{m} = \mathfrak{f}$, ψ is said to be primitive modulo $\mathfrak{m}$.

It is known that there exists a vector $r \in (\mathbb{Z}/2\mathbb{Z})^g$ such that

$$\psi(aO_F) = \mathrm{sgn}(a)^r \text{ for any } a \in O_F \text{ with } a \equiv 1 \bmod \mathfrak{m}.$$

We call r the signature of ψ. Then we can define a character $\psi_{\mathrm{f}} : (O_F/\mathfrak{m})^\times \to \mathbb{C}^\times$ associated to ψ given by $\psi_{\mathrm{f}}(a) = \psi(aO_F)\mathrm{sgn}(a)^r$. We will always regard the right-hand side as a character on $(O_F/\mathfrak{m})^\times$, without any notice.

When $\mathfrak{m} = O_F$, we write Cl_F^+ rather than $\mathrm{Cl}_F(O_F)$, and we call this group the narrow ideal class group of F. There is a canonical surjective homomorphism from Cl_F^+ to the (wide) ideal class group Cl_F of F. In particular $h = \#\mathrm{Cl}_F^+$ is a multiple of the class number of F.

We now describe the definition of parallel weight Hilbert modular forms over F. First we choose a representative fractional ideal $\mathfrak{t}_\lambda$ of each $\lambda \in \mathrm{Cl}_F^+$, and define a subgroup $\Gamma_\lambda(\mathfrak{m})$ of $GL_2^+(F)$ by

$$\Gamma_\lambda(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F) \middle| a, d \in O_F, b \in (\mathfrak{d}\mathfrak{t}_\lambda)^{-1}, c \in \mathfrak{m}\mathfrak{d}\mathfrak{t}_\lambda,\ ad - bc \in O_F^\times \right\}.$$

Definition 2.1.2 (cf. [40] Sections 1 and 2)**.** Let $k \geq 0$ be an integer, and $\mathfrak{m}, \psi$ as above. The space $M_k(\mathfrak{m}, \psi)$ of Hilbert modular forms of (parallel) weight k, level $\mathfrak{m}$ and character ψ consists of elements f such that

(1) $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+}$ is an h-tuple of holomorphic functions $f_\lambda : \mathfrak{H}^I \to \mathbb{C}$;

(2) for each $\lambda \in \mathrm{Cl}_F^+$, f_λ satisfies the following modularity property:

$$f_\lambda|_k\gamma = \psi_{\mathrm{f}}(d) f_\lambda \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\lambda(\mathfrak{m}). \tag{2.2.1}$$

Here we put

$$j(\gamma, z) = \prod_{\sigma \in I}(c^\sigma z_\sigma + d^\sigma),\ \gamma z = \left(\frac{a^\sigma z_\sigma + b^\sigma}{c^\sigma z_\sigma + d^\sigma}\right)_{\sigma \in I}$$

and $f_\lambda|_k\gamma$ is a function on $\mathfrak{H}^I$ defined by

$$(f_\lambda|_k\gamma)(z) = \mathrm{N}(\det(\gamma))^{\frac{k}{2}} j(\gamma, z)^{-k} f_\lambda(\gamma z).$$

Since each f_λ is a function on $\mathfrak{H}^I$, we regard z a g-tuple of variables z_σ. We also note that $\gamma z \in \mathfrak{H}^I$ for any $\gamma \in GL_2^+(F)$. We often omit the subscript k of $f_\lambda|_k\gamma$ when there is no ambiguity concerning the weight.

(3) when $F = \mathbb{Q}$, we also impose the holomorphy condition around each cusp: that is, for any $\gamma \in SL_2(\mathbb{Z})$, we have

$$(f|_k\gamma)(z) = \sum_{n=0}^{\infty} a\left(\frac{n}{M}, f|_k\gamma\right) \exp\left(2\pi i \frac{nz}{M}\right)$$

where M is the positive integer uniquely determined by $M\mathbb{Z} = \mathfrak{m}$.

Remark 2.1.3. The definition of the subgroup $\Gamma_\lambda(\mathfrak{m})$ depends on the choice of a representative fractional ideal $\mathfrak{t}_\lambda$. We take two representative ideals $\mathfrak{t}_{\lambda,i}$ $(i = 1, 2)$ of $\lambda \in \mathrm{Cl}_F^+$ and consider the $\mathbb{C}$-vector space $M_k(\mathfrak{m}, \psi)_i$ consisting of modular forms satisfying the modularity property (2.2.1) with respect to $\Gamma_{\mathfrak{t}_{\lambda,i}}(\mathfrak{m})$ for each i. By definition we have $\mathfrak{t}_{\lambda,2} = u\mathfrak{t}_{\lambda,1}$ for some $u \in F_+$. Then there is an isomorphism

$$M_k(\mathfrak{m}, \psi)_1 \to M_k(\mathfrak{m}, \psi)_2;\ (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+} \mapsto \left(f_\lambda|_k \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right)_{\lambda \in \mathrm{Cl}_F^+}.$$

However we will define the Fourier coefficients of f independent of the choice of a representative ideal $\mathfrak{t}_\lambda$ (see Definition 2.2.2 and Remark 2.2.3 for details).

It is known that for any k, $\mathfrak{m}$ and ψ, the space $M_k(\mathfrak{m}, \psi)$ is finite dimensional (refer to Sections 1.3 and 1.7 of [8], for instance).

2.2 Fourier coefficients of a Hilbert modular form

We define the Fourier expansion of a Hilbert modular form (cf. [40] (2.16); p. 649).

Proposition 2.2.1. *A Hilbert modular form $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+} \in M_k(\mathfrak{m}, \psi)$ has the Fourier expansion (at the cusp $\infty = (\infty, \infty, \ldots, \infty)$) of the form*

$$f_\lambda(z) = a_\lambda(0) + \sum_{b \in \mathfrak{t}_\lambda \cap F_+} a_\lambda(b) e_F(bz) \text{ for each } \lambda \in \mathrm{Cl}_F^+. \tag{2.2.2}$$

Here $a_\lambda(0)$, $a_\lambda(b)$ are complex numbers and $e_F(x) = \exp(2\pi i \sum_{\sigma \in I} x_\sigma)$ (we use this notation both for $x \in F$ and for a g-tuple of variables $x = (x_\sigma)_{\sigma \in I}$).

Proof. The assertion is well known when $F = \mathbb{Q}$. When $F \neq \mathbb{Q}$, ideas of the proof are basically the same as that for $F = \mathbb{Q}$. Namely, the modularity property (2.2.1) implies that $f_\lambda(z)$ is invariant under the translation by elements of $(\mathfrak{d}\mathfrak{t}_\lambda)^{-1}$, and since f_λ is holomorphic in z, we conclude that f is of the form

$$f_\lambda(z) = \sum_{b \in \mathfrak{t}_\lambda} a_\lambda(b) e_F(bz).$$

We need to show that $a_\lambda(b) = 0$ for all $b \in \mathfrak{t}_\lambda$ with $b \notin F_+$ and $b \neq 0$. This is so-called "Koecher's principle" (one can refer to [10] Theorem 3.3; p. 64 for instance). Note that Koecher's principle does not hold when $F = \mathbb{Q}$. □

We call the coefficients $a_\lambda(b)$ the unnormalized Fourier coefficients of f. We also define the normalized ones as follows.

Definition 2.2.2. Let f be as in Proposition 2.2.1 with the Fourier expansion (2.2.2). We define the normalized constant term $c_\lambda(0, f)$ of f by

$$c_\lambda(0, f) = a_\lambda(0)\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}$$

for each $\lambda \in \mathrm{Cl}_F^+$. For each non-zero integral ideal $\mathfrak{n}$ of F, there exist a unique $\lambda \in \mathrm{Cl}_F^+$, and $b \in F_+$ unique up to multiplication by totally positive units, such that $\mathfrak{n} = b\mathfrak{t}_\lambda^{-1}$. Then $b \in \mathfrak{t}_\lambda \cap F_+$ and the normalized Fourier coefficient $c(\mathfrak{n}, f)$ associated to $\mathfrak{n}$ is defined by

$$c(\mathfrak{n}, f) = a_\lambda(b)\mathrm{N}(\mathfrak{t}_\lambda)^{-\frac{k}{2}}.$$

Remark 2.2.3. The following two facts show why $c_\lambda(0, f)$ and $c(\mathfrak{n}, f)$ are called the "normalized" coefficients. These facts can be deduced from the modularity property (2.2.1).

1. $c_\lambda(0, f)$ and $c(\mathfrak{n}, f)$ are independent of the choice of a representative ideal $\mathfrak{t}_\lambda$.
2. $c(\mathfrak{n}, f)$ is independent of the choice of $b \in \mathfrak{t}_\lambda \cap F_+$ such that $\mathfrak{n} = b\mathfrak{t}_\lambda^{-1}$.

One can show, in exactly the same way as Proposition 2.2.1, that for every $\gamma_\lambda \in GL_2^+(F)$ we have a Fourier expansion

$$(f_\lambda|_k\gamma_\lambda)(z) = a_\lambda(0, \gamma_\lambda) + \sum_{b \in F^+} a_\lambda(b, \gamma_\lambda) e_F(bz). \tag{2.2.3}$$

of f_λ around the cusp $\gamma_\lambda(\infty)$. Moreover, for each $\lambda \in \mathrm{Cl}_F^+$, $a_\lambda(b, \gamma_\lambda)$ are supported on a fractional ideal in F determined by $\mathfrak{t}_\lambda$ and γ_λ, as it is the case in Proposition 2.2.1. We do not explain this in detail because we will not need this.

Definition 2.2.4. A Hilbert modular form $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+}$ is said to be a cusp form if for every $\lambda \in \mathrm{Cl}_F^+$, we have $a_\lambda(0, \gamma_\lambda) = 0$ for any $\gamma_\lambda \in GL_2^+(F)$.

The space of Hilbert cusp forms of (parallel) weight k, level $\mathfrak{m}$ and character ψ is denoted by $S_k(\mathfrak{m}, \psi)$.

2.3 Hilbert modular forms in the adelic language

In the next section, we will define Hecke operators acting on the space of Hilbert modular forms. In order to describe such operators, it is convenient to characterize the space under consideration as a subspace consisting of functions on the adele group of GL_2 over F with some properties such as the one corresponding to the modularity property (2.2.1). We will define the action of Hecke operators in this adelic point of view.

In what follows, we will consider a certain class of complex valued functions on $GL_2(\mathbb{A}_F)$. Notice that we have $GL_2(\mathbb{A}_F) = GL_2(\mathbb{A}_{F,\mathrm{f}}) \times GL_2(\mathbb{A}_{F,\infty})$ as a group, which is in accordance with the decomposition of rings $\mathbb{A}_F = \mathbb{A}_{F,\mathrm{f}} \times \mathbb{A}_{F,\infty}$. Also note that $\mathbb{A}_{F,\infty}$ is identical to $\mathbb{R}^{P_\infty(F)}$, since F is a totally real field. Let $GL_2(\mathbb{A}_{F,\infty})^+$ denote the connected component of $GL_2(\mathbb{A}_{F,\infty})$ containing the identity element. This group acts on the product $\mathfrak{H}^{P_\infty(F)} = \mathfrak{H}^I$ of copies of the upper-half plane, componentwisely via linear fractional transformation. Note that the action of $GL_2(\mathbb{A}_{F,\infty})^+$ on $\mathfrak{H}^{P_\infty(F)}$ is transitive. Let K_∞ be the stabilizer subgroup in $GL_2(\mathbb{A}_{F,\infty})^+$ of $\mathbf{i} = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathfrak{H}^{P_\infty(F)}$, namely, $K_\infty = \prod_{v \in P_\infty(F)}(\mathbb{R}^\times SO_2(\mathbb{R}))$.

Definition 2.3.1. Let $k \geq 0$ be an integer, K_f an open compact subgroup of $GL_2(\mathbb{A}_{F,\mathrm{f}})$ and $\chi : K_\mathrm{f} \to \mathbb{C}^\times$ a finite order character. A continuous function $\Phi : GL_2(\mathbb{A}_F) \to \mathbb{C}$ is said to be an automorphic form of parallel weight k, level K_f and character χ if

(1) for any $g \in GL_2(\mathbb{A}_F)$, $\gamma \in GL_2(F)$ and $u = (u_\mathrm{f}, u_\infty) \in K_\mathrm{f} \times K_\infty$, we have

$$\Phi(\gamma g u) = \Phi(g) j(u_\infty, \mathbf{i})^{-k} \det(u_\infty)^{k/2} \chi(u_\mathrm{f}).$$

(2) Φ is smooth, namely, for any $g_\mathrm{f} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$, the function

$$GL_2(\mathbb{A}_{F,\infty}) \to \mathbb{C};\ g_\infty \mapsto \Phi(g_\infty g_\mathrm{f})$$

is of C^∞-class, and for any $g_\infty \in GL_2(\mathbb{A}_{F,\infty})$, the function

$$GL_2(\mathbb{A}_{F,\mathrm{f}}) \to \mathbb{C};\ g_\mathrm{f} \mapsto \Phi(g_\infty g_\mathrm{f})$$

is locally constant.

(3) Φ is of moderate growth on $GL_2(\mathbb{A}_F)$, that is, for any compact subset Ω of $GL_2(\mathbb{A}_F)$ and for any $c \in \mathbb{R}_{>0}$, there exist some integer m and a constant $C_\Omega \in \mathbb{R}_{>0}$ such that

$$\left| \Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g \right) \right| \leq C_\Omega |a|_{\mathbb{A}_F}^m$$

for all $a \in \mathbb{A}_F^\times$ with $|a|_{\mathbb{A}_F} = \prod_{v \in P(F)} |a_v|_v \geq c$ and $g \in \Omega$.

(4) Φ is holomorphic, namely, for any $g_{\mathrm{f}} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$, the function

$$\mathfrak{H}^{P_\infty(F)} \to \mathbb{C};\ (\tau_v)_{v\in P_\infty(F)} \mapsto \Phi(g_\infty g_{\mathrm{f}}) j(g_\infty, \mathbf{i})^k \det(g_\infty)^{-k/2}$$

is holomorphic on $\mathfrak{H}^{P_\infty(F)}$. Here, for a given $(\tau_v)_{v\in P_\infty(F)} \in \mathfrak{H}^{P_\infty(F)}$, we choose $g_\infty = (g_v)_{v\in P_\infty(F)}$ in $GL_2(\mathbb{A}_{F,\infty})^+$ so that $g_v(i) = \tau_v$ for any $v \in P_\infty(F)$. The function does not depend on the choice of such g_v's, because of the assertion (1) above.

The space of automorphic form of weight k, level K_{f} and central character χ will be denoted by $\mathcal{A}_k(K_{\mathrm{f}}, \chi)$.

We now observe that the space $M_k(\mathfrak{m}, \psi)$ which we considered in the preceding sections can be embedded into $\mathcal{A}_k(K_{\mathrm{f}}, \chi)$ for suitably chosen K_{f} and χ. For each $v \in P_{\mathrm{f}}(F)$, let $K_v(\mathfrak{m})$ be the subgroup of $GL_2(O_{F_v})$ defined by

$$K_v(\mathfrak{m}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(O_{F_v}) \middle| c \equiv 0 \bmod \mathfrak{m}O_{F_v} \right\}.$$

Then $K_{\mathrm{f}}(\mathfrak{m}) = \prod_{v\in P_{\mathrm{f}}(F)} K_v(\mathfrak{m})$ is a compact open subgroup of $GL_2(\mathbb{A}_{F,\mathrm{f}})$. We take an element $D = (D_v) \in \mathbb{A}_{F,\mathrm{f}}^\times$ so that $D_v O_{F_v} = \mathfrak{d} O_{F_v}$ for any $v \in P_{\mathrm{f}}(F)$. Also, for each $\lambda \in \mathrm{Cl}_F^+$, choose $T_\lambda = (T_{\lambda,v}) \in \mathbb{A}_{F,\mathrm{f}}^\times$ so that $T_{\lambda,v} O_{F_v} = \mathfrak{t}_\lambda O_{F_v}$ for any $v \in P_{\mathrm{f}}(F)$. Then we see that

$$\Gamma_\lambda(\mathfrak{m}) = \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} K_{\mathrm{f}}(\mathfrak{m}) GL_2(\mathbb{A}_{F,\infty})^+ \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix} \cap GL_2(F)$$

for each $\lambda \in \mathrm{Cl}_F^+$. We will make use of the following lemma:

Lemma 2.3.2. *We have a disjoint decomposition*

$$GL_2(\mathbb{A}_F) = \coprod_{\lambda\in \mathrm{Cl}_F^+} GL_2(F) \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} K_{\mathrm{f}}(\mathfrak{m}) GL_2(\mathbb{A}_{F,\infty})^+.$$

Proof. The following proof essentially relies on the theorem of Kneser [22] from which we know that the algebraic group SL_2 defined over F has strong approximation. Let $g \in GL_2(\mathbb{A}_F)$. Take an element $a \in F^\times$ so that $a^{-1}\det(g) \in \mathbb{A}_F^\times$ is positive at every infinite place $v \in P_\infty(F)$. Since Cl_F^+ is isomorphic to $\mathbb{A}_{F,\mathrm{f}}^\times / F^\times \prod_{v\in P_{\mathrm{f}}(F)} O_{F_v}^\times$, there exists a unique $\lambda \in \mathrm{Cl}_F^+$ such that $a \cdot \det(g)^{-1} = b^{-1} u D T_\lambda$ for $u \in \prod_{v\in P_{\mathrm{f}}(F)} O_{F_v}^\times \mathbb{A}_{F,\infty}^\times$ with $u_v > 0$ for any $v \in P_\infty(F)$ and a totally positive $b \in F_+$. Let us put

$$W = \prod_{v\in P_{\mathrm{f}}(F)} (SL_2(O_{F_v}) \cap K_v(\mathfrak{m})) \times \prod_{v\in P_\infty(F)} SL_2(F_v).$$

This is an open subgroup of $SL_2(\mathbb{A}_F)$, and thus in view of Kneser's theorem we have

$$SL_2(F) \prod_{v\in P_\infty(F)} SL_2(F_v) \cap g' \begin{bmatrix} uDT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} W \begin{bmatrix} uDT_\lambda & 0 \\ 0 & 1 \end{bmatrix} \neq \emptyset$$

where $g' = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix}^{-1} g \begin{bmatrix} uDT_\lambda & 0 \\ 0 & 1 \end{bmatrix} \in SL_2(\mathbb{A}_F)$. This implies that

$$g = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \sigma \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}^{-1} h$$

for some $\gamma \in SL_2(F)$, $\sigma \in \prod_{v \in P_\infty(F)} SL_2(F_v)$ and $h \in W$, and hence

$$g \in GL_2(F) \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} GL_2(\mathbb{A}_{F,\infty})^+ K_{\mathrm{f}}(\mathfrak{m}).$$

Secondly we verify that the components in the union on the right-hand side are mutually disjoint. Suppose that we have

$$\begin{bmatrix} DT_{\lambda_1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \gamma \begin{bmatrix} DT_{\lambda_2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} g_\infty k_{\mathrm{f}}$$

for $\lambda_1, \lambda_2 \in \mathrm{Cl}_F^+$, $\gamma \in GL_2(F)$, $g_\infty \in GL_2(\mathbb{A}_{F,\infty})^+$ and $k_{\mathrm{f}} \in K_{\mathrm{f}}(\mathfrak{m})$. Taking determinants, we find that $\det(\gamma)$ is totally positive and $\mathfrak{t}_{\lambda_2} = \det(\gamma)\mathfrak{t}_{\lambda_1}$ as fractional ideals of F. This implies that $\lambda_1 = \lambda_2$. □

We let $\chi_\psi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the continuous character corresponding to ψ via global class field theory for F. To be specific, the narrow ray class group $\mathrm{Cl}_F(\mathfrak{m})$ is isomorphic to $\mathbb{A}_F^\times / F^\times U_F(\mathfrak{m}) \mathbb{A}_{F,\infty}^+$, and for each fractional ideal $\mathfrak{a} \in I(\mathfrak{m})$ we have $\psi(\mathfrak{a}) = \prod_{v \nmid \mathfrak{m}} \chi_{\psi,v}(\varpi_v^{\mathrm{ord}_v(\mathfrak{a})})$. Notice that χ_ψ is unramified at $v \in P_{\mathrm{f}}(F)$ not dividing $\mathfrak{m}$ and thus the value $\chi_{\psi,v}(\varpi_v)$ is independent of the choice of a uniformizer ϖ_v in F_v. For $u_{\mathrm{f}} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in K_{\mathrm{f}}(\mathfrak{m})$, we put $\chi_{\psi,\mathfrak{m}}(u_{\mathrm{f}}) = \prod_{v \mid \mathfrak{m}} \chi_{\psi,v}(d_v)$. Then $\chi_{\psi,\mathfrak{m}} : K_{\mathrm{f}}(\mathfrak{m}) \to \mathbb{C}^\times$ is a finite order character. Moreover, it follows that $\psi_{\mathrm{f}}(d) = \chi_{\psi,\mathfrak{m}}\left(\left[\begin{smallmatrix} DT_\lambda & 0 \\ 0 & 1 \end{smallmatrix}\right] \gamma_{\mathrm{f}}^{-1} \left[\begin{smallmatrix} DT_\lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} \right)$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_\lambda(\mathfrak{m})$.

Let $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+} \in M_k(\mathfrak{m}, \psi)$ be a Hilbert modular form. We shall construct a function $\Phi_f : GL_2(\mathbb{A}_F) \to \mathbb{C}$ out of f. It suffices to define Φ_f on each component of $GL_2(\mathbb{A}_F)$ as in Lemma 2.3.2. Let $\lambda \in \mathrm{Cl}_F^+$, $\gamma \in GL_2(F)$, $g_\infty \in GL_2(\mathbb{A}_{F,\infty})^+$, $k_{\mathrm{f}} \in K_{\mathrm{f}}(\mathfrak{m})$ and put

$$\Phi_f\left(\gamma \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} k_{\mathrm{f}} g_\infty \right) = f_\lambda(g_\infty(\mathbf{i})) j(g_\infty, \mathbf{i})^{-k} \det(g_\infty)^{k/2} \chi_{\psi,\mathfrak{m}}(k_{\mathrm{f}}).$$

It is a standard matter to check that the value on the right-hand side is independent of the way of expressing an element in $GL_2(F) \left[\begin{smallmatrix} DT_\lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} K_{\mathrm{f}}(\mathfrak{m}) GL_2(\mathbb{A}_{F,\infty})^+$ in terms of γ, k_{f} and g_∞.

Proposition 2.3.3. *We have* $\Phi_f \in \mathcal{A}_k(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$.

Proof. (1) It is straightforward to verify that

$$\Phi_f\left(\gamma g u_{\mathrm{f}} u_\infty\right) = \Phi_f(g) j(u_\infty, \mathbf{i})^{-k} \det(u_\infty)^{k/2} \chi_{\psi,\mathfrak{m}}(u_{\mathrm{f}})$$

for $\gamma \in GL_2(F)$, $u_{\mathrm{f}} \in K_{\mathrm{f}}(\mathfrak{m})$ and $u_\infty \in K_\infty$.

(2) Since each f_λ is holomorphic on $\mathfrak{H}^I$, it is of C^∞-class, and hence for each $g_{\mathrm{f}} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$, the function $g_\infty \mapsto \Phi_f(g_\infty g_{\mathrm{f}})$ is of C^∞-class as well. Let $K_{\mathrm{f},1}(\mathfrak{m})$ be the group consisting of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in K_{\mathrm{f}}(\mathfrak{m})$ such that $d_v - 1 \in \mathfrak{m}O_{F_v}$ for each $v \in P_{\mathrm{f}}(F)$. Then $K_{\mathrm{f},1}(\mathfrak{m})$ is an open subgroup of $GL_2(\mathbb{A}_{F,\mathrm{f}})$ and it follows by definition that Φ_f is right $K_{\mathrm{f},1}(\mathfrak{m})$-invariant. This implies that the function $g_{\mathrm{f}} \mapsto \Phi_f(g_\infty g_{\mathrm{f}})$ is locally constant for each $g_\infty \in GL_2(\mathbb{A}_{F,\infty})$.

(3) As each f_λ is holomorphic around any cusp, for each $R \in \mathbb{R}_{>0}$, the absolute value $|(f_\lambda|_k\gamma_\lambda)(z)|$ is bounded on the set $\left\{z \in \mathfrak{H}^I \mid \mathrm{Im}(z_v) \geq R,\ \forall v \in P_\infty(F)\right\}$. This implies that Φ_f is of moderate growth.

(4) We take $g_{\mathrm{f}} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ and write $g_{\mathrm{f}} = \gamma \left[\begin{smallmatrix} DT_\lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} k_{\mathrm{f}} g_\infty$ for $\gamma \in GL_2(F)$, $\lambda \in \mathrm{Cl}_F^+$, $k_{\mathrm{f}} \in K_{\mathrm{f}}(\mathfrak{m})$ and $g_\infty \in GL_2(\mathbb{A}_{F,\infty})^+$. Then for $u_\infty \in GL_2(\mathbb{A}_{F,\infty})^+$, we have

$$\begin{aligned}\Phi_f(g_{\mathrm{f}} u_\infty) j(u_\infty, \mathbf{i})^k \det(u_\infty)^{-k/2} &= (f_\lambda|g_\infty u_\infty)(\mathbf{i}) j(u_\infty, \mathbf{i})^k \det(u_\infty)^{-k/2} \chi_{\psi,\mathfrak{m}}(k_{\mathrm{f}}) \\ &= (f_\lambda|g_\infty)(u_\infty(\mathbf{i}))\chi_{\psi,\mathfrak{m}}(k_{\mathrm{f}}).\end{aligned}$$

Since f_λ is holomorphic on $\mathfrak{H}^I$, so is $f_\lambda|g_\infty$.

□

Conversely, for a given automorphic form $\Phi \in \mathcal{A}_k(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$, we will construct $f_\Phi = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+} \in M_k(\mathfrak{m}, \psi)$ as follows: for a point τ in $\mathfrak{H}^{P_\infty(F)}$, we choose a matrix $g_\infty \in GL_2(\mathbb{A}_{F,\infty})^+$ so that $g_\infty(\mathbf{i}) = \tau$. For each $\lambda \in \mathrm{Cl}_F^+$, we put

$$f_\lambda(\tau) = \Phi\left(\left[\begin{array}{cc} DT_\lambda & 0 \\ 0 & 1 \end{array}\right]^{-1} g_\infty\right) j(g_\infty, \mathbf{i})^k \det(g_\infty)^{-k/2}.$$

Note that the right-hand side depends only on τ and independent of the choice of g_∞. One can show the following

Proposition 2.3.4. *$f_\Phi = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+}$ belongs to $M_k(\mathfrak{m}, \psi)$ and the correspondences $f \mapsto \Phi_f$ and $\Phi \mapsto f_\Phi$ are mutually inverse.*

We leave the proof to the reader, as it is just a routine work.

Finally, we characterize the subspace $\mathcal{A}_k^{\mathrm{cusp}}(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$ of cuspidal automorphic forms, namely, the subspace of $\mathcal{A}_k(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$ that corresponds to $S_k(\mathfrak{m}, \psi)$ under the bijection between $M_k(\mathfrak{m}, \psi)$ and $\mathcal{A}_k(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$.

Lemma 2.3.5. *Let $f = (f_\lambda)_{\lambda \in \mathrm{Cl}_F^+} \in M_k(\mathfrak{m}, \psi)$ be a Hilbert modular form and Φ_f the corresponding automorphic form in $\mathcal{A}_k(K_{\mathrm{f}}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$. Then for each $\lambda \in \mathrm{Cl}_F^+$ and $\gamma_\lambda \in GL_2^+(F)$, the constant term of the Fourier expansion of $f_\lambda|\gamma_\lambda$ in* (2.2.3) *is equal to*

$$a_\lambda(0, \gamma_\lambda) = \int_{F\backslash\mathbb{A}_F} \Phi_f\left(\left[\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right] \gamma_{\lambda,\mathrm{f}}^{-1} \left[\begin{array}{cc} DT_\lambda & 0 \\ 0 & 1 \end{array}\right]^{-1}\right) dt.$$

Here, dt is the additive Haar measure on $F\backslash\mathbb{A}_F$ normalized so that $\int_{F\backslash\mathbb{A}_F} dt = 1$.

