On Extensions of Commutative Association Schemes

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Abstract

Let $(X, \mathscr{R})$ denote a commutative association scheme on s classes. For each integer $n \geqslant 2$, we construct a bigger commutative association scheme having $\binom{n+s}{s} - 1$ classes on the vertex set X^n. This new scheme is called the extension of $(X, \mathscr{R})$ with length n. The extension schemes from one-class initial schemes are in fact the Hamming association schemes, which are metric and cometric. This thesis serves as one of the systematic studies on extensions schemes obtained from initial schemes with at least two classes. Such commutative association schemes are neither metric nor cometric.

Chapter 1

Introduction

An s-class (commutative) association scheme on a finite set X of vertices is a partition $\{R_1, \ldots, R_s\}$ of the edges of the complete graph for which the adjacency matrices A_i of the subgraphs (X, R_i) satisfy certain axioms. First studied by statisticians in the context of partially balanced designs, association schemes remain to be objects of interest to this very day and are regarded as the most important unifying concept in algebraic combinatorics. A special family of association schemes known as metric and cometric schemes possesses many interesting regularity and duality properties.

Examples of association schemes that are both metric and cometric are Hamming schemes and Johnson schemes. Delsarte [18] used these examples as basis in studying codes and designs from the point of view of association schemes. He indicated how an association scheme with s classes on a finite set X can be extended (by a natural process called extension) to produce another scheme with $\binom{n+s}{s} - 1$ classes on the set X^n. In particular, the special case at $s = 1$ yields the Hamming scheme of length n over the alphabet X. The codewords of length n over X are then viewed as vertices of the extension scheme. With this approach, Delsarte was able to cover various topics such as: finding an upper bound on the number of words in codes of given length and minimum distance (the famous linear programming bound), extending the concept of duality of linear codes to duality of additive codes (generalized MacWilliams identity), and finding designs from codes (an analog of Assmus–Mattson theorem on cometric schemes).

A fundamental tool in studying association schemes is the Bose–Mesner algebra. In fact, the famous linear programming bound by Delsarte is based on the representation theory of the Bose–Mesner algebra. In the 1990s, Terwilliger [63, 64, 65] introduced a larger, non-commutative, semisimple matrix $\mathbb{C}$-algebra attached to each vertex of an association scheme. This contains the Bose–Mesner algebra, and is now known as Terwilliger algebra. Recent advances show that the representation theory of Terwilliger algebras proved to be invaluable in the study of codes (and other areas). Schrijver [51], for one, used the Terwilliger algebra of binary Hamming schemes along with semidefinite programming to improve upper bounds on the number of words in codes with given length and minimum distance. With the same approach, Gijswijt, Schrijver and Tanaka [22] used the Terwilliger algebra of Hamming schemes to improve upper bounds for the non-binary case.

In studying codes and their Hamming weight enumerators (see Section 4.4.1), we consider the Hamming association schemes. The theory of the Terwilliger algebra has been most successful when the association scheme is both metric and cometric, and Hamming schemes possess these two properties. A famous theorem of Leonard [5, p. 263], [41], states that the class of metric and cometric association schemes characterizes the univariate Askey–Wilson orthogonal polynomials and some of its limiting cases. In particular, Hamming association schemes correspond to univariate Krawtchouk polynomials. Go [23] described the irreducible modules of the Terwilliger algebras of the binary Hamming association schemes. She showed (implicitly) that the Terwilliger algebras are in this case homomorphic images of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ of the rank one Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. This relationship between univariate Krawtchouk polynomials and $\mathfrak{sl}_2(\mathbb{C})$ are discussed further by Nomura and Terwilliger [49] at a more abstract linear algebraic level. On the other hand, we consider an extension scheme called Lee association schemes over $\mathbb{Z}_4$ when dealing with $\mathbb{Z}_4$-codes and their symmetrized weight enumerators (see Section 4.4.2). Lee association schemes over $\mathbb{Z}_4$ are extensions of a 2-class commutative association scheme, and are neither metric nor cometric. The structure of the Bose–Mesner algebra of Lee association schemes is known as discussed in [24] or [44]. From the result of Mizukawa and Tanaka [44], it follows that Lee schemes over $\mathbb{Z}_4$ correspond to bivariate Krawtchouk polynomials, also known as Rahman polynomials. Recently, Iliev and Terwilliger [37] studied Rahman polynomials from the point of view of the rank two Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (see also [36]). The first objective of this paper is to describe in detail the irreducible modules of the Terwilliger algebras of Lee association schemes over $\mathbb{Z}_4$.