From Lemmas 2.3.2 and 2.3.5, we deduce that $f \in S_k(\mathfrak{m}, \psi)$ if and only if

$$\int_{F\backslash \mathbb{A}_F} \Phi_f \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g \right) dt = 0$$

holds for every $g \in GL_2(\mathbb{A}_F)$.

Definition 2.3.6. Let $k \geq 0$ be an integer, K_{f} an open compact subgroup of $GL_2(\mathbb{A}_{F,\mathrm{f}})$ and $\chi : K_f \to \mathbb{C}^\times$ a finite order character. We define $\mathcal{A}_k^{\mathrm{cusp}}(K_{\mathrm{f}}, \chi)$ to be the subspace of $\mathcal{A}_k(K_{\mathrm{f}}, \chi)$ consisting of elements Φ with the property that

$$\int_{F\backslash \mathbb{A}_F} \Phi \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g \right) dt = 0$$

for every $g \in GL_2(\mathbb{A}_F)$. The elements in $\mathcal{A}_k^{\mathrm{cusp}}(K_f, \chi)$ are said to be cuspidal.

We end this section by providing a proof of Lemma 2.3.5.

Proof. From now on, we denote by $A_\Phi(g)$ the left-hand side of the equation in Definition 2.3.6. Then for $\tau = x + \mathbf{i}y \in \mathfrak{H}^{P_\infty(F)}$ and $g_\infty = \left[\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{A}_{F,\infty})^+$, we have

$$\begin{aligned}
(f_\lambda | \gamma_\lambda)(\tau) &= j(\gamma_{\lambda,\infty}, g_\infty(\mathbf{i}))^{-k} \det(\gamma_{\lambda,\infty})^{k/2} \\
&\qquad \times \Phi \left(\begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \gamma_{\lambda,\infty} g_\infty \right) j(\gamma_{\lambda,\infty} g_\infty, \mathbf{i})^k \det(\gamma_{\lambda,\infty} g_\infty)^{-k/2} \\
&= j(g_\infty, \mathbf{i})^k \det(g_\infty)^{-k/2} \Phi \left(\begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \gamma_{\lambda,\infty} g_\infty \right) \\
&= y^{-k/2} \Phi \left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \right).
\end{aligned}$$

Hence we have $a_\lambda(0, \gamma_\lambda) = y^{-k/2} A_\Phi \left(\left[\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right] \gamma_{\lambda,\mathrm{f}}^{-1} \left[\begin{smallmatrix} DT_\lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} \right)$. By definition, we have

$$\begin{aligned}
& y^{-k/2} A_\Phi \left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \right) \\
&= \int_{F\backslash \mathbb{A}_F} y^{-k/2} \Phi \left(\begin{bmatrix} 1 & t+x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \right) dt \\
&= \int_{F\backslash \mathbb{A}_F} y^{-k/2} \Phi \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \right) dt.
\end{aligned}$$

Since Φ is holomorphic, the function

$$\tau = x + \mathbf{i}y \mapsto y^{-k/2} A_\Phi \left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} \right)$$

is holomorphic as well. By applying the Cauchy-Riemann equation to this function, we see that it is actually a constant. We put $y = 1$ to obtain

$$a_\lambda(0, \gamma_\lambda) = \int_{F\backslash\mathbb{A}_F} \Phi\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \gamma_{\lambda,\mathrm{f}}^{-1} \begin{bmatrix} DT_\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1}\right) dt,$$

as desired. □

2.4 Hecke operators on the space of Hilbert modular forms

In this section, we define the Hecke operators acting on the space $\mathcal{A}_k(K_\mathrm{f}, \chi)$ of automorphic forms. For simplicity, we assume that the open compact subgroup K_f of $GL_2(\mathbb{A}_{F,\mathrm{f}})$ is contained in $K_\mathrm{f}(\mathfrak{m})$ and the character $\chi : K_\mathrm{f} \to \mathbb{C}^\times$ is the restriction of $\chi_{\psi,\mathfrak{m}} : K_\mathrm{f}(\mathfrak{m}) \to \mathbb{C}^\times$ attached to a narrow ray class character ψ modulo $\mathfrak{m}$, as explained in the previous section.

As K_f is an open compact subgroup of $GL_2(\mathbb{A}_{F,\mathrm{f}})$, for any $\alpha_\mathrm{f} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$, the group $K_\mathrm{f} \cap \alpha_\mathrm{f} K_\mathrm{f} \alpha_\mathrm{f}^{-1}$ is a subgroup of finite index in K_f. We choose a disjoint decomposition

$$K_\mathrm{f} = \coprod_{i \in I} u_i (K_\mathrm{f} \cap \alpha_\mathrm{f} K_\mathrm{f} \alpha_\mathrm{f}^{-1}). \tag{2.2.4}$$

Here, I is a finite set and u_i belongs to K_f for each $i \in I$.

Definition 2.4.1. Let $\alpha_\mathrm{f} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ and $\Phi \in \mathcal{A}_k(K_\mathrm{f}, \chi)$. We define the Hecke operator $[K_\mathrm{f}\alpha_\mathrm{f}K_\mathrm{f}]$ on $\mathcal{A}_k(K_\mathrm{f}, \chi)$ by

$$(\Phi|[K_\mathrm{f}\alpha_\mathrm{f}K_\mathrm{f}])(g) = \sum_{i \in I} \chi(u_i)^{-1} \Phi(g u_i \alpha_\mathrm{f}). \quad (g \in GL_2(\mathbb{A}_F)) \tag{2.2.5}$$

One can verify that the right-hand side of (2.2.5) is independent of the decomposition (2.2.4).

Proposition 2.4.2. *For any $\Phi \in \mathcal{A}_k(K_\mathrm{f}, \chi)$, we have $\Phi|[K_\mathrm{f}\alpha_\mathrm{f}K_\mathrm{f}] \in \mathcal{A}_k(K_\mathrm{f}, \chi)$.*

Proof. (1) Let $g \in GL_2(\mathbb{A}_F)$, $\gamma \in GL_2(F)$ and $u = (u_\mathrm{f}, u_\infty) \in K_\mathrm{f} \times K_\infty$. For each $i \in I$, there exists a unique $i(u) \in I$ such that $u_\mathrm{f} u_i = u_{i(u)} k_{i,u}$ for some $k_{i,u} \in K_\mathrm{f} \cap \alpha_\mathrm{f} K_\mathrm{f} \alpha_\mathrm{f}^{-1}$. Then we have

$$\begin{aligned} \chi(u_i)^{-1}\Phi(\gamma g u u_i \alpha_\mathrm{f}) &= \chi(u_i)^{-1}\Phi(g u_{i(u)} \alpha_\mathrm{f} \alpha_\mathrm{f}^{-1} k_{i,u} \alpha_\mathrm{f}) j(u_\infty, \mathbf{i})^{-k} \mathrm{det}(u_\infty)^{k/2} \\ &= \chi(u_i)^{-1}\Phi(g u_{i(u)} \alpha_\mathrm{f}) \chi(\alpha_\mathrm{f}^{-1} k_{i,u} \alpha_\mathrm{f}) j(u_\infty, \mathbf{i})^{-k} \mathrm{det}(u_\infty)^{k/2}. \end{aligned}$$

Notice that the map $I \ni i \mapsto i(u) \in I$ is a bijection. Also, it follows from the definition of χ and the relation $u_\mathrm{f} u_i = u_{i(u)} k_{i,u}$ that $\chi(\alpha_\mathrm{f}^{-1} k_{i,u} \alpha_\mathrm{f}) = \chi(k_{i,u})$ is equal to $\chi(u_i)\chi(u_\mathrm{f})\chi(u_{i(u)})^{-1}$. Therefore we see that

$$\sum_{i \in I} \chi(u_i)^{-1}\Phi(\gamma g u u_i \alpha_\mathrm{f}) = \sum_{i \in I} \chi(u_{i(u)})^{-1}\Phi(g u_{i(u)} \alpha_\mathrm{f}) j(u_\infty, \mathbf{i})^{-k} \mathrm{det}(u_\infty)^{k/2} \chi(u_\mathrm{f}),$$

as desired.

(2) By definition, for any $g_{\mathrm{f}} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ and for each $i \in I$, the function

$$GL_2(\mathbb{A}_{F,\infty}) \to \mathbb{C};\ g_\infty \mapsto \Phi(g_\infty g_{\mathrm{f}} u_i \alpha_{\mathrm{f}})$$

is of C^∞-class, and hence $g_\infty \mapsto (\Phi|[K_{\mathrm{f}}\alpha_{\mathrm{f}}K_{\mathrm{f}}])(g_\infty g_{\mathrm{f}})$ is. We see in exactly the same way that the function $g_{\mathrm{f}} \mapsto (\Phi|[K_{\mathrm{f}}\alpha_{\mathrm{f}}K_{\mathrm{f}}])(g_\infty g_{\mathrm{f}})$ is locally constant on $GL_2(\mathbb{A}_{F,\mathrm{f}})$ for any fixed $g_\infty \in GL_2(\mathbb{A}_{F,\mathrm{f}})$.

(3) Let Ω be a compact subset of $GL_2(\mathbb{A}_F)$ and c a positive real number. Since Φ is of moderate growth, for each $i \in I$, there exist a constant $C_{\Omega,i} \in \mathbb{R}_{>0}$ and an integer m_i such that

$$\left|\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right)\right| \leq C_{\Omega,i}|a|_{\mathbb{A}_F}^{m_i}$$

for all $a \in \mathbb{A}_F^\times$ with $|a|_{\mathbb{A}_F} \geq c$ and $g \in \Omega u_i \alpha_{\mathrm{f}}$. Let $C_\Omega = \#I \cdot \max\{C_{\Omega,i} \mid i \in I\}$ and $m = \max\{m_i \mid i \in I\}$. Then we have

$$\left|\sum_{i\in I}\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g u_i \alpha_{\mathrm{f}}\right)\right| \leq C_\Omega |a|_{\mathbb{A}_F}^m$$

for all $a \in \mathbb{A}_F^\times$ with $|a|_{\mathbb{A}_F} \geq c$ and $g \in \Omega$, as desired.

(4) By definition, for any $g_{\mathrm{f}} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ and for each $i \in I$, the function

$$\mathfrak{H}^{P_\infty(F)} \to \mathbb{C};\ \tau = (\tau_v)_{v\in P_\infty(F)} \mapsto \Phi(g_\infty g_{\mathrm{f}} u_i \alpha_{\mathrm{f}}) j(g_\infty, \mathbf{i})^k \det(g_\infty)^{-k/2}$$

is holomorphic. Thus $\Phi|[K_{\mathrm{f}}\alpha_{\mathrm{f}}K_{\mathrm{f}}]$ is holomorphic as well.

□

With Lemma 2.3.5 in mind, we know that the subspace $\mathcal{A}_k^{\mathrm{cusp}}(K_{\mathrm{f}}, \chi)$ consisting of cuspidal automorphic forms is preserved by the action of Hecke operators.

Remark 2.4.3. In the case of $F = \mathbb{Q}$, we can easily describe the Hecke operators on $M_k(\mathfrak{m}, \psi)$ as follows: let M be the positive integer determined by $\mathfrak{m} = M\mathbb{Z}$. Since $\mathbb{Q}$ has class number one, we may take $\mathfrak{t}_\lambda$ to be $\mathbb{Z}$ and see that $\Gamma_\lambda(\mathfrak{m}) = \Gamma_0(M)$. Let ψ be a Dirichlet character modulo M with $\psi(-1) = (-1)^k$ and $\psi(\gamma) = \psi(d)$ for a matrix $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2(\mathbb{Q})$ with $d \in \mathbb{Z}$ prime to M. Take $\beta \in GL_2(\mathbb{Q})$ with positive determinant and choose a decomposition

$$\Gamma_0(M) = \coprod_{i\in I}(\Gamma_0(M) \cap \beta^{-1}\Gamma_0(M)\beta)\gamma_i$$

where I is a finite set and $\gamma_i \in \Gamma_0(M)$ for each $i \in I$. Then the Hecke operator $[\Gamma_0(M)\beta\Gamma_0(M)]$ acting on $M_k(\mathfrak{m}, \psi)$ is defined by

$$(f|[\Gamma_0(M)\beta\Gamma_0(M)])(\tau) = \sum_{i\in I}\psi(\beta\gamma_i)^{-1}(f|_k\beta\gamma_i)(\tau). \quad (\tau \in \mathfrak{H},\ f \in M_k(\mathfrak{m}, \psi))$$

Let $\chi_\psi : \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ be the idele class character corresponding to ψ by class field theory. We define the character $\chi_{\psi,\mathfrak{m}} : K_\mathrm{f} \to \mathbb{C}^\times$ as exposed in the previous section. Take $\alpha \in GL_2(\mathbb{Q})^+$ and decompose it as $\alpha = (\alpha_\mathrm{f}, \alpha_\infty)$ with $\alpha_\mathrm{f} \in GL_2(\mathbb{A}_{\mathbb{Q},\mathrm{f}})$ and $\alpha_\infty \in GL_2(\mathbb{A}_{\mathbb{Q},\infty})^+$. Then in the notation used in the previous section, we have

$$\Phi_f|[K_\mathrm{f}(\mathfrak{m})\alpha_\mathrm{f}K_\mathrm{f}(\mathfrak{m})] = \Phi_{f|[\Gamma_0(M)\alpha^\iota\Gamma_0(M)]}$$

as elements of $\mathcal{A}_k(K_\mathrm{f}(\mathfrak{m}), \chi_{\psi,\mathfrak{m}})$. Here $\alpha^\iota \in GL_2(\mathbb{Q})$ is defined by $\alpha\alpha^\iota = \det(\alpha)I_2$.

Example 2.4.4. We introduce the Hecke operators $T_\mathfrak{m}(v)$ and $S_\mathfrak{m}(v)$ associated to each finite place $v \in P_\mathrm{f}(F)$. Fix a uniformizer ϖ_v in F_v and choose $\alpha_\mathrm{f} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ to be $\left[\begin{smallmatrix}\varpi_v & 0\\ 0 & 1\end{smallmatrix}\right]$ at v and I_2 at any other finite place. Then we have

$$K_\mathrm{f}(\mathfrak{m})\alpha_\mathrm{f}K_\mathrm{f}(\mathfrak{m}) = \begin{cases} \coprod_{i_v \in O_{F_v}/\mathfrak{p}_v} \begin{bmatrix} 1 & i_v \\ 0 & 1 \end{bmatrix} \alpha_\mathrm{f} K_\mathrm{f}(\mathfrak{m}) & \text{if } v \mid \mathfrak{m}; \\ \coprod_{i_v \in O_{F_v}/\mathfrak{p}_v} \begin{bmatrix} 1 & i_v \\ 0 & 1 \end{bmatrix} \alpha_\mathrm{f} K_\mathrm{f}(\mathfrak{m}) \coprod \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \alpha_\mathrm{f} K_\mathrm{f}(\mathfrak{m}) & \text{if } v \nmid \mathfrak{m}. \end{cases}$$

This Hecke operator $[K_\mathrm{f}(\mathfrak{m})\alpha_\mathrm{f}K_\mathrm{f}(\mathfrak{m})]$ will be denoted by $T_\mathfrak{m}(v)$. When $v \mid \mathfrak{m}$, we will also write $T_\mathfrak{m}(v)$ as $U(v)$.

For each finite place $v \nmid \mathfrak{m}$, we define $\beta_\mathrm{f} \in GL_2(\mathbb{A}_{F,\mathrm{f}})$ to be $\varpi_v I_2$ at v and I_2 at any other place. The Hecke operator $[K_\mathrm{f}(\mathfrak{m})\beta_\mathrm{f}K_\mathrm{f}(\mathfrak{m})]$ will be denoted by $S_\mathfrak{m}(v)$. We put $S_\mathfrak{m}(v) = 0$ if $v \mid \mathfrak{m}$.

One can verify by direct computation that for $f \in M_k(\mathfrak{m}, \psi)$, the normalized Fourier coefficients satisfy

$$c(\mathfrak{n}, f|T_\mathfrak{m}(\mathfrak{l})) = \begin{cases} \mathrm{N}(\mathfrak{l})^{1-k/2}c(\mathfrak{n}\mathfrak{l}, f) + \mathrm{N}(\mathfrak{l})^{k/2}c(\mathfrak{n}\mathfrak{l}^{-1}, f|S_\mathfrak{m}(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid \mathfrak{m}; \\ \mathrm{N}(\mathfrak{l})^{1-k/2}c(\mathfrak{n}\mathfrak{l}, f) & \text{if } \mathfrak{l} \mid \mathfrak{m} \end{cases}$$

for every prime ideal $\mathfrak{l}$ of F and any non-zero integral ideal $\mathfrak{n}$ in O_F. Here we understand that $c(\mathfrak{n}', f) = 0$ when $\mathfrak{n}'$ does not lie in O_F. As for the constant terms, it follows that

$$c_\lambda(0, f|T_\mathfrak{m}(\mathfrak{l})) = \begin{cases} \mathrm{N}(\mathfrak{l})^{1-k/2}c_{\lambda\mu}(0, f) + \mathrm{N}(\mathfrak{l})^{k/2}c_{\lambda\mu^{-1}}(0, f|S_\mathfrak{m}(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid \mathfrak{m}; \\ \mathrm{N}(\mathfrak{l})^{1-k/2}c_{\lambda\mu}(0, f) & \text{if } \mathfrak{l} \mid \mathfrak{m} \end{cases}$$

where $\mu \in \mathrm{Cl}_F^+$ is the narrow ideal class such that $\mathfrak{l} = (b_\mathfrak{l})\mathfrak{t}_\mu^{-1}$ for some $b_\mathfrak{l} \in F_+$.

For each integral ideal $\mathfrak{n}$ in O_F, we define the Hecke operator $T_\mathfrak{m}(\mathfrak{n})$ acting on the space $M_k(\mathfrak{m}, \psi)$ so that an equality of formal Euler product

$$\sum_\mathfrak{n} T_\mathfrak{m}(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{-s} = \prod_\mathfrak{l}(1 - T_\mathfrak{m}(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{-s} + S_\mathfrak{m}(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{1-2s})^{-1}$$

holds. Here the product on the right-hand side runs over all prime ideals $\mathfrak{l}$ in F. In other words, we have

$$T_\mathfrak{m}(\mathfrak{l}^{e+1}) = T_\mathfrak{m}(\mathfrak{l})T_\mathfrak{m}(\mathfrak{l}^e) - S_\mathfrak{m}(\mathfrak{l})\mathrm{N}(\mathfrak{l})T_\mathfrak{m}(\mathfrak{l}^{e-1}) \quad (e \in \mathbb{Z}_{\geq 1})$$

for each prime ideal $\mathfrak{l}$ (here we understand that $T_{\mathfrak{m}}(O_F)$ is the identity map). Furthermore, we have $T_{\mathfrak{m}}(\mathfrak{l})T_{\mathfrak{m}}(\mathfrak{q}) = T_{\mathfrak{m}}(\mathfrak{q})T_{\mathfrak{m}}(\mathfrak{l})$ for distinct prime ideals $\mathfrak{l}$ and $\mathfrak{q}$, and

$$T_{\mathfrak{m}}\left(\prod_{j=1}^{r} \mathfrak{l}_j^{e_j}\right) = \prod_{j=1}^{r} T_{\mathfrak{m}}(\mathfrak{l}_j^{e_j})$$

for distinct prime ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_r$. We will simply write $T(\mathfrak{n})$ and $S(\mathfrak{n})$ for $T_{\mathfrak{m}}(\mathfrak{n})$ and $S_{\mathfrak{m}}(\mathfrak{n})$ if there is no ambiguity concerning the level $\mathfrak{m}$.

Remark 2.4.5. We make a remark on the central character of a Hilbert modular form. As Shimura pointed out in Proposition 2.1 (p. 649 of [40]), if a non-zero element $f \in M_k(\mathfrak{m}, \psi)$ is an eigenform for all $S_{\mathfrak{m}}(v)$ with $v \nmid \mathfrak{m}$, then there exists a finite order character $\eta_f : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ so that $\Phi_f\left(g \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]\right) = \eta_f(z)\Phi_f(g)$ for all $z \in \mathbb{A}_F^\times$ and $g \in GL_2(\mathbb{A}_F)$. This character is referred to as the central character of f. This η_f should satisfy $\eta_f(u) = \prod_{v|m} \chi_{\psi,v}(u_v)$ for any $u = (u_v)_{v \in P_{\mathrm{f}}(F)} \in \prod_{v \in P_{\mathrm{f}}(F)} O_{F_v}^\times$ and $\eta_f(x_\infty) = \mathrm{sgn}(x_\infty)^k$ for every $x_\infty \in \mathbb{A}_{F,\infty}^\times$. In particular η_f factors through the quotient $\mathbb{A}_F^\times/F^\times U_F(\mathfrak{m})\mathbb{A}_{F,\infty}^+$ which is isomorphic to $\mathrm{Cl}_F(\mathfrak{m})$.

Remark 2.4.6. Here is a remark concerning the normalzation of the Hecke operators. In p. 650 of his paper [40], Shimura introduced the modified version of the Hecke operators $T'_{\mathfrak{m}}(\mathfrak{n}) = \mathrm{N}(\mathfrak{n})^{k/2-1}T_{\mathfrak{m}}(\mathfrak{n})$ as seen in the equation (2.21) in that paper. With respect to this definition, we have

$$c(\mathfrak{n}, f|T'_{\mathfrak{m}}(\mathfrak{l})) = \begin{cases} c(\mathfrak{n}\mathfrak{l}, f) + \mathrm{N}(\mathfrak{l})^{k-1}c(\mathfrak{n}\mathfrak{l}^{-1}, f|S_{\mathfrak{m}}(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid \mathfrak{m}; \\ c(\mathfrak{n}\mathfrak{l}, f) & \text{if } \mathfrak{l} \mid \mathfrak{m} \end{cases}$$

(see the equation (2.23) in [40]) and

$$c_\lambda(0, f|T'_{\mathfrak{m}}(\mathfrak{l})) = \begin{cases} c_{\lambda\mu}(0, f) + \mathrm{N}(\mathfrak{l})^{k-1}c_{\lambda\mu^{-1}}(0, f|S_{\mathfrak{m}}(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid \mathfrak{m}; \\ c_{\lambda\mu}(0, f) & \text{if } \mathfrak{l} \mid \mathfrak{m}, \end{cases}$$

respectively. As the equation (2.26) in [40] indicates, the formal Euler product is modified as follows:

$$\sum_{\mathfrak{n}} T'_{\mathfrak{m}}(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{l}} (1 - T'_{\mathfrak{m}}(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{-s} + S_{\mathfrak{m}}(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{k-1-2s})^{-1}.$$

It is proved by Shimura that the eigenvalues of $T_{\mathfrak{m}}(\mathfrak{n})$ are algebraic numbers, and further, in the case of Hilbert modular forms with parallel weights, those of $T'_{\mathfrak{m}}(\mathfrak{n})$ are algebraic *integers* (Proposition 2.2; p. 650 of [40]).

In the following chapter, we will interpolate the Fourier coefficients $\{c(\mathfrak{n}, f)\}$, $\{c_\lambda(0, f)\}$ and the Hecke operators $T'_{\mathfrak{m}}(\mathfrak{n})$. We note that $C(\mathfrak{n}, f)$ in Shimura's paper [40] is denoted by $c(\mathfrak{n}, f)$ in Wiles' paper [46] for each non-zero integral ideal $\mathfrak{n}$ in O_F. Similarly, $T'_{\mathfrak{m}}(\mathfrak{n})$ in [40] is denoted by $T_{\mathfrak{m}}(\mathfrak{n})$ in [46]. Our $c(\mathfrak{n}, f)$ is identical to $C(\mathfrak{n}, f)$ in [40] (and thus to $c(\mathfrak{n}, f)$ in [46]).

Chapter 3

p-adic families of Hilbert modular forms

In this chapter, we will give a definition of Hilbert modular forms over a p-adically complete ring, and then introduce a p-adic family of such modular forms which will be called a Λ-adic form. We fix a prime number p once and for all.

3.1 Modular forms defined over a ring

In what follows, we will define the space of Hilbert modular forms defined over a p-adic integer ring. For this purpose, we first make a brief review of the theory of rationality for Hilbert modular forms, which was established by Shimura in [40]. As in the previous chapter, we let $k \geq 1$ be an integer, $\mathfrak{m}$ a non-zero integral ideal in O_F and $\psi : \mathrm{Cl}_F^+(\mathfrak{m}) \to \mathbb{C}^\times$ a narrow ray class character modulo $\mathfrak{m}$.

Shimura showed that for $f \in M_k(\mathfrak{m}, \psi)$ and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, there exists a form $f^\sigma \in M_k(\mathfrak{m}, \psi^\sigma)$ such that $c_\lambda(0, f^\sigma) = c_\lambda(0, f)^\sigma$ and $c(\mathfrak{n}, f^\sigma) = c(\mathfrak{n}, f)^\sigma$ for all $\lambda \in \mathrm{Cl}_F^+$ and $\mathfrak{n}$ (Proposition 1.6; p. 644 of [40]). Here $\psi^\sigma : \mathrm{Cl}_F^+(\mathfrak{m}) \to \mathbb{C}^\times$ is a narrow ray class character modulo $\mathfrak{m}$ defined by $\psi^\sigma(\mathfrak{n}) = \psi(\mathfrak{n})^\sigma$.

Definition 3.1.1. Let A be a subring of $\mathbb{C}$. A Hilbert modular form $f \in M_k(\mathfrak{m}, \psi)$ is said to be defined over A if $c_\lambda(0, f), c(\mathfrak{n}, f) \in A$ for all $\lambda \in \mathrm{Cl}_F^+$ and non-zero ideal $\mathfrak{n}$ in O_F.

The subspace of $M_k(\mathfrak{m}, \psi)$ consisting of forms defined over A is denoted by $M_k(\mathfrak{m}, \psi; A)$. We put $S_k(\mathfrak{m}, \psi; A) = S_k(\mathfrak{m}, \psi) \cap M_k(\mathfrak{m}, \psi; A)$ for the space of cusp forms defined over A. $M_k(\mathfrak{m}, \psi; A)$ and $S_k(\mathfrak{m}, \psi; A)$ admit the structure of A-modules via scalar multiplication. In view of Example 2.4.4, we have $f^\sigma | T_\mathfrak{m}(\mathfrak{n}) = (f | T_\mathfrak{m}(\mathfrak{n}))^\sigma$ and $f^\sigma | S_\mathfrak{m}(\mathfrak{n}) = (f | S_\mathfrak{m}(\mathfrak{n}))^\sigma$ for $f \in M_k(\mathfrak{m}, \psi)$ and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, namely, the spaces $M_k(\mathfrak{m}, \psi; A)$ and $S_k(\mathfrak{m}, \psi; A)$ are preserved by the Hecke operators $T_\mathfrak{m}(\mathfrak{n})$ and $S_\mathfrak{m}(\mathfrak{n})$, if A contains the values of ψ. Shimura proved that

$$M_k(\mathfrak{m}, \psi) = M_k(\mathfrak{m}, \psi; \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} \mathbb{C}$$

where $\mathbb{Z}[\psi]$ is the ring obtained by adjoining all the values of ψ to $\mathbb{Z}$ (Proposition 1.7; p. 645 and Proposition 2.6; p. 652 of [40]). We can deduce from this equality that the Hecke eigenvalues for $T'_{\mathfrak{m}}(\mathfrak{n})$ are algebraic integers (Proposition 2.2; p. 650 of [40]).

Now let p be a prime number and fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We also fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and put $\iota_\infty = \iota \circ \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\mathfrak{p}$ be the prime ideal of F lying over p determined by ι_p.