In situations where we focus on a more complicated type of weight enumerator of a block code (just like in [31], [57], [59]), we think of the code (say, of length n) in question as lying in a structure much finer than a Hamming association scheme; that is to say, codewords are vertices of an extension of an s-class commutative association scheme with $s \geqslant 2$. The resulting extension scheme is neither metric nor cometric in this case. It turns out that the representation theory of Terwilliger algebras of some extension schemes (together with available tools from multivariable polynomial interpolation) is quite handy in proving an Assmus–Mattson-type theorem that works for more complicated weight enumerators (see Theorem 4.0.1 for the original Assmus–Mattson theorem). This is our second objective in this paper.

Recently, there is a growing interest in studying spectra of graphs from the point of view of quantum probability theory. In this perspective, the adjacency algebra of a k-regular graph G with vertex set X and adjacency matrix A is viewed as an algebraic probability space with respect to the tracial state $\varphi_{\mathrm{tr}} = |X|^{-1}\mathrm{tr}$. Hence, the matrix A is treated as a real random variable (with mean 0 and variance k), and a unique probability measure ν_G on $\mathbb{R}$ exists (see (5.1)) such that

$$\varphi_{\mathrm{tr}}\left(\left[\frac{A}{\sqrt{k}}\right]^j\right) = \int_{\mathbb{R}} x^j\,\nu_G(dx) \quad (j = 0, 1, 2, \dots).$$

This ν_G is called the (normalized) spectral distribution of G, and is given by

$$\nu_G\left(\left\{\frac{\theta_j}{\sqrt{k}}\right\}\right) = \frac{m_j}{|X|} \quad (j = 0, 1, \dots, d)$$

where

$$\mathrm{spec}(G) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}.$$

We consider the limit of ν_G when G "grows", as an analogue of the classical central limit theorem. Hora [33] described various limit distributions for several growing families of distance-regular graphs (cf. [17]) including the Hamming graphs $H(n,q)$. In the case of $H(n,q)$, he obtained a Poisson distribution when $q/n \to q'$ $(n \to \infty)$ where $0 < q' < \infty$, and the standard Gaussian distribution when $q/n \to 0$ $(n \to \infty)$. Hora worked with the spectra directly in [33], but then Hora, Obata, and others revisited, simplified, and generalized these results based on the quantum decomposition

$$A = A^+ + A^\circ + A^-$$

where A^+, A°, A^- are non-commuting matrices in a larger $*$-algebra. Particularly, the matrices A^+, A°, and A^- are in the Terwilliger algebra of G (see [17, §16.6]). Besides the Poisson and Gaussian distributions, many important univariate distributions arise in this way, such as the exponential, geometric, gamma, and the two-sided Rayleigh distributions (see [34] for more details).

We give a concrete bivariate example of this sort, as an attempt towards a multivariate extension of the theory. In our context, we take another regular graph H (say, with valency ℓ) having the same vertex set as G, and assume that the adjacency matrix B of H commutes with A. This situation occurs for instance when H is the complement of graph G. Thus, the algebra generated by A and B is viewed as an algebraic probability space with respect to φ_{tr}. The pair (A, B) is treated as a pair of real random variables, and a unique probability measure $\nu_{G,H}$ on $\mathbb{R}^2$ exists (see

(5.2)) such that

$$\varphi_{\mathrm{tr}}\left(\left[\frac{A}{\sqrt{k}}\right]^j\left[\frac{B}{\sqrt{\ell}}\right]^h\right)=\int_{\mathbb{R}^2}x^jy^h\,\nu_{G,H}(dxdy)\quad(j,h=0,1,2,\dots).$$

This $\nu_{G,H}$ is called the (normalized) joint spectral distribution of G and H, and is given by

$$\nu_{G,H}\left(\left\{\left(\frac{\theta_j}{\sqrt{k}},\frac{\tau_h}{\sqrt{\ell}}\right)\right\}\right)=\frac{m_{j,h}}{|X|}\quad(j=0,1,\dots,d,\ h=0,1,\dots,e).$$

where $\tau_0>\cdots>\tau_e$ are the distinct eigenvalues of H, and $m_{j,h}$ is the dimension of the common eigenspace of (A,B) with respective eigenvalues (θ_j,τ_h). Note that the covariance $\varphi_{\mathrm{tr}}(AB)$ for A and B equals 0 if and only if G and H have no edge in common. We are interested in the limit of $\nu_{G,H}$ when G and H both grow, as an analogue of the bivariate central limit theorem.

Our third objective is to prove a bivariate version of the result of Hora [33] for the Hamming graphs mentioned above. $H(n,q)$ is defined as the n^{th} Cartesian power $K_q^{\square n}$ of the complete graph K_q on q vertices. We will instead consider the pair $(G^{\square n},\overline{G}^{\square n})$ of the n^{th} Cartesian powers of a strongly regular graph G and its complement $\overline{G}$, and obtain as limits a bivariate Poisson distribution and the standard bivariate Gaussian distribution, together with an intermediate distribution. The method of quantum decomposition is yet to be developed for the multivariate case, and hence we will deal with the spectra of these graphs directly, as was done by Hora in [33], though the discussions here become much more involved. We note that the complete graphs are the connected regular graphs with precisely two distinct eigenvalues, whereas the connected strongly regular graphs are those with precisely three distinct eigenvalues. This comparison can be made clearer when viewed in the framework of association schemes, and our choice of considering the pair $(G^{\square n}, \overline{G}^{\square n})$ above was in fact guided naturally by the work of Mizukawa and Tanaka [44] on a construction of multivariate Krawtchouk polynomials from arbitrary association schemes.

The paper is organized as follows: In Chapter 2, we review basic concepts on commutative association schemes (see Section 2.1); we recall Terwilliger algebras and the notion of the inner distribution of a code (see Section 2.2); we discuss translation association schemes and recall the notion of duality among additive codes (see Section 2.3); and we review some important properties of extensions of commutative association schemes (including Hamming association schemes) and recall a generalized MacWilliams identity (see Section 2.4).

In Chapter 3, we provide a resolution to the first objective which requires representation theory of symmetric groups and special linear Lie algebras. We give a brief introduction to the representation theory of symmetric groups in Section 3.1. We discuss irreducible modules of the special linear Lie algebra (from two points of view), and recall the Schur–Weyl duality (on symmetric groups and special linear Lie algebra) in Section 3.2.

In Chapter 4, we prove a general Assmus–Mattson-type theorem that works for more complicated kinds of weight enumerator. The proofs (provided in Section 4.3) requires some techniques from multivariate polynomial interpolation (reviewed in Section 4.1). Some examples are given in Section 4.4.

In Chapter 5, we provide a resolution to the third objective. We review basic definitions about graphs and state the main theorem in Section 5.1. We recall important properties of strongly regular graphs in Section 5.2. We prove the main theorem in Section 5.3. In Section 5.4, we demonstrate the main theorem with some specific families of strongly regular graphs.

The entireties of Chapters 3, 4, and 5, are based on [45], [47], and [46], respectively.

Chapter 2

Preliminary Concepts

In this chapter, we review some basic concepts concerning commutative association schemes and related algebras. We also review extensions of commutative association schemes and discuss some examples. We advise the reader to refer to [5, 13, 18, 19, 42] for a more thorough discussion of the topic. Throughout the thesis, $\mathbb{N}$ denotes the set of all nonnegative integers.

Let X denote a nonempty finite set. Let V denote the vector space over $\mathbb{C}$ of column vectors with coordinates indexed by X. We endow V with a standard basis $\{\hat{x} : x \in X\}$ and a Hermitian inner product $\langle \hat{x}, \hat{y} \rangle = \delta_{xy}$ $(x, y \in X)$. For every subset $C \subset X$, let $\hat{C} = \sum_{x \in C} \hat{x}$ denote its characteristic vector. We will naturally identify $\mathrm{End}(V)$ with the $\mathbb{C}$-algebra of complex matrices with rows and columns indexed by X. The adjoint (or conjugate-transpose) of $A \in \mathrm{End}(V)$ will be denoted by $A^\dagger$. Let $\mathscr{R} = \{R_0, R_1, \ldots, R_s\}$ denote a set of nonempty binary relations on X. For each integer $0 \leqslant i \leqslant s$, let $A_i \in \mathrm{End}(V)$ denote the matrix such that

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise,} \end{cases} \qquad (x, y \in X).$$

In other words, A_i is the adjacency matrix of the (directed) graph (X, R_i). We use the above notations throughout the chapter.