Definition 3.1.2. We define $M_k(\mathfrak{m}, \psi; \overline{\mathbb{Q}}_p)$ to be $M_k(\mathfrak{m}, \psi; \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} \overline{\mathbb{Q}}_p$. For a subring A of $\overline{\mathbb{Q}}_p$, the subspace $M_k(\mathfrak{m}, \psi; A)$ of $M_k(\mathfrak{m}, \psi; \overline{\mathbb{Q}}_p)$ consists of forms f with the defining data $c_\lambda(0, f), c(\mathfrak{n}, f) \in A$ for all λ and $\mathfrak{n}$. The same for the space of cusp forms $S_k(\mathfrak{m}, \psi; A)$.

Shimura has shown in Section 2 of [40] that $M_k(\mathfrak{m}, \psi; A) = M_k(\mathfrak{m}, \psi; \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} A$ for any field A in $\overline{\mathbb{Q}}_p$ or $\mathbb{C}$ containing $\mathbb{Q}[\psi]$ or $\mathbb{Q}_p[\psi]$, respectively. For a subring R of such a field A containing $\mathbb{Z}[\psi]$ or $\mathbb{Z}_p[\psi]$, we let $M_k(\mathfrak{m}, \psi; R)$ denote the subspace of $M_k(\mathfrak{m}, \psi; A)$ with the defining data in R. This definition is compatible with that in Definition 3.1.2.

3.2 The U_p-operator and the ordinary idempotent

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing the values of ψ. We will define an ordinary idempotent e on $M_k(\mathfrak{m}, \psi; \mathcal{O})$ which we now make precise. We note that $M_k(\mathfrak{m}, \psi; \mathcal{O})$ is a free $\mathcal{O}$-module of finite rank. We assume that $\mathfrak{m}$ is divisible by p and let $U_p = T'_{\mathfrak{m}}(p)$. The ordinary idempotent e is an $\mathcal{O}$-linear endomorphism on $M_k(\mathfrak{m}, \psi; \mathcal{O})$ given by

$$e = \lim_{n \to \infty} U_p^{n!}.$$

Since $M_k(\mathfrak{m}, \psi; \mathcal{O})$ is free of finite rank over $\mathcal{O}$, the limit e does exist in the endomorphism ring $\mathrm{End}_{\mathcal{O}}(M_k(\mathfrak{m}, \psi; \mathcal{O}))$ and satisfies $e^2 = e$ (see p. 537 of [46] and Lemma 7.2.1; p. 199 of [15] for details). We define $M_k^{\mathrm{o}}(\mathfrak{m}, \psi; \mathcal{O})$ to be the image of $M_k(\mathfrak{m}, \psi; \mathcal{O})$ by e and likewise $S_k^{\mathrm{o}}(\mathfrak{m}, \psi; \mathcal{O})$ for the space of cusp forms.

Definition 3.2.1. We say that a Hilbert modular form $f \in M_k(\mathfrak{m}, \psi; \mathcal{O})$ is ordinary if $f \in M_k^{\mathrm{o}}(\mathfrak{m}, \psi; \mathcal{O})$.

The space $M_k^{\mathrm{o}}(\mathfrak{m}, \psi; \mathcal{O})$ is the maximal direct summand of $M_k(\mathfrak{m}, \psi; \mathcal{O})$ on which the action of U_p is invertible.

We now introduce the notion of p-stabilized newforms. For the precise definition of a newform, we refer the reader to Section 2 of [40] where it is called a primitive form. A newform is in particular an eigenform f for all the Hecke operators $T(\mathfrak{l})$, $U(\mathfrak{l})$ and $S(\mathfrak{l})$ and is normalized so that $c(O_F, f) = 1$. We just recall that for each eigenform $f \in M_k(\mathfrak{m}, \psi)$ for all the Hecke operators, there exists a unique newform f^* of level $\mathfrak{c}$ a divisor of $\mathfrak{m}$ such that the system of eigenvalues of f and f^* for the Hecke operators $T(\mathfrak{l})$, $U(\mathfrak{l})$ and $S(\mathfrak{l})$ are the same for all prime ideals $\mathfrak{l} \nmid \mathfrak{m}\mathfrak{c}^{-1}$.

Definition 3.2.2. Assume that $\mathfrak{m}$ is divisible by p. A p-stabilized newform in $M_k(\mathfrak{m}, \psi; \overline{\mathbb{Q}}_p)$ is an eigenform for all the Hecke operators $T_{\mathfrak{m}}(\mathfrak{l})$, $U_{\mathfrak{m}}(\mathfrak{l})$ and $S_{\mathfrak{m}}(\mathfrak{l})$ such that the associated newform has a level that only differs from $\mathfrak{m}$ at prime ideals dividing p and the eigenvalue for $U_{\mathfrak{m}}(\mathfrak{q})$ is a p-adic unit for each prime ideal $\mathfrak{q}$ lying over p.

We note that any p-stabilized newform lies in $M_k^{\circ}(\mathfrak{m}, \psi)$ including in weight one. In particular, when $k \geq 2$, a p-stabilized newform is always of the shape $f|e$ for a newform $f \in M_k(\mathfrak{m}, \psi)$ of level dividing $\mathfrak{m}$ with $c(\mathfrak{q}, f)$ a p-adic unit for each prime $\mathfrak{q}$ sitting above p (cf. pp. 537–538 of [46]).

3.3 Λ-adic forms

In this section, we introduce a p-adic family of Hilbert modular forms in $M_k(\mathfrak{m}, \psi)$ which interpolates Fourier coefficients of each member of the family as k and ψ vary. We will give a exposition of such a family basically following [46].

We fix a prime number p and define $\mathbf{p}$ to be p if $p \neq 2$ and 4 if $p = 2$. Let F_∞ be the cyclotomic $\mathbb{Z}_p$-extension of F and $\mathbf{G}$ the Galois group $\mathrm{Gal}(F_\infty/F)$ which is isomorphic to $\mathbb{Z}_p$ as a topological group. This $\mathbf{G}$ will parametrize the weights and exponents of p in the levels of a p-adic family which we will deal with. Let $\mathcal{O}$ be the p-adic integer ring of a finite extension of $\mathbb{Q}_p$ and $\Lambda = \mathcal{O}[[\mathbf{G}]] \cong \varprojlim_n \mathcal{O}[\mathbb{Z}/p^n\mathbb{Z}]$ the complete group algebra. We fix a topological generator γ of $\mathbf{G}$. Then Λ can be identified with the power series ring $\mathcal{O}[[X]]$ in one-variable by sending γ to $1 + X$. Let $F(\mu_{p^\infty})$ be the Galois extension of F obtained by adjoining all the p-power roots of unity to F. Then we have an isomorphism $\mathrm{Gal}(F(\mu_{p^\infty})/F) \cong \mathbf{G} \times \Delta$, where Δ is a subgroup of the group $\mu_{\mathbf{p}-1}$ of $(\mathbf{p}-1)$-st roots of unity in $\mathbb{Z}_p^\times$. Since $\mathbf{G}$ is isomorphic to $1 + \mathbf{p}\mathbb{Z}_p$, we may regard $\mathrm{Gal}(F(\mu_{p^\infty})/F)$ as a subgroup of $\mathbb{Z}_p^\times$, and obtain the p-adic cyclotomic character

$$\chi_p : G_F \to \mathrm{Gal}(F(\mu_{p^\infty})/F) \hookrightarrow \mathbb{Z}_p^\times$$

of G_F. Namely, we have

$$a(\zeta) = \zeta^{\chi_p(a)} \quad (a \in G_F)$$

for any p-power root of unity $\zeta \in \overline{F}$. We may and do regard χ_p as a character of $\mathbf{G}$ via the isomorphism $\mathrm{Gal}(F(\mu_{p^\infty})/F) \cong \mathbf{G} \times \Delta$. Let

$$\omega = \lim_{n\to\infty} \chi_p^{p^n} : G_F \to \mathbb{Z}_p^\times \to \mu_{p-1}$$

be the Teichmüller character. For $k \in \mathbb{Z}$ and a p-power order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$, let $\varphi_{k,\varepsilon} : \Lambda \to \overline{\mathbb{Q}}_p$ be the ring homomorphism induced by $\varepsilon\chi_p^{k-1} : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$. Let $P_{k,\varepsilon}$ denote the kernel of $\varphi_{k,\varepsilon}$. Note that the character sending $a \in \mathbf{G}$ to $\varepsilon(a)\chi_p(a)^{k-1}$ is of finite order if and only if $k = 1$.

Definition 3.3.1. Let L be a finite extension of the field of fractions of Λ and Λ_L the integral closure of Λ in L. An $\mathcal{O}$-algebra homomorphism $P : \Lambda_L \to \overline{\mathbb{Q}}_p$ is called an arithmetic point of Λ_L if the restriction $P|_\Lambda$ of P to Λ is equal to $\varphi_{k,\varepsilon}$ for some integer $k \geq 2$ and a finite order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$.

Let $\chi_\Lambda : G_F \to \Lambda^\times$ be the Λ-adic cyclotomic character obtained by composing the canonical surjection $G_F \to \mathbf{G}$ with the map $\mathbf{G} \to \Lambda^\times$ taking γ to $1+X$. Throughout the paper, we will fix a finite extension L of the field of fractions of Λ.

Definition 3.3.2. Let $\mathfrak{n}_0 \subset O_F$ be a non-zero ideal prime to p and $\psi : \mathrm{Cl}_F(\mathfrak{n}_0 p) \to \mathcal{O}^\times$ a narrow ray class character that is totally odd and tamely ramified at all prime ideals $\mathfrak{p}$ of F lying over p as a character of G_F. A Λ-adic form $\mathcal{F}$ of tame level $\mathfrak{n}_0$ and nebentype ψ is a collection $\{c_\lambda(0,\mathcal{F})\}_{\lambda \in \mathrm{Cl}_F^+} \cup \{c(\mathfrak{n},\mathcal{F})\}_\mathfrak{n}$ of elements in Λ_L indexed by Cl_F^+ and the set of non-zero integral ideals $\mathfrak{n}$ in O_F, such that for all but finitely many arithmetic points P of Λ_L, if $P|_\Lambda = \varphi_{k,\varepsilon}$, then there exists a Hilbert modular form $f_P \in M_k(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$ for the order p^r of ε with the property that

$$P(c_\lambda(0,\mathcal{F})) = c_\lambda(0, f_P);$$
$$P(c(\mathfrak{n},\mathcal{F})) = c(\mathfrak{n}, f_P)$$

for all λ and $\mathfrak{n}$. This form f_P is often denoted by $\varphi_{k,\varepsilon}(\mathcal{F})$ as well (by slight abuse of notation), and we call f_P the specialization of $\mathcal{F}$ at P.

The set of Λ-adic forms of tame level $\mathfrak{n}_0$ and nebentype ψ is denoted by $\mathbf{M}(\mathfrak{n}_0,\psi)$. One can also define the notion of Λ-adic cusp forms by asserting each f_P to lie in $S_k(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$ in place of $M_k(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$. The resulting set will be denoted by $\mathbf{S}(\mathfrak{n}_0,\psi)$. In contrast to Λ-adic forms, elements in $M_k(\mathfrak{m},\psi;\overline{\mathbb{Q}}_p)$ will often be referred to as *classical* forms.

The sets $\mathbf{M}(\mathfrak{n}_0,\psi)$ and $\mathbf{S}(\mathfrak{n}_0,\psi)$ admit the structure of Λ_L-modules by multiplying an element of Λ_L simultaneously to the defining data $\{c_\lambda(0,\mathcal{F})\}_{\lambda\in\mathrm{Cl}_F^+} \cup \{c(\mathfrak{n},\mathcal{F})\}_\mathfrak{n}$. Furthermore, these spaces are equipped with Hecke operators $T(\mathfrak{l})$, $U(\mathfrak{q})$ and $S(\mathfrak{l})$ for each prime ideal $\mathfrak{l} \nmid \mathfrak{n}_0 p$ and $\mathfrak{q} \mid \mathfrak{n}_0 p$ of F that are compatible with specializations, as we now explain. For each prime ideal $\mathfrak{l}$ prime to $\mathfrak{n}_0 p$, χ_Λ and ω are unramified at $\mathfrak{l}$. Let $\mathrm{Frob}_\mathfrak{l}$ be an arithmetic Frobenius at $\mathfrak{l}$. For a Λ-adic form $\mathcal{F} \in \mathbf{M}(\mathfrak{n}_0,\psi)$, let us define

$$\mathcal{F}|S(\mathfrak{l}) = \chi_\Lambda(\mathrm{Frob}_\mathfrak{l})\psi(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{-1}\mathcal{F}.$$

For each prime ideal $\mathfrak{l} \mid \mathfrak{n}_0 p$, we put $S(\mathfrak{l}) = 0$. Then for every classical specialization $\varphi_{k,\varepsilon}(\mathcal{F})$, we have

$$\varphi_{k,\varepsilon}(\mathcal{F}|S(\mathfrak{l})) = \mathrm{N}(\mathfrak{l})^{k-2}\varphi_{k,\varepsilon}(\mathcal{F})|S_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{l}).$$

Secondly, we define $T(\mathfrak{m})$ for a non-zero ideal $\mathfrak{m}$ in O_F. For each $\lambda \in \mathrm{Cl}_F^+$, we put

$$c_\lambda(0,\mathcal{F}|T(\mathfrak{m})) = \sum_{\mathfrak{c}|\mathfrak{m}} \mathrm{N}(\mathfrak{c}) c_{\lambda\rho^2\mu^{-1}}(0,\mathcal{F}|S(\mathfrak{c}))$$

where $\rho \in \mathrm{Cl}_F^+$ (resp. $\mu \in \mathrm{Cl}_F^+$) is the narrow ideal class of $\mathfrak{c}$ (resp. $\mathfrak{m}$) and we understand that $S(O_F)$ is the identity map. The non-constant term $c(\mathfrak{n}, \mathcal{F}|T(\mathfrak{m}))$ for each non-zero ideal $\mathfrak{n}$ in O_F is defined to be

$$c(\mathfrak{n}, \mathcal{F}|T(\mathfrak{m})) = \sum_{\mathfrak{n}+\mathfrak{m}\subset\mathfrak{a}} \mathrm{N}(\mathfrak{a})c(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathcal{F}|S(\mathfrak{a})).$$

Then it is a standard matter to check that for every classical specialization $\varphi_{k,\varepsilon}(\mathcal{F})$,

$$\varphi_{k,\varepsilon}(\mathcal{F}|T(\mathfrak{m})) = \varphi_{k,\varepsilon}(\mathcal{F})|T'_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{m}).$$

Therefore $S(\mathfrak{l})$ and $T(\mathfrak{m})$ act on the space $\mathbf{M}(\mathfrak{n}_0, \psi)$ of Λ-adic forms, and the subspace $\mathbf{S}(\mathfrak{n}_0, \psi)$ is preserved by these operators as the subspace of classical cusp forms is stable under the corresponding Hecke operators.

3.4 The space of ordinary Λ-adic forms

Definition 3.4.1. A Λ-adic form $\mathcal{F} \in \mathbf{M}(\mathfrak{n}_0, \psi)$ is said to be ordinary if f_P is ordinary in the sense of Definition 3.2.1 for all but finitely many arithmetic points P of Λ_L.

The Λ_L-submodule of $\mathbf{M}(\mathfrak{n}_0, \psi)$ consisting of ordinary Λ-adic forms of tame level $\mathfrak{n}_0$ and character ψ is denoted by $\mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$. Likewise $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ for the space of ordinary Λ-adic cusp forms. An ordinary Λ-adic form is often referred to as a Hida family. The following proposition is due to Hida when $F = \mathbb{Q}$ (Proposition 7.3.1; p. 212 of [15]) and Wiles for general F (Proposition 1.2.1; p. 538 of [46]).

Proposition 3.4.2. *There exists an endomorphism of Λ_L-modules e on $\mathbf{M}(\mathfrak{n}_0, \psi)$ such that $e^2 = e$ and*

$$\varphi_{k,\varepsilon}(\mathcal{F}|e) = \varphi_{k,\varepsilon}(\mathcal{F})|e$$

for any classical specialization $\varphi_{k,\varepsilon}(\mathcal{F})$ of $\mathcal{F}$. Here, e on the right-hand side is the ordinary idempotent which was defined in Section 3.2. *Furthermore, for the Hecke operator $U_p = T(p)$ acting on $\mathbf{M}(\mathfrak{n}_0, \psi)$, we have*

$$\mathcal{F}|e = \lim_{n\to\infty} \mathcal{F}|U_p^{n!}$$

with respect to the topology induced by the maximal ideal of Λ_L.

We call this e the ordinary idempotent as in the classical case. This proposition implies that $\mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi) = \mathbf{M}(\mathfrak{n}_0, \psi)|e$ and likewise for the space of Λ-adic cusp forms. Indeed, an inclusion $\mathbf{M}(\mathfrak{n}_0, \psi)|e \subset \mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ follows by Proposition 3.4.2 which asserts that e is defined in a way compatible with specializations. In order to verify the other inclusion, we take $\mathcal{F} \in \mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$. There exist infinitely many arithmetic points P such that f_P is ordinary, namely, $f_P = f_P|e$. Then we see that $\varphi_{k,\varepsilon}(\mathcal{F}|e) = \varphi_{k,\varepsilon}(\mathcal{F})|e = f_P|e = f_P = \varphi_{k,\varepsilon}(\mathcal{F})$. Since Λ_L is of Krull dimension two and the intersection of $\ker(P)$ for infinitely many such P is zero, we have $\mathcal{F}|e = \mathcal{F}$, that is, $\mathcal{F} \in \mathbf{M}(\mathfrak{n}_0, \psi)|e$. One of the main results concerning the space of ordinary Λ-adic forms is the following

Theorem 3.4.3 (Theorem 1.2.2; p.539 of [46])). $\mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ *and* $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ *are finitely generated* Λ_L*-modules.*

3.5 The control theorem

The key observation in the proof of Theorem 3.4.3 in the previous section is that the rank of $M_k^{\mathrm{o}}(\mathfrak{n}_0 p, \psi\omega^{1-k}; \mathcal{O})$ as an $\mathcal{O}$-module is bounded *independently* of the weight $k \geq 2$ (Lemma; p. 539 in [46]). The control theorem, which we now state, asserts that we can actually say more: for each $k \geq 2$, the space of classical ordinary forms of weight k is realized exactly as the image of the specialization of $\mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ at the corresponding arithmetic point.

Theorem 3.5.1 (Theorem 3.4; p. 316 of [13]). *For each arithmetic point P of Λ_L with $P|_\Lambda = \varphi_{k,\varepsilon}$, the specialization at P induces isomorphisms of $\Lambda_L/\ker(P)$-modules:*

$$\mathbf{M}^{\mathrm{o}}(\mathfrak{n}_0, \psi) \otimes_{\Lambda_L} \Lambda_L/\ker(P) \cong M_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; P(\Lambda_L));$$
$$\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi) \otimes_{\Lambda_L} \Lambda_L/\ker(P) \cong S_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; P(\Lambda_L)).$$

Remark 3.5.2. In a sequence of his papers including [13] in the late 1980s, Hida extensively investigated the properties of the ordinary part of the universal Hecke algebra over the Iwasawa algebra, and the original statement of Theorem 3.5.1 is given in terms of Hecke algebras. It was not Hida but Wiles who first invented the notion of Λ-adic forms, namely, a p-adic interpolation of Fourier coefficients of classical Hilbert modular forms (of parallel weight) in [46]. Since there is the duality between the aforementioned Hecke algebra and the space of ordinary Λ-adic cusp forms (cf. Theorem 5.6; p. 336 of [13]), the original Theorem 3.4 in [13] can be translated into the statement as in Theorem 3.5.1.

We now explain an important implication of Theorem 3.5.1.

Definition 3.5.3. An ordinary Λ-adic cusp form $\mathcal{F} \in \mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ is said to be a primitive ordinary Λ-adic cusp form of (tame) level $\mathfrak{n}_0$ if it is normalized (i.e., $c(O_F, \mathcal{F}) = 1$) eigenform for all the Hecke operators $T(\mathfrak{l})$, $U(\mathfrak{q})$ and $S(\mathfrak{l})$ for prime ideals $\mathfrak{l} \nmid \mathfrak{n}_0 p$ and $\mathfrak{q} \mid \mathfrak{n}_0 p$, and for almost all arithmetic points P of Λ_L the specialization f_P is an ordinary, p-stabilized newform of level divisible by $\mathfrak{n}_0$.

More detailed exposition on the notion of primitive ordinary Λ-adic cusp forms can be found in p. 552 of [46]. As seen in Corollary 3.5 (p. 316) and Theorem 5.6 (p. 336) of [13], Theorem 3.5.1 has the following consequence:

Corollary 3.5.4. *For a p-stabilized newform f in $S_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \overline{\mathbb{Q}}_p)$ of weight $k \geq 2$, there exists a unique primitive ordinary Λ-adic cusp form $\mathcal{F}$ of tame level $\mathfrak{n}_0$ so that $f = f_P$ for some arithmetic point P of Λ_L.*

3.6 Weight one specializations of a Λ-adic form

In the previous section, we have observed that the spaces of classical ordinary Hilbert modular forms of weight k and level $\mathfrak{n}_0 p^{r+1}$ are very nicely controlled as weights $k \geq 2$ and exponents $r \geq 0$ vary. However, in weight one this may fail. More precisely, a weight one specialization (by this we mean a specialization at an $\mathcal{O}$-algebra homomorphism $P : \Lambda_L \to \overline{\mathbb{Q}}_p$ that restricts to $\varphi_{1,\varepsilon}$ on Λ for some finite order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^{\times}$) of an ordinary Λ-adic form $\mathcal{F}$ is in general an overconvergent Hilbert modular form of weight one, and may not be classical.

Remark 3.6.1. Relating to Corollary 3.5.4, Wiles showed that for any p-stabilized newform f in $S_k^{\circ}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \overline{\mathbb{Q}}_p)$ of weight $k \geq 1$, there exists a primitive ordinary Λ-adic cusp form $\mathcal{F}$ of tame level $\mathfrak{n}_0$ specializing to f (Theorem 3; p. 532 of [46]). However, by his method which is slightly different from that of Hida used in [13], the uniqueness of $\mathcal{F}$ is not guaranteed.

The main purpose of this thesis is to know about classical weight one specializations of an ordinary Λ-adic cusp form. We will deal with this topic in Chapters 5 through 7.

Chapter 4

Galois representations attached to classical Hilbert modular forms

In this chapter, we give a brief account of the (systems of) Galois representations attached to classical modular forms. It will be revealed that the Galois representation attached to a classical cusp form of weight one possesses quite different properties from that in the case of weight at least two. Therefore we will distinguish the descriptions for each case.

Throughout this chapter, any Hilbert cusp form is assumed to be a normalized Hecke eigenform for the Hecke operators $T'(\mathfrak{l})$ and $S(\mathfrak{l})$ for all prime ideals $\mathfrak{l}$ of F. Let $\mathfrak{m}$ be a non-zero integral ideal in F, $\psi : \mathrm{Cl}_F(\mathfrak{m}) \to \mathbb{C}^\times$ a narrow ray class character modulo $\mathfrak{m}$ and $f \in S_k(\mathfrak{m}, \psi)$ a Hilbert cusp form. Since f is a normalized Hecke eigenform, one has $c(O_F, f) = 1$ and thus $f|T'(\mathfrak{l}) = c(\mathfrak{l}, f)f$ for each prime ideal $\mathfrak{l}$ (cf. Remark 2.4.6). The field $\mathbb{Q}(\{c(\mathfrak{n}, f)\}_{\mathfrak{n}})$ obtained by adjoining all the Hecke eigenvalues of f is known to be an algebraic number field, and the coefficients $c(\mathfrak{n}, f)$ are algebraic *integers* (Proposition 2.2; p. 650 of [40]). This field is called the Hecke field of f and denoted by $\mathbb{Q}(f)$. Shimura also showed that $\mathbb{Q}(f)$ is either totally real or a totally imaginary quadratic extension of a totally real field (Proposition 2.8; p. 654 of [40]).

4.1 The case of parallel weight at least two

We will first deal with the case where the weight k is at least two. We introduce a theorem that associates to f the system of Galois representations $\{\rho_{f,v}\}_{v \in P_{\mathrm{f}}(\mathbb{Q}(f))}$ parametrized by the set $P_{\mathrm{f}}(\mathbb{Q}(f))$ of finite places of the Hecke field $\mathbb{Q}(f)$. Let $O(f)$ be the ring of integers of $\mathbb{Q}(f)$, $O(f)_v$ the completion of $O(f)$ at $v \in P_{\mathrm{f}}(\mathbb{Q}(f))$ and p the residue characteristic of v. In the case where $F = \mathbb{Q}$, the following theorem is due to Eichler-Shimura when $k = 2$ ([41] Chapter 7) and to Deligne when $k \geq 2$ ([3]).

Theorem 4.1.1 (Rogawski-Tunnell [33], Ohta [25], Wiles [46] and Taylor [45])**.** *Let $f \in S_k(\mathfrak{m}, \psi)$ be a normalized eigenform of weight $k \geq 2$. For each finite place*

$v \in P_{\mathrm{f}}(\mathbb{Q}(f))$, there exists a continuous irreducible Galois representation

$$\rho_{f,v} : G_F \to GL_2(O(f)_v)$$

which is unramified outside $\mathfrak{m}p$ and

$$\mathrm{Tr}(\rho_{f,v})(\mathrm{Frob}_{\mathfrak{l}}) = c(\mathfrak{l}, f), \ \ \det(\rho_{f,v})(\mathrm{Frob}_{\mathfrak{l}}) = \psi(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{k-1} \tag{4.4.1}$$

for every prime ideal $\mathfrak{l} \nmid \mathfrak{m}p$. Here $\mathrm{Frob}_{\mathfrak{l}}$ is an arithmetic Frobenius at $\mathfrak{l}$.

The system $\{\rho_{f,v}\}_{v\in P_{\mathrm{f}}(\mathbb{Q}(f))}$ is compatible in the sense that the trace of the representations $\rho_{f,v}$ are independent of the finite places v. Furthermore, Chebotarëv density theorem implies that $\det(\rho_{f,v}) = \psi\chi_p^{k-1}$, where $\chi_p : G_F \to \mathrm{Gal}(F(\mu_{p^\infty})/F) \cong \mathbb{Z}_p^\times$ is the p-adic cyclotomic character of G_F.

Remark 4.1.2. The way of constructing Galois representations as in Theorem 4.1.1 is quite geometric. In the case of elliptic cusp forms with weight two, Eichler and Shimura constructed an abelian variety A_f of dimension $[\mathbb{Q}(f) : \mathbb{Q}]$ out of f and the v-adic Galois representation $\rho_{f,v} : G_F \to GL_2(O(f)_v)$ in Theorem 4.1.1 is obtained as the extension of scalar to $\mathbb{Q}(f)_v$ of the p-adic Tate module attached to A_f over $\mathbb{Q}_p$. In the higher weight case, Deligne considered $(k-2)$-times self fiber product $\mathcal{E}^{k-2}$ of the generalized universal elliptic curve $\mathcal{E}$ and showed that the v-adic Galois representation in question appears as a subquotient of the p-adic étale cohomology group of the desingularization of $\mathcal{E}^{k-2}$ (so-called Kuga-Sato variety).

Contrary to Deligne's method, in his unpublished paper [42] Shimura showed congruences modulo a power of p between cusp forms of weights $k \geq 2$ and those of weight two, and reduced the construction of Galois representations in Theorem 4.1.1 to the case of weight two. However, Shimura's method does not prove Ramanujan's conjecture which is a consequence of Deligne's geometric construction of $\rho_{f,v}$ combined with Weil conjecture (more precisely, Riemann hypothesis).

Here is a historical remark concerning Theorem 4.1.1 in the case where F is a general totally real field.

Remark 4.1.3. In the case of Hilbert modular forms, Rogawski-Tunnell and Ohta proved Theorem 4.1.1 when either

(i) the degree $[F : \mathbb{Q}]$ is odd, or

(ii) for some finite place $v \in P_{\mathrm{f}}(F)$ of F, the local component π_v at v of the automorphic representation $\pi_f = \otimes'_{v\in P_{\mathrm{f}}(F)}\pi_v$ of $GL_2(\mathbb{A}_{F,\mathrm{f}})$ corresponding to f is special or supercuspidal.

Such restrictions are forced because their method of proof rely on Jacquet-Langlands correspondence ([18] Chapter III and [39]) which relates Hilbert modular forms defined over F with automorphic forms on certain quaternion algebra over F. Wiles constructed missing v-adic representations when $[F : \mathbb{Q}]$ is even under the assumption that f is ordinary at v in the sense of Definition 4.1.4 below. Finally, Taylor removed the constraint of f being ordinary at v in the case where $[F : \mathbb{Q}]$ is even.

For later use, we also mention a known result on the local behavior of $\rho_{f,v}$ at the prime ideals dividing the residue characteristic of v.

Definition 4.1.4. We say that a form f of weight $k \geq 1$ is ordinary at v if for each prime $\mathfrak{q} \mid p$ the polynomial

$$x^2 - c(\mathfrak{q}, f)x + \psi(\mathfrak{q})\mathrm{N}(\mathfrak{q})^{k-1} \tag{4.4.2}$$

has at least one root which is a unit in $O(f)_v$.

Theorem 4.1.5 (Theorem 2.1.4; p. 561 of [46])**.** *Let f be a newform of weight $k \geq 2$, level $\mathfrak{m}$, character ψ and $v \in P_{\mathrm{f}}(\mathbb{Q}(f))$ with residue characteristic p. Suppose that f is ordinary at v. Then the v-adic Galois representation $\rho_{f,v} : G_F \to GL_2(O(f)_v)$ as in Theorem* 4.1.1 *restricted to the decomposition group $D_\mathfrak{q}$ at $\mathfrak{q}$ is, up to equivalence, of the shape*

$$\rho_{f,v}|_{D_\mathfrak{q}} \cong \begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{bmatrix}$$

where $\varepsilon_2 : D_\mathfrak{q} \to O(f)_v^\times$ is an unramified character and $\varepsilon_2(\mathrm{Frob}_\mathfrak{q}) = \alpha(\mathfrak{q}, f)$. Here $\alpha(\mathfrak{q}, f)$ is a v-adic unit root of (4.4.2)*.*

Remark 4.1.6. We note that in fact Theorem 4.1.5 is valid for $k = 1$ as well. When the weight k is at least two, the polynomial (4.4.2) has at least one v-adic unit root if and only if the Fourier coefficient $c(\mathfrak{q}, f)$ is a v-adic unit. In this case exactly one root of (4.4.2) is a v-adic unit and the other is not, as $\mathrm{N}(\mathfrak{q})^{k-1}$ is divisible by v. Hence there is exactly one choice for $\alpha(\mathfrak{q}, f)$ in Theorem 4.1.5.

4.2 The case of parallel weight one

In the case of weight one, we will see that one can also associate a system of Galois representation $\{\rho_{f,v}\}_{v \in P_{\mathrm{f}}(\mathbb{Q}(f))}$ to f as in the previous section. However, in this case we can further say that for each $v \in P_{\mathrm{f}}(\mathbb{Q}(f))$ the image of $\rho_{f,v}$ is finite, and $\rho_{f,v}$ lifts to the same complex representation for any v. More precisely, we have the following theorem:

Theorem 4.2.1 (Deligne-Serre [4], Rogawski-Tunnell [33], Ohta [26] and Wiles [46])**.** *Let $f \in S_1(\mathfrak{m}, \psi)$ be a normalized eigenform for all the Hecke operators. There exists a totally odd irreducible continuous representation $\rho_f : G_F \to GL_2(\mathbb{C})$ that is unramified outside $\mathfrak{m}$ and*

$$\mathrm{Tr}(\rho_f)(\mathrm{Frob}_\mathfrak{l}) = c(\mathfrak{l}, f), \;\; \det(\rho_f)(\mathrm{Frob}_\mathfrak{l}) = \psi(\mathfrak{l}) \tag{4.4.3}$$

for each prime ideal $\mathfrak{l} \nmid \mathfrak{m}$.

Remark 4.2.2. Note that the image of ρ_f is finite since ρ_f is continuous. In general, a continuous representation $\rho : G_F \to GL_n(\mathbb{C})$ is called an Artin representation.

In the case where $F = \mathbb{Q}$, this theorem is due to Deligne-Serre. Later Rogawski-Tunnell and Ohta generalized the result of Deligne-Serre to the case of Hilbert modular forms under the assumptions (i) and (ii) explained in Remark 4.1.3. In contrast to the higher weight case, Wiles showed the existence of $\rho_{f,v}$ without the assumption that f is ordinary at v (Theorem 2.4.1; p. 571 of [46]).

Roughly speaking, the construction of ρ_f consists of two steps. To explain this first suppose that $F = \mathbb{Q}$. For each $v \in P_{\mathrm{f}}(\mathbb{Q}(f))$, let $\mathbb{F}_v$ denote the residue field at v. The first step is to construct $\bar{\rho}_{f,v} : G_F \to GL_2(\mathbb{F}_v)$ with the property that

$$\mathrm{Tr}(\bar{\rho}_{f,v})(\mathrm{Frob}_{\mathfrak{l}}) = c(\mathfrak{l}, f) \bmod \mathfrak{p}_v, \ \ \det(\bar{\rho}_{f,v})(\mathrm{Frob}_{\mathfrak{l}}) = \psi(\mathfrak{l}) \bmod \mathfrak{p}_v \tag{4.4.4}$$

for each prime ideal $\mathfrak{l} \nmid \mathfrak{m}p$. This can be done by using the Eisenstein series

$$E_m = 1 - b_m^{-1} \cdot 2m \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^{m-1} \right) q^n$$

of weight m with respect to $SL_2(\mathbb{Z})$, where $m \geq 4$ is an even integer and b_m is the m-th Bernoulli number. By von Staudt-Clausen's theorem, E_m has p-integral coefficients and is congruent to 1 modulo p when $m \equiv 0 \bmod (p-1)$. Therefore $f \cdot E_m$ is of weight $m+1 \geq 2$ and $f \cdot E_m \equiv f \bmod \mathfrak{p}_v$. We use Deligne-Serre lemma (Lemme 6.11; p. 522 of [4]) to find a normalized eigenform $g \in S_{m+1}(\mathfrak{m}, \psi)$ with p-integral coefficients such that $g \equiv f \cdot E_m \bmod \mathfrak{p}'$ for some prime ideal $\mathfrak{p}'$ dividing $\mathfrak{p}_v$. Then we obtain the $\mathfrak{p}'$-adic Galois representation $\rho_{g,\mathfrak{p}'}$ by applying Theorem 4.1.1 to g, and the residual representation $\bar{\rho}_{g,\mathfrak{p}'} = \rho_{g,\mathfrak{p}'} \bmod \mathfrak{p}'$ satisfies (4.4.4). It turns out that $\bar{\rho}_{g,\mathfrak{p}'}$ is realizable as a representation over $\mathbb{F}_v$. Hence $\bar{\rho}_{g,\mathfrak{p}'}$ is the desired representation $\bar{\rho}_{f,v}$.

The second step is to show that the image of $\bar{\rho}_{f,v}$ is bounded independently of v. If one could show this, then the order of $\mathrm{Im}(\bar{\rho}_{f,v})$ would be prime to p for sufficiently large p and one could find an integral representation ρ of G_F, defined over a finite extension of the Hecke field $\mathbb{Q}(f)$ of f which reduces to $\bar{\rho}_{f,v}$ for infinitely many v. This step strongly relies on a result of Rankin which asserts that for a system $\{c(l, f)\}_{l \nmid \mathfrak{m}}$ of Hecke eigenvalues of the Hecke operators in $S_k(\mathfrak{m}, \psi)$, one has

$$\sum_{l|\mathfrak{m}} |c(l, f)|^2 l^{-s} \leq \log(1/(s-k)) + O(1)$$

as $s \to k$ (see Section 5 of [4] for details).

Now we deal with a general totally real field F. We first need to replace the Eisenstein series E_m by "Hasse invariant" studied by Katz in [20] (this is the replacement adopted by Rogawski-Tunnell and Ohta) or the form θ whose existence was proved by Wiles (Lemma 1.4.2; p. 547 of [46]). The application of Rankin's method in the second step is justified by a theorem of Jacquet-Shalika [19] (and also Proposition 4.13; p. 669 of [40]).

4.3 Projective image of Galois representations attached to modular forms

Throughout this section, we fix an odd prime p and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We also fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and put $\iota_\infty = \iota \circ \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Let $f \in S_k(\mathfrak{m}, \psi)$ be a normalized eigenform for all the Hecke operators and $\mathbb{Q}(f)$ the Hecke field of f. We regard $\mathbb{Q}(f)$ as a subfield of $\overline{\mathbb{Q}}_p$ via the embedding ι_p, and let $\mathfrak{p}$ be the prime ideal of $\mathbb{Q}(f)$ lying over p determined by this ι_p. When $k \geq 2$, let $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O(f)_\mathfrak{p})$ be the $\mathfrak{p}$-adic representation attached to f as in Theorem 4.1.1. When the weight k is one, according to the construction of the Artin representation $\rho_f : G_F \to GL_2(\mathbb{C})$ attached to f in Theorem 4.2.1, we see that the mod-$\mathfrak{p}$ representation $\bar{\rho}_{f,\mathfrak{p}} : G_F \to GL_2(\mathbb{F}_\mathfrak{p})$ in the previous section lifts to a characteristic zero representation $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O')$, where O' is the ring of integers of some finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_\mathfrak{p}$. We have $\rho_f = \iota \circ \rho_{f,\mathfrak{p}}$. From now on, we write $O = O(f)_\mathfrak{p}$ if $k \geq 2$ and $O = O'$ if $k = 1$, $\mathbb{F}_\mathfrak{p}$ for the residue field of O, and $\bar{\rho}_{f,\mathfrak{p}}$ for $\rho_{f,\mathfrak{p}}$ mod $\mathfrak{p}$.

The main purpose of this section is to classify the image of the mod-$\mathfrak{p}$ representation $\bar{\rho}_{f,\mathfrak{p}} : G_F \to GL_2(\mathbb{F}_\mathfrak{p})$ in $PGL_2(\mathbb{F}_\mathfrak{p})$. When the weight $k = 1$, we will further specify the image of $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O)$ in $PGL_2(O)$ according to that of $\bar{\rho}_{f,\mathfrak{p}}$. To carry out this, we first recall Dickson's classification of subgroups of $PGL_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field of characteristic p and p is an odd prime.

Theorem 4.3.1 (Dickson)**.** *Let p be an odd prime and $\mathbb{F}$ a finite field of characteristic p. Let G be a subgroup of $GL_2(\mathbb{F})$ and H the image of G in $PGL_2(\mathbb{F})$. Then either of the following holds:*

- *if the order of G is divisible by p, then H is either*
  - (1) *contained in a Borel subgroup of $PGL_2(\mathbb{F}')$ where $\mathbb{F}'/\mathbb{F}$ is quadratic, or*
  - (2) *conjugate to $PSL_2(\mathbb{F}')$ or $PGL_2(\mathbb{F}')$ for a subfield $\mathbb{F}'$ of $\mathbb{F}$.*
- *if the order of G is prime to p, then H is either*
  - (3) *a cyclic group of $PGL_2(\mathbb{F})$ and G is contained in a Cartan subgroup of $GL_2(\mathbb{F})$, or*
  - (4) *a dihedral group D_{2m} of order $2m$ where $m \geq 2$ is an integer prime to p, and G is contained in the normalizer of a Cartan subgroup C of $GL_2(\mathbb{F})$ but not contained in C itself, or*
  - (5) *isomorphic to one of S_4, A_4 or A_5 where S_n is the symmetric group of order $n!$ and A_n is the alternative group in S_n.*

For a modern and clear proof of this theorem, we refer the reader to Section 3.6 of [43]. One can also find a detailed proof in Lemma 2 of [44] (p. 12) in the case where $\mathbb{F} = \mathbb{F}_p$. We take G to be the image of $\bar{\rho}_{f,\mathfrak{p}}$ in Theorem 4.3.1. Then $\bar{\rho}_{f,\mathfrak{p}}$ is irreducible if and only if one of (2), (4) or (5) happens.

Definition 4.3.2. Let $f \in S_k(\mathfrak{m}, \psi)$ be a normalized Hecke eigenform of weight $k \geq 2$. Then $\mathfrak{p}$ is said to be an exceptional prime for f if (2) in Theorem 4.3.1 does not occur for $G = \mathrm{Im}(\bar{\rho}_{f,\mathfrak{p}})$.

The following theorem, which was shown by Serre–Swinnerton-Dyer, Ribet and Momose in the case of $F = \mathbb{Q}$ and by Dimitrov for general F, tells us why prime ideals as in Definition 4.3.2 are said to be "exceptional".

Theorem 4.3.3 (Serre–Swinnerton-Dyer [44], Ribet [29], [31], [32], Momose [24], Dimitrov [5])**.** *Let $f \in S_k(\mathfrak{m}, \psi)$ be a non-CM normalized Hecke eigenform of weight $k \geq 2$. There exist only finitely many prime ideals of the Hecke field $\mathbb{Q}(f)$ which are exceptional for f. In particular, for all but finitely many prime ideals $\mathfrak{p}$ of $\mathbb{Q}(f)$, the image of $\bar{\rho}_{f,\mathfrak{p}}$ contains $SL_2(\mathbb{F}_\mathfrak{p})$.*

In contrast to higher weight case, it will be revealed that for a weight one form $f \in S_1(\mathfrak{m}, \psi)$, the image of $\bar{\rho}_{f,\mathfrak{p}}$ satisfies either (4) or (5) in Theorem 4.3.1. To observe this, we recall possible finite subgroups of $PGL_2(\mathbb{C})$.

Proposition 4.3.4 (cf. Section 3.3 of [37] or Section 2.5 of [35])**.** *A finite subgroup of $PGL_2(\mathbb{C})$ is isomorphic to either of the following:*

(i) *a cyclic group C_n of order n, where $n \geq 1$ is an integer, or*

(ii) *a dihedral group D_{2n} of order $2n$, where $n \geq 2$ is an integer, or*

(iii) *one of S_4, A_4 or A_5.*

Since the Artin representation $\rho_f : G_F \to GL_2(\mathbb{C})$ attached to a weight one form f is irreducible, the image of ρ_f in $PGL_2(\mathbb{C})$ satisfies either (ii) or (iii) in Proposition 4.3.4. We now prove the following:

Proposition 4.3.5. *Let $f \in S_1(\mathfrak{m}, \psi)$ be a normalized Hecke eigenform of weight one and ρ_f the Artin representation attached to f.*

- *If the image of ρ_f in $PGL_2(\mathbb{C})$ is isomorphic to a dihedral group D_{2n}, then the image of $\bar{\rho}_{f,\mathfrak{p}}$ in $PGL_2(\mathbb{F}_\mathfrak{p})$ is isomorphic to the dihedral group D_{2m}, where m is the prime-to-p part of n.*
- *If the image of ρ_f in $PGL_2(\mathbb{C})$ is isomorphic to S_4* (resp. A_4, A_5)*, then the image of $\bar{\rho}_{f,\mathfrak{p}}$ in $PGL_2(\mathbb{F}_\mathfrak{p})$ is isomorphic to S_4* (resp. A_4, A_5).

Proof. This proposition can be shown in the same way as Lemma 4.2; p. 674 of [6]. Let $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O)$ be the aforementioned $\mathfrak{p}$-adic representation which reduces to $\bar{\rho}_{f,\mathfrak{p}}$ modulo $\mathfrak{p}$ and satisfies $\rho_f = \iota \circ \rho_{f,\mathfrak{p}}$. We note that the kernel of the reduction map $PGL_2(O) \to PGL_2(\mathbb{F}_\mathfrak{p})$ is a pro-p group. Therefore the image of $\rho_{f,\mathfrak{p}}$ in $PGL_2(O)$ injects into $PGL_2(\mathbb{F}_\mathfrak{p})$ if it does not contain a non-trivial normal subgroup of p-power order. Since neither S_4, A_4 nor A_5 contains such a normal subgroup, the image of $\rho_{f,\mathfrak{p}}$ in $PGL_2(O)$ is isomorphic to the image of $\bar{\rho}_{f,\mathfrak{p}}$ in $PGL_2(\mathbb{F}_\mathfrak{p})$ in the case of (iii). On the other hand, the dihedral group D_{2n} contains a normal subgroup of order n/m, namely, a cyclic subgroup $C_{n/m} \subset C_n$. Therefore the image of D_{2n} in $PGL_2(\mathbb{F}_\mathfrak{p})$ is isomorphic to D_{2m} in the case of (ii). □

Definition 4.3.6. Let f be as in Proposition 4.3.5. We say that f is of dihedral type (resp. of exceptional type) if the image of ρ_f in $PGL_2(\mathbb{C})$ satisfies (ii) (resp. (iii)) of Proposition 4.3.4.

Chapter 5

Galois representations attached to Λ-adic forms

In this chapter, we observe that to each primitive ordinary Λ-adic cusp form $\mathcal{F}$ whose notion was introduced in Section 3.5, one can associate a Galois representation $\rho_{\mathcal{F}} : G_F \to GL_2(L)$ so that it p-adically interpolates the $\mathfrak{p}$-adic Galois representations attached to any arithmetic specialization of $\mathcal{F}$ in a well-defined sense.

We fix an odd prime number p and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ in the rest of this thesis. We also fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and put $\iota_\infty = \iota \circ \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We keep the notation introduced in Section 3.3. We fix a finite extension L of the field of fractions of Λ, a non-zero integral ideal $\mathfrak{n}_0 \subset O_F$ prime to p and a narrow ray class character $\psi : \mathrm{Cl}_F(\mathfrak{n}_0 p) \to \mathcal{O}^\times$ which is totally odd and tamely ramified at all prime ideals of F lying over p.

5.1 Construction of Λ-adic Galois representations

As we explained in Section 3.5, each arithmetic point P of Λ_L gives rise to the space of ordinary cusp forms $S_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$ where $P|_\Lambda = \varphi_{k,\varepsilon}$. The ordinary Hecke algebra $h_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$ associated to this space is defined to be the $\mathcal{O}[\varepsilon]$-subalgebra of $\mathrm{End}_{\mathcal{O}[\varepsilon]}(S_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon]))$ generated by Hecke operators $T'_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{l})$ and $S_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{l})$ for all prime ideals $\mathfrak{l}$ of F. Theorem 4.1.1 asserts that any normalized Hecke eigenform in $S_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon])$ produces the $\mathfrak{p}$-adic Galois representation from G_F to $GL_2(\mathcal{O}[\varepsilon])$. In terms of the ordinary Hecke algebra, this can be rephrased as follows: There exists a continuous Galois representation

$$\rho_{k,\mathfrak{n}_0 p^{r+1},\psi\varepsilon\omega^{1-k}} : G_F \to GL_2(h_k^{\mathrm{o}}(\mathfrak{n}_0 p^{r+1}, \psi\varepsilon\omega^{1-k}; \mathcal{O}[\varepsilon]))$$

that is unramified outside $\mathfrak{n}_0 p$ and

$$\det(1 - X\rho_{k,\mathfrak{n}_0 p^{r+1},\psi\varepsilon\omega^{1-k}}(\mathrm{Frob}_{\mathfrak{l}})) = 1 - T'_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{l})X + S_{\mathfrak{n}_0 p^{r+1}}(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{k-1}X^2$$

for each prime ideal $\mathfrak{l}$ not dividing $\mathfrak{n}_0 p$.

In a sequence of his papers beginning from [11], Hida showed that $\rho_{k,\mathfrak{n}_0 p^{r+1},\psi\varepsilon\omega^{1-k}}$'s can be p-adically interpolated. To be more precise, let $\mathbf{h}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ be the Λ_L-subalgebra of $\mathrm{End}_{\Lambda_L}(\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0,\psi))$ generated by Hecke operators $T(\mathfrak{l})$ and $S(\mathfrak{l})$ acting on $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0,\psi)$ for all prime ideals $\mathfrak{l}$ of F. We let $\mathbf{h}^{\mathrm{o,new}}(\mathfrak{n}_0,\psi)$ be the quotient of $\mathbf{h}^{\mathrm{o}}(\mathfrak{n}_0,\psi)$ by its nilradical. We have the following theorem:

Theorem 5.1.1 (Theorem II; p. 546 of [11]: see also [14]). *There exists a Galois representation*

$$\rho_{\mathfrak{n}_0,\psi} : G_F \to GL_2(\mathbf{h}^{\mathrm{o,new}}(\mathfrak{n}_0,\psi)\otimes_{\Lambda_L} L)$$

that is unramified outside $\mathfrak{n}_0 p$ *and*

$$\det(1 - X\rho_{\mathfrak{n}_0,\psi}(\mathrm{Frob}_{\mathfrak{l}})) = 1 - T(\mathfrak{l})X + S(\mathfrak{l})\mathrm{N}(\mathfrak{l})X^2$$

for each prime ideal $\mathfrak{l}$ *of* F *prime to* $\mathfrak{n}_0 p$. *Furthermore,* $\rho_{\mathfrak{n}_0,\psi}$ *is ordinary in the sense that for each prime ideal* $\mathfrak{q}$ *of* F *sitting above* p*, the restriction of* $\rho_{\mathfrak{n}_0,\psi}$ *to the decomposition group* $D_{\mathfrak{q}}$ *at* $\mathfrak{q}$ *is of the shape*

$$\rho_{\mathfrak{n}_0,\psi}|_{D_{\mathfrak{q}}} \cong \begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{bmatrix}$$

where ε_2 *is an unramified character and* $\varepsilon_2(\mathrm{Frob}_{\mathfrak{q}}) = U(\mathfrak{q})$.

This was the path followed by Hida. On the other hand, Wiles invented the notion of Λ-adic forms in [46], and constructed the Galois representation attached to a primitive ordinary Λ-adic cusp form by glueing together infinitely many pseudo-representations arising from the Galois representations attached to p-stabilized newforms. We refer the reader to Section 2.2 of [46] for details of his argument, particularly his use of the pseudo-representations.

Theorem 5.1.2 (Theorems 2.2.1 and 2.2.2; p. 562 of [46]). *Let* $\mathcal{F}$ *be a primitive ordinary* Λ*-adic cusp form of tame level* $\mathfrak{n}_0$ *in* $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0,\psi)$. *There exists an irreducible Galois representation*

$$\rho_{\mathcal{F}} : G_F \to GL_2(L)$$

that is continuous in the following sense: $\rho_{\mathcal{F}}$ *preserves a* Λ_L*-lattice* $\mathcal{L}$ *in* L^2*, and the resulting representation* $G_F \to \mathrm{End}_{\Lambda_L}(\mathcal{L})$ *is continuous with respect to the topology induced by the maximal ideal of* Λ_L. *Furthermore,*

- $\rho_{\mathcal{F}}$ *is unramified outside* $\mathfrak{n}_0 p$ *and*

$$\mathrm{Tr}(\rho_{\mathcal{F}})(\mathrm{Frob}_{\mathfrak{l}}) = c(\mathfrak{l},\mathcal{F}),\ \ \det(\rho_{\mathcal{F}})(\mathrm{Frob}_{\mathfrak{l}}) = \psi(\mathfrak{l})\chi_\Lambda(\mathrm{Frob}_{\mathfrak{l}}) \qquad (5.5.1)$$

  for each prime ideal $\mathfrak{l}$ *of* F *not dividing* $\mathfrak{n}_0 p$*, and*

- $\rho_{\mathcal{F}}$ *is ordinary, that is, for each prime ideal* $\mathfrak{q}$ *of* F *lying over* p*,* $\rho_{\mathcal{F}}$ *restricted to the decomposition group* $D_{\mathfrak{q}}$ *at* $\mathfrak{q}$ *is, up to equivalence, of the shape*

$$\rho_{\mathcal{F}}|_{D_{\mathfrak{q}}} \cong \begin{bmatrix} \mathcal{E}_{\mathfrak{q}} & * \\ 0 & \mathcal{D}_{\mathfrak{q}} \end{bmatrix}$$

  where $\mathcal{E}_{\mathfrak{q}}, \mathcal{D}_{\mathfrak{q}} : D_{\mathfrak{q}} \to \Lambda_L^\times$ *are characters with the property that* $\mathcal{D}_{\mathfrak{q}}$ *is unramified and* $\mathcal{D}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}) = c(\mathfrak{q},\mathcal{F})$.

Remark 5.1.3. We have two remarks concerning the definition of the continuity of $\rho_{\mathcal{F}}$ in Theorem 5.1.2. Firstly, the definition of continuity does not depend on the choice of a lattice $\mathcal{L}$, due to Artin-Rees theorem. Secondly, we do not know in general whether a G_F-stable lattice $\mathcal{L}$ can be taken so that $\mathcal{L} \cong \Lambda_L^2$ or not (see p. 533 and p. 562 of [46]). In spite of this fact, the notion of the reduction of $\rho_{\mathcal{F}}$ modulo each prime ideal of Λ_L makes sense, as we will see in the next section.

We mention the relation between Theorems 5.1.1 and 5.1.2. By duality theorem (Theorem 5.3; p. 336 of [13]), a primitive ordinary Λ-adic cusp form $\mathcal{F}$ of tame level $\mathfrak{n}_0$ in $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ corresponds to a Λ_L-algebra homomorphism $\lambda_{\mathcal{F}} : \mathbf{h}^{\mathrm{o,new}}(\mathfrak{n}_0, \psi) \to \Lambda_L$ such that $\lambda_{\mathcal{F}}(T(\mathfrak{l})) = c(\mathfrak{l}, \mathcal{F})$ and $\lambda_{\mathcal{F}}(S(\mathfrak{l})) = \psi(\mathfrak{l})\mathrm{N}(\mathfrak{l})^{-1}\chi_\Lambda(\mathrm{Frob}_\mathfrak{l})$ for each prime $\mathfrak{l}$ prime to $\mathfrak{n}_0 p$, and $\lambda_{\mathcal{F}}(U(\mathfrak{q})) = c(\mathfrak{q}, \mathcal{F})$ for each prime $\mathfrak{q}$ of F dividing $\mathfrak{n}_0 p$. These equalities imply that $\rho_{\mathcal{F}} = \lambda_{\mathcal{F}} \circ \rho_{\mathfrak{n}_o, \psi}$.

Remark 5.1.4. In his paper [14], Hida obtained results similar to Theorem 5.1.1 in the case of nearly ordinary (namely, ordinary but not necessarily parallel weights) Hecke algebras as well. Since we will content ourselves in the case of Hilbert modular forms with parallel weights, we do not pursue this.

5.2 Reductions and the residual representation

In this section, as we declared in Remark 5.1.3, we will associate the residual representation of $\rho_{\mathcal{F}}$ modulo P to each prime ideal P of Λ_L. Recall that the coefficient ring Λ_L is local and of Krull dimension two. Let $\mathfrak{m} = \mathfrak{m}_{\Lambda_L}$ be the maximal ideal of the local ring Λ_L and $\mathbb{F} = \Lambda_L/\mathfrak{m}$ which is a finite field of characteristic p.

Proposition 5.2.1. *For each prime ideal P of Λ_L, there exists a unique continuous semi-simple representation $\rho_{\mathcal{F}} \bmod P$ of G_F which takes values in $GL_2(Q(\Lambda_L/P))$, where $Q(\Lambda_L/P)$ is the quotient field of Λ_L/P, characterized as follows: $\rho_{\mathcal{F}} \bmod P$ is unramified outside $\mathfrak{n}_0 p$ with*

$$\det(1 - X(\rho_{\mathcal{F}} \bmod P)(\mathrm{Frob}_\mathfrak{l})) = \det(1 - X\rho_{\mathcal{F}}(\mathrm{Frob}_\mathfrak{l})) \bmod P$$

for each prime ideal $\mathfrak{l}$ of F not dividing $\mathfrak{n}_0 p$.