2.1 Commutative association schemes

The pair $(X, \mathscr{R})$ is called a *commutative association scheme* with s classes if each of the following conditions is satisfied:

(AS1) $A_0 = I$, the identity matrix;

(AS2) $\sum_{i=0}^{s} A_i = J$, the all-ones matrix;

(AS3) $A_i^\dagger \in \{A_0, A_1, \ldots, A_s\}$ for $0 \leqslant i \leqslant s$;

(AS4) $A_iA_j = A_jA_i \in M := \sum_{k=0}^{s} \mathbb{C}A_k$ for $0 \leqslant i, j \leqslant s$.

Unless otherwise stated, we assume that $(X, \mathscr{R})$ is a commutative association scheme with s classes. It follows from (AS1), (AS2), and (AS4) that M is an $(s+1)$-dimensional linear subspace of $\mathrm{End}(V)$ which is a commutative $\mathbb{C}$-algebra. We call M the *Bose–Mesner algebra* of $(X, \mathscr{R})$. By (AS3), M is closed under †, and so it is semisimple. Consequently, M has a basis $\{E_i\}_{i=0}^{s}$ consisting of the primitive idempotents, that is, $E_iE_j = \delta_{ij}E_i$, $\sum_{i=0}^{s} E_i = I$. We will always set

$$E_0 = |X|^{-1}J.$$

We note that the E_i are Hermitian positive semidefinite matrices. By (AS2), M is also closed under entrywise (or *Hadamard* or *Schur*) multiplication, denoted $\circ$. The A_i are the primitive idempotents of M with respect to this multiplication, that is,

$A_i \circ A_j = \delta_{ij} A_i$, $\sum_{i=0}^{s} A_i = J$.

The *intersection numbers* p_{ij}^k and the *Krein parameters* q_{ij}^k $(0 \leqslant i, j, k \leqslant s)$ of $(X, \mathscr{R})$ are defined by the equations

$$A_i A_j = \sum_{k=0}^{s} p_{ij}^k A_k, \qquad E_i \circ E_j = |X|^{-1} \sum_{k=0}^{s} q_{ij}^k E_k.$$

Clearly, the p_{ij}^k are non-negative integers. On the other hand, since $E_i \circ E_j$ (being a principal submatrix of $E_i \otimes E_j$) is positive semidefinite, it follows that the q_{ij}^k are real and non-negative.

Since both the A_i and the E_i are bases for M, we find change-of-basis matrices $P = [P_{ji}]$ and $Q = [Q_{ji}]$ such that

$$A_i = \sum_{j=0}^{s} P_{ji} E_j, \qquad E_i = |X|^{-1} \sum_{j=0}^{s} Q_{ji} A_j. \tag{2.1}$$

This leads to

$$PQ = QP = |X| I. \tag{2.2}$$

We refer to P and Q as the *first* and the *second eigenmatrix* of $(X, \mathscr{R})$, respectively. Note that P_{0i} is the degree (both in and out) of the regular graph (X, R_i), and that Q_{0i} is equal to the rank of E_i. Moreover, we have

$$P_{i0} = Q_{i0} = 1 \quad (0 \leqslant i \leqslant s). \tag{2.3}$$

2.2 Terwilliger algebras

We recall the Terwilliger algebra. Fix a *base vertex* $x_0 \in X$, and define the diagonal matrices $E_i^* = E_i^*(x_0)$, $A_i^* = A_i^*(x_0)$ $(0 \leqslant i \leqslant s)$ in $\mathrm{End}(V)$ by

$$(E_i^*)_{xx} = (A_i)_{x_0x}, \quad (A_i^*)_{xx} = |X|(E_i)_{x_0x} \quad (x \in X).$$

Note that $E_i^* E_j^* = \delta_{ij} E_i^*$, $\sum_{i=0}^s E_i^* = I$, and moreover

$$A_i^* A_j^* = \sum_{k=0}^s q_{ij}^k A_k^*, \qquad A_i^* = \sum_{j=0}^s Q_{ji} E_j^*.$$