Proof. One can find the following argument in the proof of Corollary 7.5.1; p. 229 of [15]. When $P = \{0\}$, this is nothing but Theorem 5.1.2. Therefore we may assume $P \neq \{0\}$. We proceed our argument by induction on the height of the non-zero prime ideal P. Suppose first that P is of height one. Let $\mathcal{L}$ be a Λ_L-lattice in L^2 that is preserved by the action of G_F via $\rho_{\mathcal{F}}$. We regard $\rho_{\mathcal{F}}$ as $\rho_{\mathcal{F}} : G_F \to \mathrm{End}_{\Lambda_L}(\mathcal{L})$. The localization $\Lambda_{L,P}$ of Λ_L at P is a valuation ring. Let $\mathcal{L}_P = \mathcal{L} \otimes_{\Lambda_L} \Lambda_{L,P}$. Since $\mathcal{L}$ is a Λ_L-lattice, we have $\mathcal{L}_P \otimes_{\Lambda_{L,P}} L \cong L^2$. Thus $\mathcal{L}_P$ is free of rank two over $\Lambda_{L,P}$ and we obtain $\rho_{\mathcal{F}} : G_F \to GL_2(\Lambda_{L,P})$. Reducing $\rho_{\mathcal{F}}$ modulo P, we have a continuous representation $\rho_{\mathcal{F}} \bmod P : G_F \to GL_2(\Lambda_{L,P}/P\Lambda_{L,P}) = GL_2(Q(\Lambda_L/P))$. The semi-simplification of $\rho_{\mathcal{F}} \bmod P$ satisfies the desired properties.

Secondly, let Λ'_L be the integral closure of $\Lambda/(P \cap \Lambda)$ in $\Lambda_{L,P}/P\Lambda_{L,P}$ (here, P is an arbitrary height one prime ideal). Then Λ_L/P is contained in Λ'_L which is a valuation ring and $\Lambda_{L,P}/P\Lambda_{L,P}$ is the fraction field of Λ'_L. The inverse image of the maximal ideal of Λ'_L under the map $\Lambda_L \to \Lambda_L/P \subset \Lambda'_L$ is $\mathfrak{m} = \mathfrak{m}_{\Lambda_L}$. By inductive hypothesis on the height of P, we have a representation $\bar{\rho}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$. Since the trace of $\bar{\rho}_{\mathcal{F}}$ is characterized by (5.5.1), its semi-simplification $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$ does not depend on the choice of $\mathcal{L}$. This completes the proof. □

We note that a semi-simple representation is characterized by the values of trace and hence the residual representations in Proposition 5.2.1 is independent of the choice of a lattice $\mathcal{L}$. Furthermore, we see that if P is an arithmetic point of Λ_L, then $\rho_{\mathcal{F}}$ mod P is the $\mathfrak{p}$-adic representation attached to f_P in the sense of Section 4.3.

Definition 5.2.2. We call $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$ the residual representation of $\mathcal{F}$.

With the proof of Proposition 5.2.1 in mind, the $\mathfrak{p}$-adic Galois representation $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O)$ attached to any classical specialization of $\mathcal{F}$ gives rise to the same mod-$\mathfrak{p}$ representation, namely, the residual representation $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}} : G_F \to GL_2(\mathbb{F})$.

5.3 Classical weight one specializations and the image of the residual representation

As we have observed in Section 4.3, the Artin representation attached to a classical weight one form has several properties different from that of higher weight case. In this section, we investigate the image of the residual representation $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ when $\mathcal{F}$ admits a classical weight one specialization. Let f be a classical weight one form that is obtained as a specialization of $\mathcal{F}$ and $\rho_{f,\mathfrak{p}}$ the $\mathfrak{p}$-adic representation attached to f as in Section 4.3 so that $\rho_f = \iota \circ \rho_{f,\mathfrak{p}}$. Since the mod-$\mathfrak{p}$ representation $\bar{\rho}_{f,\mathfrak{p}}$ is irreducible and isomorphic to $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$, we deduce from Proposition 4.3.5 the following:

Proposition 5.3.1. *Let $\mathcal{F}$ and f as above. Then f is of dihedral type if and only if the image of $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is a dihedral group. Similarly, the image of ρ_f in $PGL_2(\mathbb{C})$ is isomorphic to S_4 (resp. A_4, A_5) if and only if the image of $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is isomorphic to S_4 (resp. A_4, A_5).*

In particular, the type of the classical weight one specializations in $\mathcal{F}$ is determined by $\mathcal{F}$ itself, namely, if one of the classical weight one specializations of $\mathcal{F}$ is of dihedral type, then any classical weight one specialization of $\mathcal{F}$ is of dihedral type, and likewise for S_4, A_4 and A_5 types.

Another consequence of this proposition is that if $\mathcal{F}$ admits a classical weight one specialization, then $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ is irreducible, and hence it is isomorphic to $\bar{\rho}_{\mathcal{F}}$ in the proof of Proposition 5.2.1. Therefore $\bar{\rho}_{\mathcal{F}}$ itself is unique up to isomorphism.

Definition 5.3.2. A primitive ordinary Λ-adic cusp form $\mathcal{F}$ is said to be residually of dihedral (resp. exceptional) type if the image of $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is isomorphic to a dihedral group (resp. one of S_4, A_4 or A_5).

5.4 Representations of dihedral type and induced representations

In this section, we observe that if $\mathcal{F}$ is residually of dihedral type, the residual representation $\bar{\rho}_{\mathcal{F}}$ is induced by a character $\bar{\varphi}$ of the Galois group of a quadratic extension of F. Further, the Artin representation attached to any classical weight one specialization of $\mathcal{F}$ (if exists) is induced by a character φ which is a lift of $\bar{\varphi}$. To observe this, we make use of the following two lemmas.

Lemma 5.4.1. *Let $E = \mathbb{F}$ or $\mathbb{C}$, and $\rho : G_F \to GL_2(E)$ be a continuous representation such that its image in $PGL_2(E)$ is a dihedral group D. Then there exists a quadratic extension K of F so that ρ is isomorphic to $\rho \otimes \varepsilon_{K/F}$, where $\varepsilon_{K/F} : G_F \to \{\pm 1\}$ is the quadratic character corresponding to the extension K/F, that is,* $\ker(\varepsilon_{K/F}) = G_K$.

Proof. Let C be a cyclic subgroup of index two in the dihedral group D. Let $\pi : GL_2(E) \to PGL_2(E)$ be the natural surjection. Then the inverse image $\pi^{-1}(C)$ is a Cartan subgroup in $GL_2(E)$, and the image $\mathrm{Im}(\rho) = \pi^{-1}(D)$ is the normalizer of $\pi^{-1}(C)$ in $GL_2(E)$. Therefore, by replacing E with a quadratic extension of E if necessary, the realization of ρ with respect to the basis corresponding to the Cartan subgroup $\pi^{-1}(C)$ is diagonal on $\pi^{-1}(C)$ and is anti-diagonal on $\pi^{-1}(N) \backslash \pi^{-1}(C)$. Let K be the quadratic extension of F corresponding to the kernel of the homomorphism

$$G_F \xrightarrow{\ \rho\ } \pi^{-1}(D) \xrightarrow{\ \pi\ } D \longrightarrow D/C \cong \{\pm 1\}\,.$$

Then we see that the trace of $\rho(g)$ is zero if $g \in G_F$ is not contained in G_K. This implies that the trace of ρ and $\rho \otimes \varepsilon_{K/F}$ are the same. Since both of ρ and $\rho \otimes \varepsilon_{K/F}$ are irreducible, we conclude that $\rho \cong \rho \otimes \varepsilon_{K/F}$. □

Remark 5.4.2. In Lemma 5.4.1, if $D = D_{2n}$ for an integer $n \geq 3$, then the cyclic group C is unique and hence the quadratic extension K is uniquely determined by ρ. When $D = D_4$ (Klein's group), there are three distinct cyclic subgroups C of order two in D, and hence there are exactly three possibilities for K.

We shall show that any representation as in Lemma 5.4.1 is induced by a character of $\mathrm{Gal}(\overline{F}/K)$.

Lemma 5.4.3. *Let O be either a finite field of characteristic p, the ring of integers of a finite extension of $\mathbb{Q}_p$ or a finite extension L of the field of fractions of Λ. Let $\rho : G_F \to GL_2(O)$ be a continuous irreducible representation, K a quadratic extension of F, $\varepsilon_{K/F} : G_F \to \{\pm 1\}$ the quadratic character corresponding to K/F and $G_K =$ $\mathrm{Gal}(\overline{F}/K)$. When $O = L$, we understand that ρ is continuous in the sense of Theorem* 5.1.2. *Then we have $\rho \cong \rho \otimes \varepsilon_{K/F}$ if and only if there exists a character $\varphi : G_K \to O'^{\times}$ that takes values in a finite integral extension O' of O such that ρ is isomorphic to the induced representation* $\mathrm{Ind}_K^F(\varphi)$ *over O'.*

Remark 5.4.4. Lemma 5.4.3 is a special case of Lemma 3.2 (p. 570 of [7]), but for reader's convenience, we give a proof of this lemma.

Proof. If ρ is induced by a character $\varphi : G_K \to O'^{\times}$ of K, then it follows by definition of an induced representation that $\rho(g) = \begin{bmatrix} \varphi(g) & 0 \\ 0 & \varphi(\sigma g \sigma^{-1}) \end{bmatrix}$ for $g \in G_K$ and $\rho(g) = \begin{bmatrix} 0 & \varphi(\sigma g) \\ \varphi(g\sigma^{-1}) & 0 \end{bmatrix}$ for $g \in G_F$ not contained in G_K. Here, $\sigma \in G_F$ is any element which does not lie in G_K and the values of ρ is independent of the choice of such σ. In particular, the trace of $\mathrm{Ind}_K^F(\varphi)(g)$ is zero if $g \in G_F$ is not contained in G_K. Therefore ρ and $\rho \otimes \varepsilon_{K/F}$ have the same trace. Since both representations are irreducible by our assumption, we conclude that $\rho \cong \rho \otimes \varepsilon_{K/F}$.

Conversely, we assume that $\rho \cong \rho \otimes \varepsilon_{K/F}$. By definition there exists a matrix $M \in GL_2(O)$ such that

$$M\rho(g)M^{-1} = \varepsilon_{K/F}(g)\rho(g)$$

for all $g \in G_F$. We enlarge O to O', change the basis of ρ if necessary and assume that M is upper-triangular over O' (replace M by its Jordan canonical form). Since $\varepsilon_{K/F}^2 = 1$, we have $M^2\rho(g)M^{-2} = \varepsilon_{K/F}(g)^2\rho(g) = \rho(g)$. Irreducibility of ρ implies that M^2 is a scalar matrix. Since $\varepsilon_{K/F}$ is non-trivial, M itself is not a scalar matrix, and thus the trace of M has to be zero. Therefore we have $M = \begin{bmatrix} \alpha & \beta \\ 0 & -\alpha \end{bmatrix}$. In particular, M has two distinct eigenvalues α and $-\alpha$. Again we change the basis of ρ and assume that $M = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$. Let V denote the representation space of ρ. Then M acts semi-simply on V. Let $V(\alpha)$ (resp. $V(-\alpha)$) be the α-eigenspace (resp. $(-\alpha)$-eigenspace) of M. If $v \in V(\pm\alpha)$, then $M\rho(g)v = \pm\alpha\varepsilon_{K/F}(g)\rho(g)v$. Therefore if $g \in G_K$, then $\rho(g)$ preserves each of $V(\pm\alpha)$, and if $g \notin G_K$, then $\rho(g)$ permutes $V(\alpha)$ and $V(-\alpha)$. Now let $\varphi : G_K \to O'^{\times}$ be the character defined by $\rho(g)v = \varphi(g)v$ for all $v \in V(\alpha)$. Then a direct calculation shows that ρ is induced by φ. $\square$

Now, suppose that $\mathcal{F}$ is of residually of dihedral type and f is a classical weight one specialization of $\mathcal{F}$. We apply Lemmas 5.4.1 and 5.4.3 to $\bar{\rho}_{\mathcal{F}}$ to find a quadratic extension K of F and a character $\bar{\varphi} : G_K \to \mathbb{F}^{\times}$ such that $\bar{\rho}_{\mathcal{F}} \cong \mathrm{Ind}_K^F(\bar{\varphi})$. As the $\mathfrak{p}$-adic representation $\rho_{f,\mathfrak{p}}$ attached to f reduces to $\bar{\rho}_{\mathcal{F}}$ modulo $\mathfrak{p}$, if the image of $\rho_{f,\mathfrak{p}}$ in $PGL_2(O)$ is D_{2n}, we can choose a cyclic subgroup C_n of order n inside D_{2n} so that the image of C_n in $PGL_2(\mathbb{F})$ is $\bar{\rho}_{\mathcal{F}}(G_K)$. With this choice of C_n, we see that $\rho_{f,\mathfrak{p}}$ is isomorphic to $\rho_{f,\mathfrak{p}} \otimes \varepsilon_{K/F}$ and $\rho_{f,\mathfrak{p}}$ is induced by a character $\varphi : G_K \to O^{\times}$ that lifts $\bar{\varphi}$. This observation is summarized as follows:

Lemma 5.4.5. *Let $\mathcal{F}$ be a primitive ordinary Λ-adic cusp form which is residually of dihedral type and $\bar{\rho}_{\mathcal{F}} \cong \mathrm{Ind}_K^F(\bar{\varphi})$ for some quadratic extension K of F and a character $\bar{\varphi} : G_K \to \mathbb{F}^{\times}$. Then for each classical weight one specialization f of $\mathcal{F}$, the associated $\mathfrak{p}$-adic representation $\rho_{f,\mathfrak{p}}$ in the sense of Section* 4.3 *is induced by a character $\varphi : G_K \to O^{\times}$ that lifts $\bar{\varphi}$.*

5.5 Conductor of a representation

In the next section, we will study the local behavior of induced representations. To carry out it, we introduce the notion of the (local and global) conductors associated to a representation in this section.

We begin with one-dimensional representations, namely, characters. Let K be a quadratic extension of F, O either a finite field of characteristic p, the ring of integers of a finite extension of $\mathbb{Q}_p$ or Λ_L for a finite extension L of the field of fractions of Λ and $\varphi : G_K \to O^\times$ a continuous character. In view of class field theory, it gives rise to a continuous character $\varphi : \mathbb{A}_F^\times/K^\times \to O^\times$ of the idele class group of K. For each finite place $v \in P_{\mathrm{f}}(K)$ prime to p, the restriction of φ to $O_{K_v}^\times$ is of finite order. Let $U_{K_v}(\mathfrak{p}_{K_v}^{n_v(\varphi)})$ be the kernel of this restriction, where $n_v(\varphi)$ is a non-negative integer.

Definition 5.5.1. Let K, O and φ as above and v a finite place of K not dividing p. The v-conductor $\mathrm{cond}_v(\varphi)$ of the character ϕ is the ideal $\mathfrak{p}_{K_v}^{n_v(\varphi)}$ in O_K.

Now suppose that there exists a non-zero ideal $\mathfrak{C}$ in O_K so that φ factors through $\mathbb{A}_K^\times/K^\times U_K^{(p)}(\mathfrak{C})$, where $U_K^{(p)}(\mathfrak{C}) = \prod_{v \in P_{\mathrm{f}}(K),\ v \nmid p} U_{K_v}(\mathfrak{C})$. Then φ is unramified outside $\mathfrak{C}p$, namely, for any $v \in P_{\mathrm{f}}(K)$ prime to $\mathfrak{C}p$ we have $n_v(\varphi) = 0$. Hence the product $\prod_{v \in P_{\mathrm{f}}(K), v \nmid p} \mathrm{cond}_v(\varphi)$ is a well-defined integral ideal in O_K.

Definition 5.5.2. Let K, O and φ as above and in addition suppose that φ factors through $\mathbb{A}_K^\times/K^\times U_K^{(p)}(\mathfrak{C})$ for some non-zero ideal $\mathfrak{C}$ in O_K. We call the product $\mathrm{cond}^{(p)}(\varphi) = \prod_{v \in P_{\mathrm{f}}(K), v \nmid p} \mathrm{cond}_v(\varphi)$ the prime-to-p conductor of φ.

If the character $\varphi : G_K \to O^\times$ itself is of finite order (this is the case when O is a finite field or $\mathrm{Ind}_K^F(\varphi)$ is the Artin representation attached to a classical weight one form, for instance), we can define the v-conductor of φ for each place $v \in P_{\mathrm{f}}(K)$ lying over p as well, and the product $\mathrm{cond}(\varphi) = \prod_{v \in P_{\mathrm{f}}(K)} \mathrm{cond}_v(\varphi)$ is well-defined.

Definition 5.5.3. Let K, O and φ as above and suppose that φ is of finite order. We call $\mathrm{cond}(\varphi)$ the (Artin) conductor of a finite order character φ.

Secondly we wish to do the same thing as above for two-dimensional representations. Let O be either a finite field of characteristic p, the ring of integers of a finite extension of $\mathbb{Q}_p$ or a finite extension L of the field of fractions of Λ and $\rho : G_F \to GL_2(O)$ a continuous representation with the representation space V. When $O = L$, we understand that ρ is continuous in the sense of Theorem 5.1.2. For each finite place $v \in P_{\mathrm{f}}(F)$ prime to p, let us choose a decomposition group D_{F_v} at v in G_F. This choice amounts to fixing an embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_l$ via which we identify D_{F_v} with the absolute Galois group $\mathrm{Gal}(\overline{F_v}/F_v)$, where l is the residue characteristic of v. For any real number $u \geq -1$, we let $D_{F_v}^u$ denote the u-th ramification group in the upper numbering in D_{F_v}. We refer the reader to Chapter IV Section 3 of [38] for the definition of the ramification groups, particularly to p. 74 and to Remarks in p. 75. Note that this defines a decreasing filtration in D_{F_v}, and we have $D_{F_v}^{-1} = D_{F_v}$ and $D_{F_v}^0 = I_{F_v}$ the inertia group at v.

Definition 5.5.4. Let O and $\rho : G_F \to GL_2(O)$ be as above. For each finite place $v \in P_{\mathrm{f}}(F)$ not dividing p, we define the v-conductor exponent $n_v(\rho)$ of ρ as follows:

$$n_v(\rho) = \int_{-1}^{\infty} \mathrm{codim}(V^{\rho(D_{F_v}^u)})du.$$

Here $V^{\rho(D_{F_v}^u)}$ is the subspace of V on which $D_{F_v}^u$ acts trivially via ρ.

It is proved essentially due to a theorem of Hasse-Arf (Theorem in p. 76 of [38]) that the exponent $n_v(\rho)$ is a non-negative integer. Therefore we can define the v-conductor $\mathrm{cond}_v(\rho)$ as $\mathfrak{p}_{F_v}^{n_v(\rho)}$ and it is an integral ideal in O_F. If ρ is unramified at v, then we have $V^{\rho(D_{F_v}^u)} = V$ for any $u > -1$ and thus $n_v(\rho) = 0$. One can easily see that the converse is also true. This leads us to pose the following

Definition 5.5.5. Let O and $\rho : G_F \to GL_2(O)$ as above and suppose that ρ is unramified outside a finite set of primes in $P_{\mathrm{f}}(F)$. The (Artin) prime-to-p conductor $\mathrm{cond}^{(p)}(\rho)$ of ρ is the product $\prod_{v \in P_{\mathrm{f}}(F), v \nmid p} \mathrm{cond}_v(\rho)$.

When the representation $\rho : G_F \to GL_2(O)$ has finite image, one can also define the v-conductor exponent $n_v(\rho)$ at each finite place $v \in P_{\mathrm{f}}(F)$ lying over p as in Definition 5.5.4, and it is a non-negative integer. The resulting v-conductor $\mathfrak{p}_{F_v}^{n_v(\rho)}$ will be denoted by $\mathrm{cond}_v(\rho)$ as well.

Definition 5.5.6. Let O and ρ as above that suppose that $\rho : G_F \to GL_2(O)$ has finite image. The Artin conductor $\mathrm{cond}(\rho)$ associated with ρ is defined to be the product $\prod_{v \in P_{\mathrm{f}}(F)} \mathrm{cond}_v(\rho)$.

Note that since the image of ρ is finite, we have $n_v(\rho) = 0$ for all but finitely many $v \in P_{\mathrm{f}}(F)$ and the product in Definition 5.5.6 is in fact an integral ideal in O_F. In the rest of the thesis, we will extensively investigate the conductor of the Artin representation attached to each classical weight one form.

5.6 More on induced representations

In this section, we summarize several facts on induced representations which will be used afterwards. As in Lemma 5.4.3, let O be either a finite field of characteristic p, the ring of integers of a finite extension of $\mathbb{Q}_p$ or a finite extension L of the field of fractions of Λ. We begin with a simple lemma concerning the irreducibility of an induced representation.

Lemma 5.6.1 (Mackey's irreducibility criterion)**.** *Let K be a quadratic extension of F and $\varphi : G_K \to O^\times$ a character. Then the induced representation $\mathrm{Ind}_K^F(\varphi)$ is irreducible if and only if $\varphi \neq \varphi^\sigma$. Here $\sigma \in G_F$ is an element which does not lie in G_K and $\varphi^\sigma : G_K \to O^\times$ is the character defined by $\varphi^\sigma(g) = \varphi(\sigma g \sigma^{-1})$. We note that the character φ^σ does not depend on the choice of $\sigma \in G_F \setminus G_K$.*

As indicated above, this lemma is a very special case of Mackey's irreducibility criterion (Proposition 23 in [36]). For details of Mackey's theory as well as the proof of Lemma 5.6.1, we refer the reader to Section 7.2 of [36].

Secondly we compute the determinant of an induced representation as in Lemma 5.6.1.

Definition 5.6.2. Let K be a quadratic extension of F, $G_K = \mathrm{Gal}(\overline{F}/K)$ and G_F^{ab} (resp. G_K^{ab}) the maximal abelian quotient of G_F (resp. G_K). The transfer map

$\mathrm{ver}_{K/F} : G_F^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ associated to the extension K/F is defined as follows:

$$\mathrm{ver}_{K/F}(g) = \begin{cases} g\sigma g\sigma^{-1} & \text{if } g \in G_K; \\ g^2 & \text{if } g \in G_F \setminus G_K. \end{cases}$$

It should be mentioned that the definition of the transfer map can actually be given purely group-theoretically, and is applicable to more general setting. See Chapter VI §8 of [38] for the details. Now we are ready to compute the determinant of $\rho = \mathrm{Ind}_K^F(\varphi)$. By definition one has $\det(\rho)(g) = \varphi(g\sigma g\sigma^{-1}) = (\varphi \circ \mathrm{ver}_{K/F})(g)$ for $g \in G_K$ and $\det(\rho)(g) = -\varphi(g^2) = -(\varphi \circ \mathrm{ver}_{K/F})(g)$ for $g \in G_F \setminus G_K$ (see the proof of Lemma 5.4.3). Hence we have

$$\det(\mathrm{Ind}_K^F(\varphi)) = \varepsilon_{K/F} \cdot (\varphi \circ \mathrm{ver}_{K/F}). \tag{5.5.2}$$

We now move on to a study of the local behavior of induced representations.

Lemma 5.6.3. *Let $\rho : G_F \to GL_2(O)$ be a continuous representation (in the sense of Theorem* 5.1.2 *if $O = L$) that is induced by a continuous character $\varphi : G_K \to O^\times$ for a quadratic extension K of F. Let $\mathfrak{l}$ be a prime ideal in F. We have the following:*

(1) *Suppose that $\mathfrak{l}$ splits or is inert in K. Then φ is unramified at every prime ideal of K lying over $\mathfrak{l}$ if and only if ρ is unramified at $\mathfrak{l}$.*

(2) *If $\mathfrak{l}$ ramifies in K, then ρ is ramified at $\mathfrak{l}$.*

Proof. For each prime ideal $\mathfrak{l}$ in F, let us fix a decomposition group $D_{F_\mathfrak{l}}$ at $\mathfrak{l}$ and the inertia group $I_{F_\mathfrak{l}}$ inside $D_{F_\mathfrak{l}}$. We note the following consequence of the local class field theory:

- $\mathfrak{l}$ splits in K if and only if $D_{F_\mathfrak{l}}$ is contained in G_K,
- $\mathfrak{l}$ is inert in K if and only if $I_{F_\mathfrak{l}}$ is contained in G_K but $D_{F_\mathfrak{l}}$ is not,
- $\mathfrak{l}$ ramifies in K if and only if $I_{F_\mathfrak{l}}$ is not contained in G_K.

Now suppose that $\mathfrak{l}$ is unramified in K and choose a prime ideal $\mathfrak{L}$ of K sitting above $\mathfrak{l}$ so that $I_{F_\mathfrak{l}} = I_{K_\mathfrak{L}}$ as subgroups of G_K. Then the induced representation $\mathrm{Ind}_K^F(\varphi)$ restricted to $I_{F_\mathfrak{l}}$ is of the shape

$$\mathrm{Ind}_K^F(\varphi)|_{I_{F_\mathfrak{l}}} = \begin{bmatrix} \varphi|_{I_{K_\mathfrak{L}}} & 0 \\ 0 & \varphi^\sigma|_{I_{K_\mathfrak{L}}} \end{bmatrix}.$$

Therefore, if ρ is unramified at $\mathfrak{l}$, then we see that φ is unramified both at $\mathfrak{L}$ and at $\mathfrak{L}^\sigma$ (note that $\mathfrak{L} = \mathfrak{L}^\sigma$ if $\mathfrak{l}$ is inert in K), and the reverse implication is also true.

On the other hand, if $\mathfrak{l}$ ramifies in K, then there exists an element $g \in I_{F_\mathfrak{l}}$ which does not lie in G_K. For such g we have

$$\mathrm{Ind}_K^F(\varphi)(g) = \begin{bmatrix} 0 & \varphi(\sigma g) \\ \varphi(g\sigma^{-1}) & 0 \end{bmatrix}$$

and this can not be the identity matrix. Therefore the induced representation is not trivial on $I_{F_\mathfrak{l}}$, namely, ρ is ramified at $\mathfrak{l}$. □

The following proposition gives us more precise information on the local behavior of an induced representation at ramified primes. For a finite extension K/F of fields, we denote by $d_{K/F}$ (resp. by $\mathrm{N}_{K/F}$) the relative ideal norm (resp. the relative discriminant) of K/F. Let $\mathfrak{d}_{K/F}$ be the different ideal of K/F so that we have $\mathrm{N}_{K/F}(\mathfrak{d}_{K/F}) = d_{K/F}$.

Proposition 5.6.4. *Let $\rho : G_F \to GL_2(O)$, K and $\varphi : G_K \to O^\times$ be as in Lemma* 5.6.3*. For every finite place w of K prime to p, we have*

$$\mathrm{cond}_v(\rho) = \mathfrak{p}_{F_v}^{\mathrm{ord}_w(\mathfrak{d}_{K/F})}\mathrm{N}_{K/F}(\mathrm{cond}_w(\varphi)), \tag{5.5.3}$$

where v is the finite place of F lying below w. Moreover, if ρ has finite image, that is, if φ is of finite order, then the equality (5.5.3) *holds for every $w \in P_{\mathrm{f}}(K)$ dividing p as well.*

We refer the reader to [23] for a rigorous proof and confine ourselves to making some comments. In order to verify Proposition 5.6.4, it suffices to prove

$$n_v(\rho) = \mathrm{ord}_w(\mathfrak{d}_{K/F}) + f(K_w/F_v)n_w(\varphi)$$

for every finite place w of K prime to p (and for w dividing p as well when ρ has finite image), where $f(K_w/F_v)$ is the relative degree of the extension K_w/F_v of local fields. This equality can be shown basically by using Mackey's formula on induced representations which can be found in Proposition 22 of [36]. It should be noticed that Proposition 5.6.4 holds true in a more general context: for instance, the extension K/F can be replaced by an extension of arbitrary finite degree and $\varphi : G_K \to O^\times$ can be a representation of any finite dimension.

The following is a consequence of Proposition 5.6.4 in a global point of view.

Corollary 5.6.5. *Let $\rho : G_F \to GL_2(O)$, K and $\varphi : G_K \to O^\times$ be as in Lemma* 5.6.3*. Suppose that ρ is ramified at only finitely many prime ideals in F. Then we have the following formula on the prime-to-p conductors*

$$\mathrm{cond}^{(p)}(\rho) = \mathrm{N}_{K/F}(\mathrm{cond}^{(p)}(\varphi))d_{K/F}^{(p)}. \tag{5.5.4}$$

Here, $d_{K/F}^{(p)}$ is the prime-to-p part of $d_{K/F}$. Furthermore, if ρ has finite image, that is, if φ is of finite order, then the equality

$$\mathrm{cond}(\rho) = \mathrm{N}_{K/F}(\mathrm{cond}(\varphi))d_{K/F} \tag{5.5.5}$$

of Artin conductors holds as well as (5.5.4)*.*

One can now easily see that Lemma 5.6.3 is obtained as a consequence of Corollary 5.6.5 (and hence of Proposition 5.6.4). In the following chapters, the equality (5.5.5) will be referred to as the conductor-discriminant formula and frequently used.

Chapter 6

Local indecomposability of modular Galois representations

In this chapter, we will give a characterization of primitive ordinary Λ-adic cusp forms $\mathcal{F}$ such that the associated Galois representations $\rho_{\mathcal{F}}$ split at each prime ideal lying over p. It will be revealed that such Λ-adic forms should have complex multiplication (CM for short).

6.1 Construction of CM families

In this section, we will first give the definition of a primitive ordinary Λ-adic cusp form with CM, and then explain how to construct such a Λ-adic form in an explicit manner. We keep the notation in Chapter 5.

Definition 6.1.1. A primitive ordinary Λ-adic cusp form $\mathcal{F}$ of tame level $\mathfrak{n}_0$ in $\mathbf{S}^{\circ}(\mathfrak{n}, \psi)$ is said to be a CM family if there exists a totally imaginary quadratic extension K of F such that $\rho_{\mathcal{F}}$ is isomorphic to $\rho_{\mathcal{F}} \otimes \varepsilon_{K/F}$, where $\varepsilon_{K/F} : G_F \to \{\pm 1\}$ is the quadratic character corresponding to K/F. We also say that such a Λ-adic form has CM by K.

According to Lemma 5.4.3, if $\mathcal{F}$ is a CM family, then $\rho_{\mathcal{F}}$ is induced by a character $\Psi_{\mathcal{F}}$ of G_K which takes values in (a finite integral extension of) Λ_L. We will explain how to construct such a character $\Psi_{\mathcal{F}}$ explicitly. As a consequence of it, we will see at the end of this section that a CM family admits infinitely many classical weight one specializations, and hence the residual representation of a CM family is irreducible. In particular, any CM family is residually of dihedral type.

If $\mathcal{F}$ is a CM family, then for each arithmetic point P of Λ_L, the $\mathfrak{p}$-adic representation $\rho_{f_P,\mathfrak{p}}$ in the sense of Section 4.3 associated to the specialization f_P satisfies $\rho_{f_P,\mathfrak{p}} \cong \rho_{f_P,\mathfrak{p}} \otimes \varepsilon_{K/F}$. In view of Lemma 5.4.3, for each arithmetic point P, there exists a character $\varphi_P : G_K \to O_P^{\times}$ for the ring of integers O_P of a suitable finite extension of $\mathbb{Q}_p$ so that $\rho_{f_P,\mathfrak{p}}$ is induced by φ_P. We combine Theorem 4.1.1 and Lemma 5.6.3 to see that for each prime ideal $\mathfrak{l}$ of F prime to $\mathfrak{n}_0 p$, we have

$$c(\mathfrak{l}, f_P) = \begin{cases} \varphi_P(\mathrm{Frob}_{\mathfrak{L}}) + \varphi_P(\mathrm{Frob}_{\mathfrak{L}^{\sigma}}) & \text{if } \mathfrak{l} \text{ splits in } K, \text{ say } \mathfrak{l}O_K = \mathfrak{L}\mathfrak{L}^{\sigma}, \\ 0 & \text{if } \mathfrak{l} \text{ is inert in } K. \end{cases}$$

Therefore, the newform associated with the p-stabilized newform f_P of level $\mathfrak{n}_0$ has CM in the classical sense (namely, it is constructed out of a Hecke character of the totally imaginary quadratic extension K of F). For later use, we prove the following lemma:

Lemma 6.1.2. *If a p-stabilized newform f in $S_k^{\text{o}}(\mathfrak{m}, \psi)$ of weight $k \geq 2$ has CM by a totally imaginary quadratic extension K of F, then any prime ideal $\mathfrak{q}$ of F lying over p splits in K, say $\mathfrak{q}O_K = \mathfrak{Q}\mathfrak{Q}^\sigma$. Furthermore, if the $\mathfrak{p}$-adic representation $\rho_{f,\mathfrak{p}}$ attached to f is induced by a character φ of G_K, then φ is ramified at exactly one of the prime ideals $\mathfrak{Q}$ or $\mathfrak{Q}^\sigma$.*

Proof. Let $\rho_{f,\mathfrak{p}}$ be the $\mathfrak{p}$-adic Galois representation attached to f and $\mathfrak{q}$ a prime ideal of F sitting above p. Since f is ordinary, by Theorem 4.1.5, $\rho_{f,\mathfrak{p}}$ restricted to the decomposition group $D_\mathfrak{q}$ at $\mathfrak{q}$ is, up to equivalence, of the shape

$$\rho_{f,\mathfrak{p}}|_{D_\mathfrak{q}} \cong \begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{bmatrix}$$

where $\varepsilon_2 : D_\mathfrak{q} \to O(f)_\mathfrak{p}^\times$ is an unramified character. Since f has CM by K, we have $\rho_{f,\mathfrak{p}} \cong \rho_{f,\mathfrak{p}} \otimes \varepsilon_{K/F}$ and hence

$$\begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{bmatrix}\Bigg|_{D_\mathfrak{q}} \cong \begin{bmatrix} \varepsilon_{K/F}\varepsilon_1 & * \\ 0 & \varepsilon_{K/F}\varepsilon_2 \end{bmatrix}\Bigg|_{D_\mathfrak{q}}.$$

This implies that $\varepsilon_1 = \varepsilon_{K/F}\varepsilon_1$ or $\varepsilon_1 = \varepsilon_{K/F}\varepsilon_2$ on $D_\mathfrak{q}$. In the first case, the quadratic character $\varepsilon_{K/F}$ is trivial on $D_\mathfrak{q}$, namely, $\mathfrak{q}$ splits in K. In the second case, as ε_2 is unramified, we have $\varepsilon_1 = \varepsilon_{K/F}$ on the inertia group $I_\mathfrak{q}$ at $\mathfrak{q}$. On the other hand, by taking determinant we see that $\psi\chi_p^{k-1} = \varepsilon_1$ on $I_\mathfrak{q}$. Since $k \geq 2$, χ_p^{k-1} is of infinite order and thus so is ε_1. This contradicts $\varepsilon_{K/F}^2 = 1$.

As for the second assertion, we note that since $\mathfrak{q}$ splits in K, the decomposition group $D_\mathfrak{q}$ is contained in G_K and

$$\begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{bmatrix}\Bigg|_{D_\mathfrak{q}} \cong \begin{bmatrix} \varphi & 0 \\ 0 & \varphi^\sigma \end{bmatrix}\Bigg|_{D_\mathfrak{q}},$$

where σ is any element of G_F not contained in G_K and φ^σ is the character of G_K defined by $\varphi^\sigma(g) = \varphi(\sigma g \sigma^{-1})$. This implies that one has either $(\varphi, \varphi^\sigma) = (\varepsilon_1, \varepsilon_2)$ or $(\varphi, \varphi^\sigma) = (\varepsilon_2, \varepsilon_1)$. As ε_1 restricted to $I_\mathfrak{q}$ is equal to $\psi\chi_p^{k-1}$ which is ramified at $\mathfrak{q}$ and ε_2 is unramified at $\mathfrak{q}$, we see that φ is ramified at exactly one of $\mathfrak{Q}$ and $\mathfrak{Q}^\sigma$. This completes the proof. □

Remark 6.1.3. We note that this proof may not work when f is of weight one, as the determinant $\psi\chi_p^{k-1}$ of $\rho_{f,\mathfrak{p}}$ is of finite order if and only if $k = 1$. In Lemma 7.1.2, we will show the same result in the case of weight one under the assumption that $\rho_{f,\mathfrak{p}}(I_\mathfrak{q})$ has order at least three.

In what follows, we will explicitly construct a primitive ordinary Λ-adic cusp form which has CM by a totally imaginary quadratic extension K of F. To be specific, we will construct a character $\Psi : G_K \to \Lambda_L^\times$ for a suitable L that p-adically interpolates the characters φ_P above. The exposition below is based on §4 of [12], Section 3 of [16] and Section 4 of [1].

By Lemma 6.1.2, we may and do assume that every prime ideal $\mathfrak{q}$ of F lying over p splits in K. For each such prime ideal $\mathfrak{q}$, we choose a prime ideal $\mathfrak{Q}$ of K lying over $\mathfrak{q}$ and put $\mathcal{Q} = \prod_{\mathfrak{q}|p} \mathfrak{Q}$. Let $\mathfrak{C}$ be an integral ideal of K prime to p and $W_K(\mathfrak{C})$ the group $\mathbb{A}_K^\times / K^\times U_K^{(\mathcal{Q})}(\mathfrak{C}) \mathbb{A}_{K,\infty}^\times$, where we denote by $U_K^{(\mathcal{Q})}(\mathfrak{C})$ the product $\prod_{v \in P_{\mathrm{f}}(K), v \nmid \mathcal{Q}} U_{K_v}(\mathfrak{C})$. From now on we fix a continuous idele character $\varphi_0 : W_K(\mathfrak{C}) \to \overline{\mathbb{Q}}_p^\times$ with the property that $\varphi_0(a) = a$ for any $a \in \prod_{\mathfrak{q}|p} U_{K_\mathfrak{Q}}(\mathfrak{Q})$. Then the Galois representation $\mathrm{Ind}_K^F(\varphi_0)$ is attached to an ordinary cusp form f_0 in $S_2(d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C}\mathcal{Q}), \eta\varepsilon_{K/F}; \overline{\mathbb{Q}}_p)$ that has CM by K. Here, η is the ray class character of $\mathrm{Cl}_F(\mathrm{N}_{K/F}(\mathfrak{C}\mathcal{Q}))$ obtained by restricting the finite order character $\mathbb{A}_K^\times / K^\times U_K(\mathfrak{C}\mathcal{Q}) \mathbb{A}_{K,\infty}^\times \to \overline{\mathbb{Q}}^\times$ that maps $x \in \mathbb{A}_K^\times$ to $\varphi_0(x) \prod_{\mathfrak{q}|p} x_\mathfrak{Q}^{-1}$, to $\mathbb{A}_F^\times / F^\times U_F(\mathrm{N}_{K/F}(\mathfrak{C}\mathcal{Q})) \mathbb{A}_{F,\infty}^+$. Note that the latter quotient is isomorphic to $\mathrm{Cl}_F(\mathrm{N}_{K/F}(\mathfrak{C}\mathcal{Q}))$. We will construct a CM family passing through f_0. Without loss of generality, we may assume that the values of φ_0 is contained in $\mathcal{O}$, where $\Lambda = \mathcal{O}[[\mathbf{G}]]$.

Lemma 6.1.4. *Let W_K be a maximal p-profinite torsion-free subgroup of $W_K(\mathfrak{C})$ that contains $(\prod_{\mathfrak{q}|p} U_{F_\mathfrak{q}}(\mathfrak{q}))/\overline{U_F(p)}$, where $U_F(p)$ is the subgroup of $O_F^\times$ consisting of units congruent to 1 modulo $\mathfrak{q}$ for any prime ideal $\mathfrak{q}$ lying over p and $\overline{U_F(p)}$ is the closure of the image of $U_F(p)$ in $(\prod_{\mathfrak{q}|p} U_{F_\mathfrak{q}}(\mathfrak{q}))/\overline{U_F(p)}$. Then W_K is isomorphic to $\mathbb{Z}_p^{1+\delta}$, where δ is the defect in Leopoldt's conjecture for F and p.*

Proof. We note that $W_K(\mathfrak{C})$ is isomorphic to the projective limit $\varprojlim_{j \geq 0} \mathrm{Cl}_K(\mathfrak{C}\mathcal{Q}^j)$ of ray class groups of $\mathrm{Cl}_K(\mathfrak{C}\mathcal{Q}^j)$ of K modulo $\mathfrak{C}\mathcal{Q}^j$. In view of class field theory, $W_K(\mathfrak{C})$ sits in the following exact sequence:

$$\begin{array}{ccccc} 1 \longrightarrow & \overline{O_K^\times} & \longrightarrow & \prod_{\mathfrak{q}|p} O_{K_\mathfrak{Q}}^\times \times (O_K/\mathfrak{C})^\times & \\ \longrightarrow & W_K(\mathfrak{C}) & \longrightarrow & \mathrm{Cl}_K & \longrightarrow 1 \end{array} \tag{6.6.1}$$

where $\overline{O_K^\times}$ is the closure of the image of $O_K^\times$ in $\prod_{\mathfrak{q}|p} O_{K_\mathfrak{Q}}^\times$. Let g denote the degree of F over $\mathbb{Q}$. Then it follows that the $\mathbb{Z}_p$-rank of $\prod_{\mathfrak{q}|p} O_{K_\mathfrak{Q}}^\times$ is g, since $K_\mathfrak{Q}$ is identical to $F_\mathfrak{q}$ for each prime ideal $\mathfrak{q}$ of F lying over p. We also note that the global unit group $O_F^\times$ is a subgroup of finite index in $O_K^\times$, as K/F is a totally imaginary quadratic extension and thus they have the same $\mathbb{Z}$-rank $g-1$. In conclusion, the $\mathbb{Z}_p$-rank of W_K is equal to that of $(\prod_{\mathfrak{q}|p} O_{F_\mathfrak{q}}^\times)/\overline{O_F^\times}$, as desired. □

A byproduct of this lemma is that W_K is independent of an ideal $\mathfrak{C}$ prime to p and we have a (non-canonical) decomposition $W_K(\mathfrak{C}) = W_K \times \Delta_\mathfrak{C}$ where $\Delta_\mathfrak{C}$ is a finite group. Let $\mathbf{G}'$ be the torsion-free part of the Galois group of the maximal abelian pro-p extension of F unramified outside p. By an exact sequence analogous to (6.6.1) with K replaced by F, one sees that $\mathbf{G}'$ is contained in $(\prod_{\mathfrak{q}|p} U_{F_\mathfrak{q}}(\mathfrak{q}))/\overline{U_F(p)}$

and is a subgroup of W_K of finite index. In particular, the completed group algebra $\mathcal{O}[[W_K]]$ is a finite integral extension of $\mathcal{O}[[\mathbf{G}']]$.

Let L be the field of fractions of $\mathcal{O}[[W_K]] \otimes_{\mathcal{O}[[\mathbf{G}']],\pi} \Lambda$, where the $\mathcal{O}$-algebra homomorphism $\pi : \mathcal{O}[[\mathbf{G}']] \to \Lambda$ is induced by a canonical surjection $\mathbf{G}' \to \mathbf{G}$ of Galois groups. It follows that the integral closure Λ_L of Λ in L is a finite integral extension of Λ. Take a primitive character $\chi : \Delta_{\mathfrak{C}} \to \mathcal{O}^\times$ (by a primitive character we mean that χ does not factor through $\Delta_{\mathfrak{C}'}$ for any proper divisor $\mathfrak{C}'$ of $\mathfrak{C}$). Let $\widetilde{\Psi} : G_K \to \mathcal{O}[[W_K]]^\times$ be the character obtained by composing a canonical surjection $G_K \to W_K(\mathfrak{C})$ and

$$W_K(\mathfrak{C}) = W_K \times \Delta_{\mathfrak{C}} \to \mathcal{O}[[W_K]]^\times;\ w = (w_0, w_{\mathrm{t}}) \mapsto \chi(w_{\mathrm{t}})[w_0],$$

where $[w_0]$ is the image of $w_0 \in W_K$ in $\mathcal{O}[[W_K]]^\times$. We denote by Ψ the composite of $\widetilde{\Psi}$ and the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[[W_K]] \to \Lambda_L$ induced by π.

Proposition 6.1.5. *Let ρ be the induced representation $\mathrm{Ind}_K^F(\Psi)$. Then ρ is isomorphic to the Galois representation associated with a primitive ordinary Λ-adic cusp form $\mathcal{F}_\Psi$ of tame level $d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C})$ and the central character $\varepsilon_{K/F} \cdot (\mathrm{ver}_{K/F} \circ \chi)$. In particular, $\mathcal{F}_\Psi$ has CM by K.*

Proof. We first show that $\mathcal{F}_\Psi$ is a primitive Λ-adic cusp form of the prescribed tame level and the central character. For each non-zero integral ideal $\mathfrak{m}$ in O_F, let

$$c(\mathfrak{m}, \mathcal{F}_\Psi) = \sum_{\substack{\mathfrak{M} \subset O_K:\ \text{ideal},\ \mathrm{N}_{K/F}(\mathfrak{M}) = \mathfrak{m}, \\ \mathfrak{M} \text{ is prime to } \mathfrak{C}\mathcal{Q}}} \Psi^*(\mathfrak{M}).$$

Here Ψ^* is the ideal character associated to $\Psi : \mathbb{A}_K^\times / K^\times \to W_K(\mathfrak{C}) \to \Lambda_L^\times$, namely, one has $\Psi^*(\mathfrak{A}) = \prod_{v \in P_{\mathrm{f}}(K), v \nmid \mathfrak{C}\mathcal{Q}} \Psi_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{A})})$ for each fractional ideal $\mathfrak{A}$ in K prime to $\mathfrak{C}\mathcal{Q}$.

For each integer $k \geq 1$ and a finite order character $\tilde{\varepsilon} : W_K \to \overline{\mathbb{Q}}_p^\times$, we denote by $P_{k,\tilde{\varepsilon}} : \mathcal{O}[[W_K]] \to \overline{\mathbb{Q}}_p$ the $\mathcal{O}$-algebra homomorphism induced by $\tilde{\varepsilon}\varphi_0^{k-1} : W_K \to \overline{\mathbb{Q}}_p^\times$. Then we have $(P_{k,\tilde{\varepsilon}} \circ \widetilde{\Psi})(w) = \chi(w_{\mathrm{t}})\tilde{\varepsilon}(w_0)\varphi_0^{k-1}(w_0)$, and we see that for each finite order character $\varepsilon : \mathbf{G}' \to \overline{\mathbb{Q}}_p^\times$, there is $\tilde{\varepsilon} : W_K \to \overline{\mathbb{Q}}_p^\times$ so that $P_{k,\tilde{\varepsilon}} \circ \widetilde{\Psi} : \mathcal{O}[[W_K]] \to \overline{\mathbb{Q}}_p$ factors through Λ_L by some $\mathcal{O}$-algebra homomorphism $P : \Lambda_L \to \overline{\mathbb{Q}}_p$ that lifts $\varphi_{k,\varepsilon}$. For such P and every prime ideal $\mathfrak{l}$ of F not dividing $d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C})$, we now compute the Fourier coefficients $P(c(\mathfrak{l}, \mathcal{F}_\Psi))$. For a idele character χ of $\mathbb{A}_K^\times / K^\times$, let χ^* denote the corresponding ideal character. Then we have

$$P(c(\mathfrak{l}, \mathcal{F}_\Psi)) = (\chi\tilde{\varepsilon}\varphi_0^{k-1})^*(\mathfrak{L}) + (\chi\tilde{\varepsilon}\varphi_0^{k-1})^*(\mathfrak{L}^\sigma)$$

if $\mathfrak{l}$ splits in K, say $\mathfrak{l}O_K = \mathfrak{L}\mathfrak{L}^\sigma$, and

$$P(c(\mathfrak{l}, \mathcal{F}_\Psi)) = 0$$

if $\mathfrak{l}$ is inert in K. Notice that when $\mathfrak{l} = \mathfrak{q}$ lies over p we have

$$P(c(\mathfrak{q}, \mathcal{F}_\Psi)) = (\chi\tilde{\varepsilon}\varphi_0^{k-1})^*(\mathfrak{Q}^\sigma)$$

where $\mathfrak{q}O_K = \mathfrak{Q}\mathfrak{Q}^\sigma$. Therefore, the specialization f_P of $\mathcal{F}_\Psi$ at P is a classical Hilbert cusp form of weight k and level $d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C})p^{r+1}$ that has CM by K, and the level of the newform associated to f_P differs from that of f_P only by prime ideals lying over p. Also, it follows by definition that $\mathcal{F}_\Psi$ is a normalized eigenform. Therefore $\mathcal{F}_\Psi$ is a primitive Λ-adic cusp form of tame level $d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C})$.

Since Ψ is ramified at $\mathfrak{Q}$ and unramified at $\mathfrak{Q}^\sigma$ for each $\mathfrak{q} \mid p$ by construction, we have $\Psi \neq \Psi^\sigma$. Therefore the induced representation $\mathrm{Ind}_K^F(\Psi)$ is irreducible, according to Mackey's criterion (Lemma 5.6.1). This also implies that $\mathrm{Ind}_K^F(\Psi)$ is ordinary in the sense of Theorem 5.1.2. We know that the Galois representation $\rho_{\mathcal{F}_\Psi}$ attached to $\mathcal{F}_\Psi$ is irreducible, as seen in Theorem 5.1.2. It is clear that the trace of $\rho_{\mathcal{F}_\Psi}$ and $\mathrm{Ind}_K^F(\Psi)$ at $\mathrm{Frob}_\mathfrak{l}$ are the same for any prime ideal $\mathfrak{l}$ outside $d_{K/F}\mathrm{N}_{K/F}(\mathfrak{C})p$. In conclusion, we see that $\rho_{\mathcal{F}_\Psi}$ is isomorphic to $\mathrm{Ind}_K^F(\Psi)$, and hence $\mathcal{F}_\Psi$ has CM by K, as desired. □

We emphasize two important consequences of this construction:

Corollary 6.1.6. *A CM family admits infinitely many classical specialization including in weight one, and all of them have CM in the classical sense.*

Corollary 6.1.7. *The residual representation $\bar{\rho}_{\mathcal{F}}^{\mathrm{ss}}$ of a CM family $\mathcal{F}$ is irreducible, and its projective image is isomorphic to a dihedral group. In other words, a CM family is residually of dihedral type in the sense of Definition* 5.3.2.

Remark 6.1.8. Any arithmetic specialization of a non-CM family does not have CM in the classical sense. Indeed, if f is a p-stabilized newform of weight $k \geq 2$, then Hida's control theorem (Corollary 3.5.4) tells us that there exists a unique primitive ordinary Λ-adic cusp form $\mathcal{F}$ that specializes to f. On the other hand, we have just observed that if in addition f has CM in the classical sense, then one can explicitly construct a CM family passing through f. By the uniqueness, this CM family is nothing but $\mathcal{F}$.

6.2 Ordinary Galois representations and local indecomposability

Lemma 6.1.2 implies that, if a p-stabilized newform f of weight at least two has CM by a totally imaginary quadratic extension K of F, then the $\mathfrak{p}$-adic Galois representation $\rho_{f,\mathfrak{p}}$ attached to f is isomorphic to a direct sum of two 1-dimensional characters of G_K on the decomposition group $D_\mathfrak{q}$ for each prime ideal $\mathfrak{q}$ of F lying over p. Then how about the converse? In the case of $F = \mathbb{Q}$, Greenberg proposed the following question:

Question 6.2.1 (Question 1; p. 2144 of [9]). Let f be a primitive ordinary elliptic cusp form of weight at least two. When is the representation $\rho_{f,p}|_{D_p}$ splits?

To address this question, Ghate and Vatsal posed a similar question for Λ-adic forms, and obtained the following result (Theorem 6.2.3) in the case of elliptic modular forms. Recall that the Galois representation $\rho_{\mathcal{F}}$ attached to a primitive ordinary Λ-adic cusp form $\mathcal{F}$ is ordinary in the sense of Theorem 5.1.2. For each prime ideal $\mathfrak{q}$ lying over p, let

$$\rho_{\mathcal{F}}|_{D_{\mathfrak{q}}} \cong \begin{bmatrix} \mathcal{E}_{\mathfrak{q}} & * \\ 0 & \mathcal{D}_{\mathfrak{q}} \end{bmatrix}. \tag{6.6.2}$$

where $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ are as in the theorem.

Definition 6.2.2. $\mathcal{F}$ is said to be p-distinguished if $\bar{\mathcal{E}}_{\mathfrak{q}} \neq \overline{\mathcal{D}}_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q}$ lying over p.

Theorem 6.2.3 (Proposition 14; p. 2152 of [9]). *Let p be an odd prime and $\mathcal{F}$ be a primitive ordinary Λ-adic cusp form such that*

(1) *$\mathcal{F}$ is p-distinguished, and*

(2) *the residual representation $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ of $\mathcal{F}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$*, where M is the quadratic field with discriminant $(-1)^{(p-1)/2}p$.*

Then the following statements are equivalent:

(i) *$\rho_{\mathcal{F}}|_{D_p}$ splits.*

(ii) *$\mathcal{F}$ admits infinitely many classical weight one specializations.*

(iii) *$\mathcal{F}$ admits infinitely many classical weight one specializations that have CM.*

(iv) *$\mathcal{F}$ is a CM family.*

According to this theorem, the number of classical weight one specializations inside a non-CM family is finite. In their paper [6], Dimitrov and Ghate gave an explicit upper bound on the number of such forms.

The result of Ghate-Vatsal is generalized to the case of totally real fields by Balasubramanyam, Ghate and Vatsal in [1], which contains an outline of a proof even for nearly ordinary Hida families of not necessarily parallel weight. The convention of Hida families adopted by them is different from that of our Λ-adic forms. Most importantly, the Iwasawa algebra $\mathcal{O}[[\mathbf{G}']]$ in their paper is the complete group algebra which corresponds to the torsion-free part $\mathbf{G}'$ of the Galois group of the maximal abelian extension of F unramified outside p. As seen in the proof of Lemma 6.1.4, $\mathcal{O}[[\mathbf{G}']]$ is isomorphic to a power series ring of $(1+\delta)$-variables, where δ is the defect in Leopoldt's conjecture for F and p. We refer the reader to Section 2 of [1] for details of their setting. Their main result is the following:

Theorem 6.2.4 (Theorem 3; p. 517 of [1]). *Let p be an odd prime that splits completely in F and let $\mathcal{F}$ be a primitive ordinary Hida family such that*

(1) *$\mathcal{F}$ is p-distinguished, and*

(2) *the residual representation $\bar{\rho}^{\mathrm{ss}}_{\mathcal{F}}$ of $\mathcal{F}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\overline{F}/F(\zeta_p))$*, where ζ_p is a primitive p-th root of unity in $\overline{F}$.*

Then $\rho_{\mathcal{F}}|_{D_{\mathfrak{q}}}$ splits for each prime ideal $\mathfrak{q}$ of F lying over p if and only if $\mathcal{F}$ is a CM family.

Note that in their convention, we do not know in general whether a primitive ordinary Hida family that admits infinitely many classical weight one specializations is a CM family or not. This is due to the difference of Iwasawa algebras. More precisely, their Iwasawa algebra $\mathcal{O}[[\mathbf{G}']]$ may be of Krull dimension greater than two, and an intersection of infinitely many height one prime ideals of Λ may not be zero. See Lemma 1 (p. 518 of [1]) for more details.

Remark 6.2.5. It should be mentioned that Zhao in [48] and Hida in [17] also have results on the local indecomposability of a p-adic Galois representation attached to a p-ordinary non-CM Hilbert cusp form of parallel weight two (or the p-adic Tate module of an abelian variety with real multiplication) restricted to the decomposition group at each prime lying over p. By applying their results, one can show that $\rho_{\mathcal{F}}|_{D_{\mathfrak{q}}}$ being split implies that $\mathcal{F}$ being a CM family, without the assumptions in Theorem 6.2.4, at least when the degree of F over $\mathbb{Q}$ is odd (when the degree is even, one needs to put some technical condition on the Hilbert cusp form: see p. 1522 of [48]).

6.3 Finiteness result for non-CM families

In this section, we show the following result which is an analogue to Theorem 6.2.3 in our convention of Λ-adic forms.