The E_i^* and the A_i^* are called the *dual idempotents* and the *dual adjacency matrices* of $(X, \mathscr{R})$ with respect to x_0, respectively. They form two bases of the *dual Bose–Mesner algebra* $M^* = M^*(x_0)$ of $(X, \mathscr{R})$ with respect to x_0. The *Terwilliger algebra* $T = T(x_0)$ (also known as *subconstituent algebra*) of $(X, \mathscr{R})$ with respect to x_0 is the $\mathbb{C}$-subalgebra of $\mathrm{End}(V)$ generated by M and M^* [63, 64, 65]. The following are relations in T (cf. [63, Lemma 3.2]):

$$E_i^* A_j E_k^* = 0 \iff p_{ij}^k = 0; \qquad E_i A_j^* E_k = 0 \iff q_{ij}^k = 0. \tag{2.4}$$

Since T is closed under †, it is semisimple and any two non-isomorphic irreducible T-modules in the *standard module* V are orthogonal. Define a partition

$$X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_s \tag{2.5}$$

where

$$X_i = \{x \in X : (x_0, x) \in R_i\} \quad (0 \leqslant i \leqslant s).$$

Observe that $\hat{X}_i = A_i^{\dagger}\hat{x}_0 = E_i^*\hat{X}$ for every $0 \leqslant i \leqslant s$, and so it is immediate to see that the $(s+1)$-dimensional subspace

$$\sum_{i=0}^{s} \mathbb{C}\hat{X}_i = M\hat{x}_0 = M^*\hat{X}$$

is an irreducible T-module, called the *primary T-module*. It has a basis consisting of the vectors $v_i = A_i^{\dagger}\hat{x}_0 \in E_i^*V$ for $0 \leqslant i \leqslant s$ and another basis consisting of the vectors $v_j^* = A_j^{*\dagger}\hat{X} \in E_jV$ for $0 \leqslant j \leqslant s$.

For every irreducible T-module $W \subset V$, define the sets

$$W_s = \{0 \leqslant i \leqslant s \ : \ E_i^*W \neq 0\} \text{ and } W_s^* = \{0 \leqslant i \leqslant s \ : \ E_iW \neq 0\}.$$

We call W_s and W_s^* the *support* and the *dual support* of W, respectively. We say W is *thin* (resp. *dual thin*) if dim $E_i^*W \leqslant 1$ for all i (resp. dim $E_jW \leqslant 1$ for all j). Since the one-dimensional subspaces E_0V and E_0^*V are contained in the primary T-module, it follows that the primary T-module is the unique irreducible T-module that has support and dual support both equal to $\{0, 1, \ldots, s\}$.

We end this section with concepts from coding theory. We shall be using this information in the succeeding sections of the chapter. Let C denote a subset of X. For convenience, we call C a *code* if $1 < |C| < |X|$. For the moment, assume that C is a code. The *inner distribution* of C is the vector $a = (a_0, a_1, \ldots, a_s) \in \mathbb{R}^{s+1}$

defined by

$$a_i = |C|^{-1}\langle \hat{C}, A_i\hat{C}\rangle = |C|^{-1} \cdot |R_i \cap (C \times C)| \quad (0 \leqslant i \leqslant s).$$

Observe that (cf. (2.2), (2.3))

$$a_0 = 1, \quad \sum_{i=0}^{s} a_i = |C|, \quad (aQ)_0 = |C|, \quad \sum_{i=0}^{s}(aQ)_i = |X|.$$

Clearly, the a_i are non-negative. On the other hand, from (2.1) it follows that

$$\langle \hat{C}, E_i\hat{C}\rangle = |X|^{-1}|C|\,(aQ)_i \quad (0 \leqslant i \leqslant s). \tag{2.6}$$

Since the E_i are positive semidefinite, it follows that the $(aQ)_i$ are also non-negative. Delsarte's famous *linear programming bound* [18] on the sizes of codes is based on this simple observation. The vector $aQ \in \mathbb{R}^{s+1}$ is often referred to as the *MacWilliams transform* of a. We remark the following:

$$(aQ)_i = 0 \iff E_i\hat{C} = 0.$$

2.3 Translation association schemes

Suppose X is endowed with the structure of a finite abelian group (written additively) with identity element 0. We call $(X, \mathscr{R})$ a *translation association scheme* [13, §2.10] if for all $0 \leqslant i \leqslant s$ and $z \in X$, $(x, y) \in R_i$ implies $(x+z, y+z) \in R_i$.