Theorem 6.3.1. *A primitive ordinary Λ-adic cusp form $\mathcal{F}$ admits infinitely many classical weight one specializations if and only if $\mathcal{F}$ is has CM.*

Proof. The argument proceeds in a way little different from that of Proposition 3.1; p. 673 of [6]. It follows from the construction in Section 6.1 that a CM family admits infinitely many classical weight one specializations. We prove the converse. As explained in Section 4.3, the Galois representation attached to a classical weight one form is either of dihedral type or of exceptional type. By the same reasoning as in [9] p. 2155, we see that only finitely many classical weight one specializations of $\mathcal{F}$ are exceptional. Therefore $\mathcal{F}$ has infinitely many classical weight one specializations f such that the associated Galois representation ρ_f satisfies $\rho_f \cong \varepsilon_{K/F} \otimes \rho_f$ for some quadratic extension K of F.

Since the conductor of ρ_f is $\mathfrak{n}_0 p^{r+1}$ for some integer $r \geq 0$, we see by the conductor-discriminant formula (5.5.5) that there is a bound on the discriminant $d_{K/F}$. Hence there exists a quadratic extension K of F so that infinitely many classical weight one specializations f of $\mathcal{F}$ satisfy $\rho_f \cong \varepsilon_{K/F} \otimes \rho_f$. It follows that $\mathrm{Tr}\rho_f(\mathrm{Frob}_{\mathfrak{l}}) = 0$ for all primes $\mathfrak{l}$ of F prime to $\mathfrak{n}_0 p$ and inert in K. Since the intersection of infinitely many height one prime ideals of Λ_L is zero, we have $\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{l}}) = 0$ and thus

$$\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{l}}) = \varepsilon_{K/F}(\mathrm{Frob}_{\mathfrak{l}})\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{l}})$$

for all such primes $\mathfrak{l}$. As for each prime ideal $\mathfrak{l}$ that splits in K, this equality is unconditional. Hence $\rho_{\mathcal{F}}$ and $\varepsilon_{K/F} \otimes \rho_{\mathcal{F}}$ have the same trace. Since $\rho_{\mathcal{F}}$ is irreducible and thus determined by the trace, this implies that $\rho_{\mathcal{F}} \cong \varepsilon_{K/F} \otimes \rho_{\mathcal{F}}$. If K/F is not totally imaginary, this yields a contradiction, since the automorphic representation associated to any specialization of $\mathcal{F}$ at arithmetic points of weight $k \geq 2$ is a holomorphic discrete series (see §5 of [18] or §4 of [2] for details of discrete series). Therefore K/F is totally imaginary and the Λ-adic form $\mathcal{F}$ has CM by K. □

Remark 6.3.2. In Theorem 6.3.1, we do not need to assume neither that p splits completely in F, $\mathcal{F}$ is p-distinguished, nor that $\overline{\rho}_{\mathcal{F}}^{\mathrm{ss}}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$. Balasubramanyam, Ghate and Vatsal had to impose these assumptions because they used modularity lifting theorem (Theorem 3; p. 999 of [34]) to observe that any weight one specialization of $\mathcal{F}$ is classical if $\rho_{\mathcal{F}}|_{D_{\mathfrak{q}}}$ splits. More precisely, p-distinguishability of $\mathcal{F}$ is a condition that is assumed in the modularity lifting theorem (see the remark in p. 518 of [1]), and the absolute irreducibility of $\overline{\rho}_{\mathcal{F}}^{\mathrm{ss}}$ enables them to apply modularity lifting theorem to the Galois representation attached to each weight one specialization of $\mathcal{F}$. We do not need modularity lifting theorem to establish the equivalence in Theorem 6.3.1, and hence we assume neither of those.

In view of Theorem 6.3.1, a non-CM Λ-adic cusp form admits only finitely many classical weight one specializations. The goal of this paper is to give an upper bound on the number of such forms.

Chapter 7

The number of classical weight one specializations of a Λ-adic form

In this chapter, we give an explicit upper bound on the number of classical weight one specializations obtained from a primitive ordinary non-CM Λ-adic form. As explained in the introduction, the way we give such an upper bound varies according to the residual type of the Λ-adic form whose definition was given in Definition 5.3.2. Therefore, we will distinguish the arguments for each case.

Throughout this chapter, we keep the notation in Section 3.3 and the convention in Chapter 5. We fix a finite extension L of the field of fractions of Λ, a non-zero integral ideal $\mathfrak{n}_0 \subset O_F$ prime to p and a narrow ray class character $\psi : \mathrm{Cl}_F(\mathfrak{n}_0 p) \to \mathcal{O}^\times$ which is totally odd and tamely ramified at all prime ideals of F lying over p. Let $\mathcal{F}$ be a primitive ordinary Λ-adic cusp form of tame level $\mathfrak{n}_0$ in $\mathbf{S}^{\mathrm{o}}(\mathfrak{n}_0, \psi)$ in the sense of Definition 3.5.3, with coefficients in Λ_L.

7.1 Residually of dihedral case: Main Theorem 1.2.1

In this section, we will deal with the case where $\mathcal{F}$ is residually of dihedral type. In view of Lemmas 5.4.1 and 5.4.3, there exist a quadratic extension K of F and a character $\bar{\varphi} : G_K = \mathrm{Gal}(\overline{F}/K) \to \mathbb{F}^\times$ such that $\bar{\rho}_{\mathcal{F}}$ is equivalent to the induced representation $\mathrm{Ind}_K^F(\bar{\varphi})$. We consider the following condition on $\mathcal{F}$:

(P) $\mathcal{F}$ has a classical weight one specialization f such that the associated $\mathfrak{p}$-adic representation $\rho_{f,\mathfrak{p}} : G_F \to GL_2(O)$ (O is the ring of integers of a suitable finite extension of $\mathbb{Q}_p$) has the property that $\rho_f(I_{\mathfrak{q}})$ has order at least three for each prime $\mathfrak{q}$ of F lying over p.

Remark 7.1.1. Since $\mathcal{F}$ is residually of dihedral type, $\bar{\rho}_{\mathcal{F}}$ is irreducible (cf. Section 4.3), and hence one has $\bar{\varphi} \neq \bar{\varphi}^\sigma$ according to Mackey's irreducibility criterion (Lemma 5.6.1). Here $\sigma \in G_F$ is an element not contained in G_K and $\bar{\varphi}^\sigma$ is the character of G_K defined by $\bar{\varphi}^\sigma(g) = \bar{\varphi}(\sigma g \sigma^{-1})$ for each $g \in G_K$.

From now on, we fix an element $\sigma \in G_F$ which does not lie in G_K. Also, we denote simply by ρ_f the $\mathfrak{p}$-adic Galois representation $\rho_{f,\mathfrak{p}}$ attached to f by abuse of notation. According to Lemma 5.4.5, the representation ρ_f in (P) is of the shape $\mathrm{Ind}_K^F(\varphi)$ for a finite order character $\varphi : G_K \to O^\times$ which is a lift of $\bar{\varphi}$. In order to state the main theorem in this case, we need the following lemma:

Lemma 7.1.2. *Suppose that* (P) *is true. Then each prime ideal* $\mathfrak{q}$ *of* F *lying over* p *splits in* K*, say* $\mathfrak{q}O_K = \mathfrak{Q}\mathfrak{Q}^\sigma$*. Moreover,* φ *is ramified at exactly one of* $\mathfrak{Q}$ *and* $\mathfrak{Q}^\sigma$*.*

Proof. The way we verify the lemma is very close to that of Lemma 6.3 (1); p. 679 of [6]. Let $\rho_f = \mathrm{Ind}_K^F(\varphi)$ as above and D_{2n} the projective image of ρ_f. Since $\mathcal{F}$ is ordinary, so is f and we have

$$\rho_f|_{I_\mathfrak{q}} \cong \begin{bmatrix} \psi\varepsilon & * \\ 0 & 1 \end{bmatrix} \tag{7.7.1}$$

for each prime ideal $\mathfrak{q}$ of F sitting above p. Here $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$ is the character such that $\varphi_{1,\varepsilon}(\mathcal{F}) = f$. Therefore $\rho_f(I_\mathfrak{q})$ is a finite cyclic group and it injects into $PGL_2(O)$. By our assumption that $\rho_f(I_\mathfrak{q})$ has order at least three, we know that $n \geq 3$ and the projective image of $\rho_f(I_\mathfrak{q})$ is contained in the unique cyclic group C_n of order n in D_{2n} (cf. Remark 5.4.2). Therefore $I_\mathfrak{q}$ is contained in G_K, that is, $\mathfrak{q}$ is unramified in K. This implies that

$$\rho_f|_{I_\mathfrak{q}} \cong \begin{bmatrix} \varphi & 0 \\ 0 & \varphi^\sigma \end{bmatrix}, \tag{7.7.2}$$

where φ^σ is the character of G_K defined by $\varphi^\sigma(g) = \varphi(\sigma g \sigma^{-1})$ for $g \in G_K$. It follows from (7.7.1) and (7.7.2) that either φ or φ^σ is unramified at $\mathfrak{Q}$. In the formula (5.5.5) of Artin conductors

$$\mathrm{cond}(\rho_f) = d_{K/F}\mathrm{N}_{K/F}(\mathrm{cond}(\varphi)),$$

the left-hand side is divisible by $\mathfrak{q}$ and $d_{K/F}$ is not. Therefore $\mathfrak{q}$ should divide $\mathrm{N}_{K/F}(\mathrm{cond}(\varphi))$. This cannot be satisfied if $\mathfrak{q}$ is inert in K. Hence $\mathfrak{q}$ splits in K and φ is ramified at exactly one of $\mathfrak{Q}$ or $\mathfrak{Q}^\sigma$. □

Remark 7.1.3. In contrast to the higher weight case treated in Lemma 6.1.2, when f is of weight one the determinant of ρ_f is a finite order character. Thus we do not know without the assumption (P) whether or not $\mathfrak{q}$ is unramified in K.

For the prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ of F lying over p, let $\mathfrak{q}_i O_K = \mathfrak{Q}_i\mathfrak{Q}_i^\sigma$ so that φ is ramified at $\mathfrak{Q}_i$ and unramified at $\mathfrak{Q}_i^\sigma$ for each $i = 1, \ldots, t$. We put $\mathcal{Q} = \prod_{i=1}^t \mathfrak{Q}_i$ and $U_K(\mathcal{Q}) = \left\{u \in O_K^\times \mid u \equiv 1 \bmod \mathcal{Q}\right\}$. For each $i = 1, \ldots, t$, $U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)$ denotes the principal unit group of the completion $K_{\mathfrak{Q}_i}$ of K at $\mathfrak{Q}_i$. We denote by $\overline{U_K(\mathcal{Q})}$ the closure (taken in $\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)$) of the image of $U_K(\mathcal{Q})$ under the diagonal map $U_K(\mathcal{Q}) \to \prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)$.

Let $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^r)$ be the narrow ray class group of K of modulus $\mathfrak{n}_0\mathcal{Q}^r$ for each integer $r \geq 1$. Our main theorem is related to the finiteness of the projective limit

$\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty) = \varprojlim_r \mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^r)$. In view of class field theory, $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is a finite group if and only if $(\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i))/\overline{U_K(\mathcal{Q})}$ is a finite group (see Section 7.1.2 for details). If this is the case, we put

$$M' = |\mathrm{Cl}_K| \cdot \left| \left(\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i) \right) / \overline{U_K(\mathcal{Q})} \right| \cdot \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{split in } K}} (q_\mathfrak{l} - 1) \cdot \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{inert in } K}} (q_\mathfrak{l} + 1)$$

and $M(\mathcal{F}, K, f) = p^{\mathrm{ord}_p(M')}$ the p-part of M'. Here ord_p is the p-adic valuation normalized so that $\mathrm{ord}_p(p) = 1$, $|\mathrm{Cl}_K|$ is the class number of K, and for a prime $\mathfrak{l}$ of F, $q_\mathfrak{l}$ is the order of the residue field $O_F/\mathfrak{l}$.

7.1.1 Statement of the main theorem

In order to state the main theorem, it will be convenient to consider a Hida community, rather than each primitive ordinary Λ-adic cusp form.

Definition 7.1.4. A Hida community is the set $\{\mathcal{F}\}$ of primitive ordinary Λ-adic cusp forms having the same tame level and the same residual representation.

We note that a Hida community is always a finite set, which is a consequence of Theorem 3.4.3. By definition, it makes sense to speak of the residual representation and p-distinguishability (Definition 6.2.2) of a Hida community.

The main result in the dihedral case is the following:

Theorem 7.1.5. *Let p be an odd prime and $\mathcal{F}$ a primitive ordinary Λ-adic cusp form of tame level $\mathfrak{n}_0$ prime to p. Suppose that $\mathcal{F}$ has the property* (P) *above. Then the following two statements hold true:*

(1) *Assume further that Leopoldt's conjecture for F and p is true. If $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is an infinite group, then there exists a primitive ordinary Λ-adic cusp form $\mathcal{G}$ that has CM by K and belongs to the same Hida community as $\mathcal{F}$. In particular, K/F is a totally imaginary extension.*

(2) *If $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is of finite order, then $\mathcal{F}$ is not a CM family and the number of classical weight one specializations of $\mathcal{F}$ is bounded by $M(\mathcal{F}, K, f)$.*

Remark 7.1.6.

1. Recall that g is the degree of F over $\mathbb{Q}$. When K/F is totally imaginary, the rank of the torsion-free part of $U_K(\mathcal{Q})$ as a $\mathbb{Z}$-module is $g-1$, and hence the $\mathbb{Z}_p$-rank of $\overline{U_K(\mathcal{Q})}$ is at most $g-1$. On the other hand, the $\mathbb{Z}_p$-rank of $\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)$ is g. Therefore the group $\left(\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)\right)/\overline{U_K(\mathcal{Q})}$ cannot be finite.

2. If K/F is totally imaginary and $\mathcal{F}$ is not a CM family, one can also give an upper bound. This estimate will be discussed at the end of this section (Proposition 7.1.14), exactly in the same manner as Lemma 6.5; p. 682 of [6].

The outline of the proof is as follows. Suppose that f' is another classical weight one specialization of $\mathcal{F}$ (if any) and let $\rho_{f'} = \mathrm{Ind}_K^F(\varphi')$. Our strategy is counting the number of characters $\xi = \varphi/\varphi'$. Let $\mathrm{ver}_{K/F} : G_F^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ be the transfer map as in Definition 5.6.2. Our analysis on ξ is close to the arguments going into the proof of Theorem 6.4 (p. 680 of [6]), as seen in the following

Lemma 7.1.7. *The above $\xi : G_K \to \overline{\mathbb{Q}}_p^\times$ satisfies the following properties:*

(1) *ξ is a p-power order character,*

(2) *ξ is unramified outside $\mathfrak{n}_0\mathcal{Q}$ and the infinite places of K, and $N_{K/F}(\mathrm{cond}^{(p)}(\xi))$ divides $\mathfrak{n}_0$, and*

(3) *$\xi \circ \mathrm{ver}_{K/F} : G_F \to \overline{\mathbb{Q}}_p^\times$ is unramified outside p.*

Proof. Since both φ and φ' reduce to $\bar{\varphi}$ modulo $\mathfrak{p}$, they differ by a p-power order character. As for the second statement, by an argument similar to the proof of Lemma 7.1.2, we may assume that φ' is ramified at $\mathfrak{Q}_i$'s and unramified at $\mathfrak{Q}_i^\sigma$'s. Then ξ is unramified at $\mathfrak{Q}_i^\sigma$ for all i. We note that the conductor of ρ_f is equal to the level of f which is equal to $\mathfrak{n}_0 p^{r+1}$ for some integer $r \geq 0$. Then the conductor-discriminant formula (5.5.4) implies that $N_{K/F}(\mathrm{cond}^{(p)}(\varphi))$ divides $\mathfrak{n}_0$ (the same for φ'). Therefore $N_{K/F}(\mathrm{cond}^{(p)}(\xi))$ divides $\mathfrak{n}_0$ and ξ is unramified outside $\mathfrak{n}_0\mathcal{Q}$ and the infinite places. Now we prove the third assertion. As computed in (5.5.2), we have $\det(\rho_f) = \varepsilon_{K/F} \cdot (\varphi \circ \mathrm{ver}_{K/F})$ (the same for f' and φ'), and from these equalities we deduce $\xi \circ \mathrm{ver}_{K/F} = \det(\rho_f)/\det(\rho_{f'}) = \varepsilon/\varepsilon'$ (here $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$ is the character such that $f = \varphi_{1,\varepsilon}(\mathcal{F})$. The same for f' and ε'). Therefore $\xi \circ \mathrm{ver}_{K/F}$ is unramified outside p and the infinite places. Moreover $\det(\rho_f)$ and $\det(\rho_{f'})$ are both totally odd, since f and f' are weight one forms. Hence $\xi \circ \mathrm{ver}_{K/F}$ is totally even, as desired. □

Hence ξ can be regarded as a p-power order character of $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$. The restriction of such a character to the product $\prod_{i=1}^t U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i) \times (O_K/\mathfrak{n}_0)^\times$ factors through the quotient (7.7.5) below (Section 7.1.2). Hence we obtain the upper bound in Theorem 7.1.5 (2) if $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is finite.

Secondly we assume that $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is infinite. We first show that, provided Leopoldt's conjecture for F and p is true, there are infinitely many classical weight one forms that arise from the Hida community containing $\mathcal{F}$ (Section 7.1.3). Since a Hida community is a finite set, there exists a member $\mathcal{G}$ in the community from which infinitely many such specializations occur. In view of Theorem 6.3.1, K/F has to be totally imaginary and $\mathcal{G}$ has CM by K (Section 7.1.4).

7.1.2 Global class field theory

We begin with some observation on the narrow ray class group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$. Let $\overline{O_K^\times}$ be the closure (taken in $\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times$) of the image of $O_K^\times$ under the diagonal embedding $O_K^\times \to \prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times$. By global class field theory, we know that $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$

is sitting in the exact sequence:

$$\begin{aligned} 1 \longrightarrow \quad & \overline{O_K^\times} \longrightarrow \prod_{i=1}^{t} O_{K_{\mathfrak{Q}_i}}^\times \times (O_K/\mathfrak{n}_0)^\times \\ \longrightarrow \quad & \mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty) \longrightarrow \mathrm{Cl}_K^+ \longrightarrow 1. \end{aligned} \tag{7.7.3}$$

Here, for a number field E we denote by Cl_E the ideal class group of E and by Cl_E^+ the narrow ideal class group of E. Notice that the p-part of $|\mathrm{Cl}_K^+|$ and $|\mathrm{Cl}_K|$ are equal, since p is odd. In particular we see that $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is a finite group if and only if the $\mathbb{Z}_p$-rank of $\overline{O_K^\times}$ is equal to $g = [F : \mathbb{Q}]$. Let $\xi : \mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$ be a p-power order character. Then ξ restricted to the product $\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times \times (O_K/\mathfrak{n}_0)^\times$ factors through the quotient

$$\left(\left(\prod_{i=1}^{t} U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)\right) / \overline{U_K(\mathcal{Q})}\right) \times (O_K/\mathfrak{n}_0)^\times. \tag{7.7.4}$$

Suppose further that $\xi \circ \mathrm{ver}_{K/F} : G_F^{\mathrm{ab}} \to \overline{\mathbb{Q}}_p^\times$ is unramified outside p. Note that $\xi \circ \mathrm{ver}_{K/F}$ is unramified outside $\mathfrak{n}_0 p$ because ξ is unramified outside $\mathfrak{n}_0\mathcal{Q}$. However it is not always the case that ξ is unramified at primes dividing $\mathfrak{n}_0$. Therefore this additional assertion should cause some constraint on the behavior of ξ at the primes of K lying over $\mathfrak{n}_0$. We describe this constraint explicitly in the following lemma:

Lemma 7.1.8. *Let ξ be a character satisfying the properties* (1) *through* (3) *in Lemma* 7.1.7. *Then ξ restricted to $\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times \times (O_K/\mathfrak{n}_0)^\times$ factors through the quotient*

$$\left(\left(\prod_{i=1}^{t} U_{K_{\mathfrak{Q}_i}}(\mathfrak{Q}_i)\right) / \overline{U_K(\mathcal{Q})}\right) \times \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{split in } K}} \mathbb{F}_\mathfrak{l}^\times \times \prod_{\substack{\mathfrak{l}|\mathfrak{n}_0 \\ \text{inert in } K}} \mathbb{F}_{\mathfrak{l}^2}^\times / \mathbb{F}_\mathfrak{l}^\times \tag{7.7.5}$$

of (7.7.4). *Here for a prime ideal $\mathfrak{l}$ of F dividing $\mathfrak{n}_0$, we denote by $\mathbb{F}_\mathfrak{l}$ the residue field $O_F/\mathfrak{l}$. As for a prime factor $\mathfrak{l}$ of $\mathfrak{n}_0$ that is inert in K, $\mathbb{F}_{\mathfrak{l}^2}$ is the residue field $O_K/\mathfrak{l}O_K$ which is the unique quadratic extension of $\mathbb{F}_\mathfrak{l}$.*

Proof. In view of class field theory, the transfer map $\mathrm{ver}_{K/F} : G_F^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ corresponds to a canonical map from $\mathbb{A}_F^\times/F^\times$ to $\mathbb{A}_K^\times/K^\times$. With this in mind, what we have to do is to characterize the set of p-power order characters $\xi : \prod_{\mathfrak{L}|\mathfrak{l}} O_{K_\mathfrak{L}}^\times \to \overline{\mathbb{Q}}_p^\times$ whose restriction to $O_{F_\mathfrak{l}}^\times$ is trivial, for each prime ideal $\mathfrak{l}$ of F dividing $\mathfrak{n}_0$. Since $\mathfrak{n}_0$ is prime to p and ξ is of p-power order, ξ is trivial on the principal unit group $U_{K_\mathfrak{L}}(\mathfrak{L})$ in $O_{K_\mathfrak{L}}^\times$, and it is enough to investigate the quotient group

$$\left(\prod_{\mathfrak{L}|\mathfrak{l}} (O_{K_\mathfrak{L}}/\mathfrak{L})^\times\right) / (O_{F_\mathfrak{l}}/\mathfrak{l})^\times \tag{7.7.6}$$

Suppose first that $\mathfrak{l}$ splits in K, say $\mathfrak{l}O_K = \mathfrak{L}\mathfrak{L}^\sigma$, for distinct prime ideals $\mathfrak{L}$ and $\mathfrak{L}^\sigma$ of K. Then σ induces an isomorphism $\sigma : O_{K_\mathfrak{L}}^\times \cong O_{K_{\mathfrak{L}^\sigma}}^\times$ and the group (7.7.6) is isomorphic to $(O_{K_{\mathfrak{L}^\sigma}}/\mathfrak{L}^\sigma)^\times$ by the map taking $(x, y) \bmod (O_{F_\mathfrak{l}}/\mathfrak{l})^\times$ to $y \cdot (x^\sigma)^{-1}$. Secondly if $\mathfrak{l}$ is inert in K, the group (7.7.6) is nothing but $\mathbb{F}_{\mathfrak{l}^2}^\times/\mathbb{F}_\mathfrak{l}^\times$, as desired. Finally the group (7.7.6) is trivial if $\mathfrak{l}$ is ramified in K. □

Now we shall prove Theorem 7.1.5 (2). If $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is a finite group, so is the group (7.7.5) and the number of classical weight one specializations of $\mathcal{F}$ is bounded by the order of the p-Sylow group of (7.7.5). This is the desired assertion of Theorem 7.1.5 (2).

We now move on to Theorem 7.1.5 (1). To prove it, we need to take a closer look at the narrow ray class groups. Let $\widetilde{F_\infty}$ be the composite of the $\mathbb{Z}_p$-extensions of F and $\mathbf{G}'$ the Galois group $\mathrm{Gal}(\widetilde{F_\infty}/F)$. As explained in Section 2 of [1] where this group is indicated by $\mathbf{G}$ instead of $\mathbf{G}'$, $\mathbf{G}'$ is isomorphic to the torsion-free part of $\mathrm{Gal}(M/F)$ where M is the maximal abelian extension of F unramified outside p, and we have $\mathrm{Gal}(M/F) = \mathbf{G}' \times \mathrm{Gal}(F'/F)$ so that $\mathrm{Gal}(F'/F)$ is a quotient of the narrow ray class group $\mathrm{Cl}_F(p^\alpha)$ of modulus p^α for some integer $\alpha \geq 1$. Also, $\mathbf{G}'$ is isomorphic to the $(1+\delta)$-copies of $\mathbb{Z}_p$, where δ is the defect in Leopoldt's conjecture for F and p.

Put $\mathrm{Cl}_K(\mathcal{Q}^\infty) = \varprojlim_r \mathrm{Cl}_K(\mathcal{Q}^r)$. In view of the exact sequence (7.7.3), $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ is infinite if and only if $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is infinite. To be more precise, this group lies in the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Cl}_K(\mathcal{Q}^\infty) & & & & \\
 & & & & \| & & & & \\
1 & \longrightarrow & \left(\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times\right)/\overline{O_K^\times} & \longrightarrow & \mathbb{A}_K^\times/K^\times U_K^{(\mathcal{Q})}(O_K)\mathbb{A}_{K,\infty}^\times & \longrightarrow & \mathrm{Cl}_K & \longrightarrow & 1, \\
 & & \uparrow \mathrm{ver}_{K/F} & & \uparrow \mathrm{ver}_{K/F} & & \uparrow \mathrm{ver}_{K/F} & & \\
1 & \longrightarrow & \left(\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times\right)/\overline{O_F^\times} & \longrightarrow & \mathbb{A}_F^\times/F^\times U_F^{(p)}(O_F)\mathbb{A}_{F,\infty}^+ & \longrightarrow & \mathrm{Cl}_F^+ & \longrightarrow & 1 \\
 & & & & \| & & & & \\
 & & & & \mathrm{Cl}_F(p^\infty) & & & &
\end{array} \tag{7.7.7}$$

whose horizontal lines are exact. Here, for $E = F$ or K and an integral ideal $\mathfrak{m}$ in O_E, we denote by $U_E^{(\mathfrak{m})}(O_E)$ the product $\prod_{v \in P_\mathrm{f}(E), v \nmid \mathfrak{m}} O_{E_v}^\times$. Note that the left vertical map is surjective, since $K_{\mathfrak{Q}_i}$ is identical to $F_{\mathfrak{q}_i}$ for each $i = 1, \ldots, t$.

We know that the $\mathbb{Z}_p$-rank of $\mathrm{Cl}_F(p^\infty)$ and the quotient $(\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times)/\overline{O_F^\times}$ is $1+\delta$. Moreover, the Galois group $\mathbf{G}'$ is known to be a direct summand of $(\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times)/\overline{O_F^\times}$, and we have $\mathrm{Cl}_F(p^\infty) \cong \mathbf{G}' \times \Delta$ where Δ is a finite group. Hence the $\mathbb{Z}_p$-rank of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ and $(\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times)/\overline{O_K^\times}$ are the same and at most $1+\delta$.

Now assume that $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ is infinite, and let $1+s$ be its $\mathbb{Z}_p$-rank where s is an integer so that $0 \leq s \leq \delta$. We choose topological generators $\tilde{\gamma}_0, \ldots, \tilde{\gamma}_\delta$ of $\mathbf{G}'$ so that the kernel of $\mathrm{ver}_{K/F} : (\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times)/\overline{O_F^\times} \to (\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times)/\overline{O_K^\times}$ restricted to $\mathbf{G}'$ is topologically generated by $\tilde{\gamma}_{s+1}, \ldots, \tilde{\gamma}_\delta$. Let $\mathbf{H}$ (resp. $\mathbf{H}'$) be the subgroup of $\mathbf{G}'$ topologically generated by $\tilde{\gamma}_0, \ldots, \tilde{\gamma}_s$ (resp. $\tilde{\gamma}_{s+1}, \ldots, \tilde{\gamma}_\delta$) so that $\mathbf{G}' = \mathbf{H} \times \mathbf{H}'$.

It is the following lemma that forces us to assume $\delta = 0$ in Theorem 7.1.5 (1).

Lemma 7.1.9. *For any p-power order character $\varepsilon : \mathbf{H} \to \overline{\mathbb{Q}}_p^\times$, there exists a p-power order character $\xi : \mathrm{Cl}_K(\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$ so that $\xi \circ \mathrm{ver}_{K/F} = \varepsilon$ on $\mathbf{H}$. In particular, any p-power order character $\varepsilon : \mathbf{G}' \to \overline{\mathbb{Q}}_p^\times$ is of the shape $\xi \circ \mathrm{ver}_{K/F}$ on $\mathbf{G}'$ for some character $\xi : \mathrm{Cl}_K(\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$ if and only if the $\mathbb{Z}_p$-rank of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ is equal to $1 + \delta$.*

Proof. Since $\mathrm{ver}_{K/F} : (\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times)/\overline{O_F^\times} \to (\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times)/\overline{O_K^\times}$ is surjective, we may assume that $x_i = \mathrm{ver}_{K/F}(\tilde{\gamma}_i)$ for $i = 0, \ldots, s$ topologically generate the torsion-free part of $(\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times)/\overline{O_K^\times}$, and hence that of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$. We define the p-power order character $\xi : \mathrm{Cl}_K(\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$ by putting $\xi(x_i) = \varepsilon(\tilde{\gamma}_i)$ for each $i = 0, \ldots, s$. This certainly defines a character of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ because the subgroup topologically generated by x_i for $i = 0, \ldots, s$ is a direct summand of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$. Then it is clear that $\xi \circ \mathrm{ver}_{K/F} = \varepsilon$ on $\mathbf{H}$. □

Remark 7.1.10. It should be noticed that $\tilde{\gamma}_0$ may not correspond to the cyclotomic variable in this choice of topological generators $\tilde{\gamma}_i$. Therefore, with Lemma 7.1.9 in mind, we do not know in general whether any p-power order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$ is of the shape $\xi \circ \mathrm{ver}_{K/F}$ for some character $\xi : \mathrm{Cl}_K(\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$. In other words, there may not exist infinitely many classical weight one forms f_ξ as in Lemma 7.1.12 whose nebentypus are of the shape $\psi\varepsilon$ for some fixed $\psi : \mathrm{Cl}_F(\mathfrak{n}_0 p^\alpha) \to \overline{\mathbb{Q}}^\times$ and (varying) $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$.

7.1.3 Characters and classical weight one forms

In this subsection, we attach a classical weight one cuspidal Hecke eigenform to each character ξ treated in the previous subsection.

Lemma 7.1.11. *Let ξ be a character satisfying the properties* (1) *through* (3) *in Lemma* 7.1.7 *and put $\rho_\xi = \mathrm{Ind}_K^F(\varphi\xi)$. Then ρ_ξ is an irreducible, totally odd representation and is unramified outside $\mathfrak{n}_0 p d_{K/F}$.*

Proof. Since $\mathcal{F}$ is residually of dihedral type, the reduction $\bar{\rho}_\xi = \bar{\rho}_{\mathcal{F}}$ is irreducible, and so is ρ_ξ. We have $\det(\rho_\xi) = \varepsilon_{K/F} \cdot ((\varphi\xi) \circ \mathrm{ver}_{K/F})$ as a character of G_F^{ab} since ρ_ξ is induced by $\varphi\xi$. As p is odd and ξ is a p-power order character, we have $(\xi \circ \mathrm{ver}_{K/F})(c) = 1$ for any complex conjugate c in G_F. Therefore the parity ρ_ξ and ρ_f are the same and thus ρ_ξ is totally odd. The last assertion follows from the conductor-discriminant formula (5.5.5). □

Lemma 7.1.12. *Let ρ_ξ be the induced representation in Lemma* 7.1.11. *Then there exists a holomorphic Hilbert cusp form f_ξ over F of parallel weight one so that $\rho_{f_\xi} \cong \rho_\xi$.*

Proof. Let τ_ξ be the automorphic form on $GL_1(\mathbb{A}_K)$ associated to ξ via class field theory. Let $\pi = \mathrm{AI}_K^F(\tau_\xi)$ be the automorphic induction of τ_ξ which is a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ so that $L(s,\pi) = L(s,\rho_\xi)$. Note that the irreducibility of ρ_ξ implies the cuspidality of π. It follows from Lemma 7.1.11 that ρ_ξ is totally odd and then we see that the Weil-Deligne representation $WD_\infty(\rho_\xi)$ associated to ρ_ξ at ∞ is described as

$$WD_\infty(\rho_\xi) : W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \{1, j\} \longrightarrow GL_2(\mathbb{C}),\ \mathbb{C} \ni z \mapsto I_2,\ j \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(see the argument in Proposition 2.1 of [21] and also [2]). It follows from this that for each infinite place ∞ of F, one has

$$\pi_\infty \simeq \pi(1, \mathrm{sgn}) := \mathrm{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(1 \otimes \mathrm{sgn})$$

where $B(\mathbb{R})$ is the Borel sugroup consisting of upper-triangular matrices in $GL_2(\mathbb{R})$, $1 \otimes \mathrm{sgn} : B(\mathbb{R}) \to \mathbb{R}^\times$ is the group homomorphism defined by $\left[\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right] \mapsto \mathrm{sgn}(d)$ and $\mathrm{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(1 \otimes \mathrm{sgn})$ is the parabolic induction by $1 \otimes \mathrm{sgn}$. This implies π generated by a holomorphic Hilbert cusp form of parallel weight one. □

By Lemmas 7.1.11 and 7.1.12, we know that there exists a classical weight one cuspidal Hecke eigenform f_ξ such that ρ_ξ is equivalent to the representation ρ_{f_ξ} associated to f_ξ. We shall observe that the p-stabilization(s) of f_ξ can be obtained from the Hida community containing $\mathcal{F}$. More precisely, we are going to establish the following

Proposition 7.1.13. *Let f_ξ be as in Lemma* 7.1.12*. Then there exists a primitive ordinary Λ-adic cusp form $\mathcal{F}_\xi$ which specializes to the p-stabilization(s) of f_ξ. Moreover, if ξ is unramified at the prime ideals of K lying over $\mathfrak{n}_0$, then the tame level of $\mathcal{F}_\xi$ is $\mathfrak{n}_0$ and hence $\mathcal{F}_\xi$ belongs to the same Hida community as $\mathcal{F}$.*

Proof. For each $i = 1, \ldots, t$, since φ and ξ are unramified at $\mathfrak{Q}_i^\sigma$, $\varphi\xi$ is ramified at $\mathfrak{Q}_i$ if and only if $N_{K/F}(\mathrm{cond}(\varphi\xi))$ is divisible by $\mathfrak{q}_i$. The conductor-discriminant formula (5.5.5) $\mathrm{cond}(\rho_{f_\xi}) = N_{K/F}(\mathrm{cond}(\varphi\xi))d_{K/F}$ implies that these equivalent conditions hold true if and only if $\mathrm{cond}(\rho_{f_\xi})$ is divisible by $\mathfrak{q}_i$. As the conductor of ρ_{f_ξ} is equal to the level of f_ξ, $\varphi\xi$ is ramified at $\mathfrak{Q}_i$ if and only if the level of f_ξ is divisible by $\mathfrak{q}_i$. If this is the case, we have $(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i}) = 0$ and $(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i^\sigma}) \neq 0$, and thus the normalized Fourier coefficient $c(\mathfrak{q}_i, f_\xi) = (\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i^\sigma})$ is a root of unity. Therefore f_ξ is stabilized and ordinary at $\mathfrak{q}_i$. Suppose, on the contrary, that $\varphi\xi$ is unramified at $\mathfrak{Q}_i$. Then ρ_{f_ξ} is unramified at $\mathfrak{q}_i$ we have

$$\det(I_2 - X\rho_{f_\xi}(\mathrm{Frob}_{\mathfrak{q}_i})) = (1 - X(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i}))(1 - X(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i^\sigma}))$$

and both $(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i})$ and $(\varphi\xi)(\mathrm{Frob}_{\mathfrak{Q}_i^\sigma})$ are roots of unity. Therefore the two $\mathfrak{q}_i$-stabilizations of f_ξ are both ordinary at $\mathfrak{q}_i$. In any case we can conclude that there is a p-stabilized newform f_ξ^* such that the associated representation is equivalent to ρ_ξ. Then by a theorem of Wiles (Theorem 3; p. 532 of [46]) there exists a primitive ordinary Λ-adic cusp form $\mathcal{F}_\xi$ specializing to f_ξ^*. In particular the residual representation $\bar{\rho}_{\mathcal{F}_\xi}$ is equivalent to $\bar{\rho}_{\mathcal{F}}$ by construction.

The tame level of $\mathcal{F}_\xi$ is equal to the prime-to-p part of the level of f_ξ. If ξ is unramified at the prime ideals of K sitting above $\mathfrak{n}_0$, then $N_{K/F}(\mathrm{cond}^{(p)}(\varphi\xi))$ is equal to $N_{K/F}(\mathrm{cond}^{(p)}(\varphi))$. We apply the conductor-discriminant formula (5.5.4) to see that $\mathrm{cond}^{(p)}(\rho_{f_\xi})$ and $\mathrm{cond}^{(p)}(\rho_f)$ coincide. Thus the prime-to-p part of the level of f_ξ and that of f are the same. Since $\mathcal{F}$ is primitive of tame level $\mathfrak{n}_0$, the prime-to-p part of the level of f is $\mathfrak{n}_0$. Therefore the prime-to-p part of the level of f_ξ is also $\mathfrak{n}_0$, which shows the assertion. □

7.1.4 The proof of the main theorem

Now we complete the proof of Theorem 7.1.5 (1).

Proof. Suppose that the ray class group $\mathrm{Cl}_K(\mathfrak{n}_0\mathcal{Q}^\infty)$ is an infinite group and $\delta = 0$. Then $\mathbf{G}'$ is identical to $\mathbf{G}$ by definition and a Λ-adic cusp form in Definition 2 of [1] is nothing but the one introduced by Wiles in [46] and adopted in this paper. It follows by the argument in Section 7.1.2 that the $\mathbb{Z}_p$-rank of $\mathrm{Cl}_K(\mathcal{Q}^\infty)$ is equal to one, and the kernel of the transfer map $\mathrm{ver}_{K/F} : (\prod_{i=1}^t O_{F_{\mathfrak{q}_i}}^\times)/\overline{O_F^\times} \to (\prod_{i=1}^t O_{K_{\mathfrak{Q}_i}}^\times)/\overline{O_K^\times}$ is a finite group. Then Lemma 7.1.9 tells us that any p-power order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$ is of the shape $\xi \circ \mathrm{ver}_{K/F}$ for some character $\xi : \mathrm{Cl}_K(\mathcal{Q}^\infty) \to \overline{\mathbb{Q}}_p^\times$. In particular, there are infinitely many characters $\xi : G_K \to \overline{\mathbb{Q}}_p^\times$ of p-power order that are unramified outside $\mathcal{Q}$ and the infinite places, and such that the composite $\xi \circ \mathrm{ver}_{K/F} : G_F \to \overline{\mathbb{Q}}_p^\times$ is unramified outside p. Each of such characters ξ produces a p-stabilized newform f_ξ^* of weight one which arises from the Hida community $\{\mathcal{F}\}$ and such that the Galois representation ρ_ξ associated to f_ξ^* satisfies $\rho_\xi \cong \varepsilon_{K/F} \otimes \rho_\xi$. Namely, the community $\{\mathcal{F}\}$ has infinitely many classical weight one specializations whose associated Galois representations are induced by characters of K. Since a Hida community is a finite set, there exists a member $\mathcal{G}$ of $\{\mathcal{F}\}$ which admits infinitely many such specializations. With Theorem 6.3.1 in mind, we conclude that the family $\mathcal{G}$ has CM by K and hence K/F is totally imaginary. This concludes the proof of Theorem 7.1.5 (1). □

7.1.5 Families residually of CM type

As we have declared in Remark 7.1.6 (2), we end our analysis in the dihedral case by giving an upper bound when $\mathcal{F}$ is residually of CM type.

Let $\mathcal{F}$ be a non-CM primitive ordinary Λ-adic cusp form of tame level $\mathfrak{n}_0$ such that $\bar{\rho}_{\mathcal{F}} \cong \mathrm{Ind}_K^F(\bar{\varphi})$ for some totally imaginary quadratic extension K/F and a character $\bar{\varphi} : G_K \to \mathbb{F}^\times$. Since $\mathcal{F}$ is not of CM type, there is at least one prime ideal $\mathfrak{l}$ in F that is inert in K, prime to $\mathfrak{n}_0 p$ and the trace $\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_\mathfrak{l})$ is non-zero (see the proof of Theorem 6.3.1). Therefore $\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_\mathfrak{l})$ is contained in only finitely many height one prime ideals of Λ_L. In particular, there are only finitely many height one prime ideals of Λ_L that contain $\mathrm{Tr}\rho_{\mathcal{F}}(\mathrm{Frob}_\mathfrak{l})$ and sit above the kernel of $\varphi_{1,\varepsilon}$ for some p-power order character $\varepsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$. Let $\lambda_{\mathcal{F},\mathfrak{l}}$ denote this number and put $\lambda_{\mathcal{F}} = \min\{\lambda_{\mathcal{F},\mathfrak{l}} \mid \mathfrak{l} \text{ is inert in } K \text{ and prime to } \mathfrak{n}_0 p\}$.

Proposition 7.1.14 (cf. Lemma 6.5; p. 682 of [6]). *Let $\mathcal{F}$ be a non-CM family that is residually of CM type by K in the above sense. Then the number of classical weight one specializations of $\mathcal{F}$ is bounded by $\lambda_{\mathcal{F}}$.*

7.2 Residually of exceptional case: Main Theorem 1.2.3

In this section, we will give an upper bound on the number of classical weight one forms in a primitive ordinary Λ-adic cusp form $\mathcal{F}$ which is residually of exceptional type. Note that $\mathcal{F}$ can not be a CM family, since any CM family is residually of dihedral type (cf. Section 6.1). As the image in $PGL_2(\mathbb{C})$ of the Artin representation attached to each classical weight one specialization of $\mathcal{F}$ has bounded order, say 24, 12 or 60, our analysis in this case is much simpler that the dihedral case.

As in Section 7.1, we denote simply by ρ_f the $\mathfrak{p}$-adic Galois representation $\rho_{f,\mathfrak{p}}$ attached to a classical weight one form f, by abuse of notation.

7.2.1 Statement and the proof of the main theorem

We now state the main theorem in the exceptional case. Let p^r be the p-part of the class number $|\mathrm{Cl}_F|$ of F and $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ the prime ideals of F lying over p.

Theorem 7.2.1. *If $\mathcal{F}$ is residually of exceptional type, then $\mathcal{F}$ has at most $a \cdot b$ classical weight one specializations, where*

- *$a = 1$, and*
- *$b = p^r$, except $p = 3$ or 5, in which case $b = 2^t \cdot p^r$.*

In particular, $\mathcal{F}$ has at most one weight one specialization if $p \geq 7$ and the class number $|\mathrm{Cl}_F|$ is not divisible by p.

Remark 7.2.2. In the original paper [27] on which this thesis is based, the definition of a in Theorem 7.2.1 is the following:

- $a = 1$, except $p = 5$ and the type of $\mathcal{F}$ is A_5, in which case $a = 2$.

However, according to the classification of Dickson in Theorem 4.3.1, the image of $\bar{\rho}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is exceptional only if the order of $\mathrm{Im}(\bar{\rho}_{\mathcal{F}})$ in $GL_2(\mathbb{F})$ is prime to p. Therefore, as explained in the proof, if $p = 5$, then $\mathcal{F}$ cannot be of A_5 type.

Proof. The proof relies on that of Theorem 5.1; p. 676 of [6]. We note that the $\mathfrak{p}$-adic Galois representation $\rho_f : G_F \to GL_2(O)$ attached to a classical weight one specialization f of $\mathcal{F}$ (if any) is irreducible and hence determined by the trace. Therefore it is enough to show that there is at most one choice for the projective trace $(\mathrm{Tr}\rho_f)^2/\det\rho_f$, and there are at most b choices for the determinant of ρ_f. Indeed, since $\mathrm{Tr}\rho_f$ is congruent to $\mathrm{Tr}\bar{\rho}_{\mathcal{F}}$ and p is odd, $\mathrm{Tr}\rho_f$ is uniquely determined by $(\mathrm{Tr}\rho_f)^2$.

Since $\mathcal{F}$ is residually of A_4, S_4 or A_5 type, the image of $\bar{\rho}_{\mathcal{F}}(g)$ in $PGL_2(\mathbb{F})$ has order at most five. A standard computation shows that the projective trace $\mathrm{Tr}\bar{\rho}_{\mathcal{F}}(g)^2/\det\bar{\rho}_{\mathcal{F}}(g)$ of $\bar{\rho}_{\mathcal{F}}(g)$ varies as in **Table 7.1**, according to the order of $\bar{\rho}_{\mathcal{F}}(g)$ in $PGL_2(\mathbb{F})$:

Table 7.1: Order and Projective Trace

order of $\bar{\rho}_{\mathcal{F}}(g)$ in $PGL_2(\mathbb{F})$	1	2	3	4	5
projective trace of $\bar{\rho}_{\mathcal{F}}(g)$	4	0	1	2	a root of X^2-3X+1

Therefore, if $\mathcal{F}$ is residually of A_4 or S_4 type, there is at most one choice for the projective trace $(\mathrm{Tr}\rho_f)^2/\det\rho_f$ of f. As for A_5 case, the two roots of X^2-3X+1 are congruent modulo p if and only if $p=5$. Since the image of $\bar{\rho}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ is of exceptional type, its order should be prime to p (cf. Theorem 4.3.1). Hence if $p=5$, then the image of $\bar{\rho}_{\mathcal{F}}$ in $PGL_2(\mathbb{F})$ cannot be isomorphic to A_5.

It remains to be shown that there are at most b choices for the determinant of $\det(\rho_f)$ with a given projective trace. Since $\mathcal{F}$ is ordinary, we have

$$\rho_f|_{I_{\mathfrak{q}}} \cong \begin{pmatrix} \det(\rho_f) & * \\ 0 & 1 \end{pmatrix}$$

for each prime $\mathfrak{q}$ of F lying over p. This implies that $\rho_f(I_{\mathfrak{q}})$ injects into $PGL_2(O)$. Therefore the order of $\det(\rho_f)=\psi\varepsilon$ restricted to $I_{\mathfrak{q}}$ is at most five, where $\varepsilon:\mathbf{G}\to\overline{\mathbb{Q}}_p^\times$ is the p-power order character such that $\varphi_{1,\varepsilon}(\mathcal{F})=f$. Recall that the character ψ of $\mathcal{F}$ is tamely ramified at $\mathfrak{q}$ and thus $\psi|_{I_{\mathfrak{q}}}$ has order prime to p. If $p\geq 7$, this implies that $\varepsilon|_{I_{\mathfrak{q}}}=1$ for each prime ideal $\mathfrak{q}$ lying over p. Consequently ε is unramified everywhere and hence is a character of the ideal class group of F. This proves the assertion when $p\geq 7$. Throughout the rest of the proof, we assume $p=3$ or 5. Suppose that $f=\varphi_{1,\varepsilon}(\mathcal{F})$ and $f'=\varphi_{1,\varepsilon'}(\mathcal{F})$ are two classical weight one forms in $\mathcal{F}$ having the same projective trace. Then there exists a p-power order character $\eta:G_F\to\overline{\mathbb{Q}}_p^\times$ so that $\rho_f\cong\eta\otimes\rho_{f'}$. This immediately yields the equality $\varepsilon=\eta^2\varepsilon'$. Note that η is uniquely determined by f'. Since ε and ε' are unramified outside p, so is η^2. Moreover η is of p-power order and p is odd. Thus η is unramified outside p. We investigate the behavior of η at each prime $\mathfrak{q}$ lying over p. The relation $\rho_f\cong\eta\otimes\rho_{f'}$ restricted to $I_{\mathfrak{q}}$ implies that at least one of the following occurs:

1. $\eta=1$ and $\varepsilon'=\varepsilon$ on $I_{\mathfrak{q}}$;

2. $\eta=\varepsilon=(\varepsilon')^{-1}$ and $\psi=1$ on $I_{\mathfrak{q}}$.

Let S be the set of prime ideals $\mathfrak{q}$ lying over p such that $\varepsilon'|_{I_{\mathfrak{q}}}\neq\varepsilon|_{I_{\mathfrak{q}}}$. Then η is unramified outside S and $\eta|_{I_{\mathfrak{q}}}=\varepsilon|_{I_{\mathfrak{q}}}$ for each $\mathfrak{q}\in S$. Let $\zeta:G_F\to\overline{\mathbb{Q}}_p^\times$ be a p-power order character and suppose that $\eta\zeta$ satisfies the same ramification condition as that of η. Then ζ is unramified everywhere so there are p^r choices for ζ (recall that p^r is the p-part of the class number $|\mathrm{Cl}_F|$). Obviously the set S depends on ε' and there are at most 2^t choices for S. Hence $\mathcal{F}$ has at most $2^t \cdot p^r$ weight one specializations, which proves the theorem. □

7.2.2 Classical weight one specializations of a Hida community

Recall that $\mathcal{F}$ is said to be p-distinguished if $\overline{\mathcal{E}}_{\mathfrak{q}} \neq \overline{\mathcal{D}}_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q}$ lying over p, where $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ are as in (6.6.2). It makes sense to speak of the residual representation and p-distinguishability of a Hida community. The following proposition is a generalization of Proposition 5.2; p. 678 of [6].

Proposition 7.2.3. *Let $p \geq 7$ be a prime number that splits completely in F and $\{\mathcal{F}\}$ is a Hida community of exceptional type which is p-distinguished and the residual representation $\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\overline{F}/F(\zeta_p))$*. Assume further that the tame level $\mathfrak{n}_0$ of $\{\mathcal{F}\}$ is the same as the conductor of $\bar{\rho}_{\mathcal{F}}$. Then $\{\mathcal{F}\}$ has at least one classical weight one specialization f. Moreover, any other classical weight one specialization of $\{\mathcal{F}\}$ can be written as $f \otimes \eta$, where $\eta : G_F \to \mathbb{Q}_p^\times$ is a p-power order character of conductor dividing $\mathfrak{n}_0$.*

Proof. Since $p \geq 7$ the order of the image of $\bar{\rho}_{\mathcal{F}}$ in $GL_2(\mathbb{F})$ is prime to p. Therefore we can take the Teichmüller lift $\tilde{\rho} : G_F \to GL_2(W(\mathbb{F}))$ of $\bar{\rho}_{\mathcal{F}}$, where $W(\mathbb{F})$ is the ring of Witt vectors of $\mathbb{F}$. The reduction map $GL_2(W(\mathbb{F})) \to GL_2(\mathbb{F})$ induces an isomorphism of $\tilde{\rho}(G_F)$ onto $\bar{\rho}_{\mathcal{F}}(G_F)$. This implies that $\tilde{\rho}$ is an Artin representation and $\tilde{\rho}(G_F)$ is a semi-simple group. As $\mathcal{F}$ is ordinary, $\bar{\rho}_{\mathcal{F}}|_{D_{\mathfrak{q}}}$ is a direct sum of the characters $\overline{\mathcal{E}}_{\mathfrak{q}}$ and $\overline{\mathcal{D}}_{\mathfrak{q}}$ for each prime $\mathfrak{q}$ lying over p. Therefore $\tilde{\rho}|_{D_{\mathfrak{q}}}$ is also a direct sum of two characters. We apply the modularity lifting theorem below to obtain a classical weight one form f such that $\rho_f \cong \tilde{\rho}$:

Theorem 7.2.4 (Theorem 3; p. 999 of [34])**.** *Let p be an odd prime that splits completely in F. Let E be a finite extension of $\mathbb{Q}_p$ with the ring of integers O and the maximal ideal $\mathfrak{m}$, and $\rho : G_F \to GL_2(O)$ a continuous totally odd representation satisfying the following conditions:*

- *ρ ramifies at only finitely many primes;*
- *$\bar{\rho} = (\rho \bmod \mathfrak{m})$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$, and has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of F above p;*
- *For every prime $\mathfrak{q}$ of F above p, the restriction $\rho|_{D_{\mathfrak{q}}}$ is the direct sum of 1-dimensional characters $\chi_{\mathfrak{q},1}$ and $\chi_{\mathfrak{q},2}$ of $D_{\mathfrak{q}}$ such that the images of the inertia subgroup at $\mathfrak{q}$ are finite and $(\chi_{\mathfrak{q},1} \bmod \mathfrak{m}) \not\equiv (\chi_{\mathfrak{q},2} \bmod \mathfrak{m})$.*

Then there exist an embedding $\iota : E \to \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and a classical cuspidal Hilbert modular eigenform f of weight one such that $\iota \circ \rho : G_F \to GL_2(\mathbb{C})$ is isomorphic to the representation associated to f by Rogawski-Tunnell [33].

By a theorem of Wiles (Theorem 3; p. 532 of [46]) there exists a Hida family $\mathcal{G}$ specializing to (the p-stabilization(s) of) f. The tame level of $\mathcal{G}$ is equal to the prime-to-p part of the level of f. By our assumption, $\mathrm{cond}(\rho_f) = \mathrm{cond}(\tilde{\rho}) = \mathrm{cond}(\bar{\rho}_{\mathcal{F}})$ is equal to $\mathfrak{n}_0$.

As for the second claim, let f and g be classical weight one specializations of the community $\{\mathcal{F}\}$, say $f = \varphi_{1,\varepsilon}(\mathcal{F})$ and $g = \varphi_{1,\varepsilon'}(\mathcal{G})$ for members $\mathcal{F}, \mathcal{G} \in \{\mathcal{F}\}$ and p-power order characters $\varepsilon, \varepsilon' : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$. As the projective image of ρ_f and ρ_g are equivalent, there exists a p-power order character $\eta : G_F \to \overline{\mathbb{Q}}_p^\times$ so that $\rho_g \cong \eta \otimes \rho_f$. By determinant considerations, one can show, in a fashion similar to the proof of Theorem 7.2.1, that η is unramified outside $\mathfrak{n}_0 p$. Let $\mathfrak{q}$ be a prime of F lying over p. The relation $\rho_g \cong \eta \otimes \rho_f$ restricted to $I_\mathfrak{q}$ implies that at least one of the following holds true:

1. $\eta = 1, \psi_{\mathcal{F}} = \psi_{\mathcal{G}}$ and $\varepsilon' = \varepsilon$ on $I_\mathfrak{q}$;
2. $\eta = \varepsilon^{-1} = \varepsilon'$ and $\psi_{\mathcal{F}} = \psi_{\mathcal{G}} = 1$ on $I_\mathfrak{q}$.

Here, the nebentype of $\mathcal{F}$ (resp. of $\mathcal{G}$) is denoted by $\psi_{\mathcal{F}}$ (resp. $\psi_{\mathcal{G}}$). Recall that $\varepsilon|_{I_\mathfrak{q}} = \varepsilon'|_{I_\mathfrak{q}} = 1$ provided $p \geq 7$ (see the proof of Theorem 7.2.1). Thus in both cases we have $\eta|_{I_\mathfrak{q}} = 1$. Therefore η is unramified outside $\mathfrak{n}_0$, and the conductor of η divides $\mathfrak{n}_0$ since we have $\psi_{\mathcal{G}}\varepsilon' = \eta^2 \psi_{\mathcal{F}}\varepsilon$. □

We end this thesis by making two remarks on Proposition 7.2.3.

Remark 7.2.5.

1. In the proposition, we assumed that p splits completely in F, $\{\mathcal{F}\}$ is p-distinguished, and the residual representation $\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$. This is due to our application of modularity lifting theorem (Theorem 7.2.4) in the course of the proof.
2. Under the assumptions of Proposition 7.2.3, if the order of the narrow ray class group $\mathrm{Cl}_F(\mathfrak{n}_0)$ of F of modulus $\mathfrak{n}_0$ is prime to p, then any p-power order character of $\mathrm{Cl}_F(\mathfrak{n}_0)$ is trivial and so is η in the proof of Proposition 7.2.3. Hence the community $\{\mathcal{F}\}$ has a unique classical weight one specialization.

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