For the rest of this section, assume that $(X, \mathscr{R})$ is a translation association scheme. For convenience, we will *always* choose 0 as the base vertex. (Note that the automorphism group of $(X, \mathscr{R})$ is transitive on X.) Observe that

$$R_i = \{(x,y) \in X \times X : y - x \in X_i\} \quad (0 \leqslant i \leqslant s).$$

Let X^* denote the character group of X with identity element ι. To each $\varepsilon \in X^*$ we associate the vector

$$\hat{\varepsilon} = |X|^{-1/2} \sum_{x \in X} \overline{\varepsilon(x)}\, \hat{x} \in V,$$

so that

$$\langle \hat{x}, \hat{\varepsilon} \rangle = |X|^{-1/2} \varepsilon(x) \quad (x \in X,\, \varepsilon \in X^*). \tag{2.7}$$

Note that the $\hat{\varepsilon}$ form an orthonormal basis of V by the orthogonality relations for the characters. Moreover, it follows that

$$A_i \hat{\varepsilon} = \left(\sum_{x \in X_i} \overline{\varepsilon(x)} \right) \hat{\varepsilon} \quad (0 \leqslant i \leqslant s,\ \varepsilon \in X^*).$$

This shows that each of the $\hat{\varepsilon}$ is an eigenvector for M, and hence belongs to one of the $E_i V$. Thus, we have a partition

$$X^* = X_0^* \sqcup X_1^* \sqcup \cdots \sqcup X_s^*,$$

given by

$$X_i^* = \{\varepsilon \in X^* : \hat{\varepsilon} \in E_i V\} \quad (0 \leqslant i \leqslant s).$$

Note that $X_0^* = \{\iota\}$, and that

$$E_i = \sum_{\varepsilon \in X_i^*} \hat{\varepsilon}\,\hat{\varepsilon}^\dagger \quad (0 \leqslant i \leqslant s). \tag{2.8}$$

Define the set $\mathscr{R}^* = \{R_0^*, R_1^*, \ldots, R_s^*\}$ of nonempty binary relations on X^* by

$$R_i^* = \{(\varepsilon, \eta) \in X^* \times X^* : \eta\varepsilon^{-1} \in X_i^*\} \quad (0 \leqslant i \leqslant s).$$

Then it follows from the orthogonality relations and (2.8) that

$$A_i^* = \sum_{(\varepsilon,\eta) \in R_i^*} \hat{\varepsilon}\,\hat{\eta}^\dagger \quad (0 \leqslant i \leqslant s).$$

In other words, the matrix representing A_i^* with respect to the orthonormal basis $\{\hat{\varepsilon} : \varepsilon \in X^*\}$ of V is precisely the adjacency matrix of the graph (X^*, R_i^*). It turns out that the pair $(X^*, \mathscr{R}^*)$ is again a translation association scheme, called the *dual* of $(X, \mathscr{R})$. In particular, the q_{ij}^k are the intersection numbers of $(X^*, \mathscr{R}^*)$, so that these are again non-negative integers in this case. We also note that $(X^*, \mathscr{R}^*)$ has eigenmatrices $P^* = Q$ and $Q^* = P$, and that

$$P_{ji} = \sum_{x \in X_i} \overline{\varepsilon(x)} \quad (\varepsilon \in X_j^*), \qquad Q_{ji} = \sum_{\varepsilon \in X_i^*} \varepsilon(x) \quad (x \in X_j).$$

We will view V together with the basis $\{\hat{\varepsilon} : \varepsilon \in X^*\}$ as the standard module for $(X^*, \mathscr{R}^*)$, and choose ι as the base vertex.

A code C in X is called an *additive code* if it is a subgroup of X. Assume for the moment that C is an additive code, and let $a = (a_0, a_1, \ldots, a_s)$ be its inner distribution. Observe that

$$a_i = |X_i \cap C| \quad (0 \leqslant i \leqslant s),$$

and hence a is also called the *weight distribution* of C in this case. The *dual code* of C is the subgroup $C^\perp$ in X^* defined by

$$C^\perp = \{\varepsilon \in X^* : \varepsilon(x) = 1 \text{ for all } x \in C\}.$$

From (2.7) it follows that

$$\hat{C} = |X|^{-1/2}\,|C| \sum_{\varepsilon \in C^\perp} \hat{\varepsilon}. \tag{2.9}$$

In other words, $\hat{C}$ is a scalar multiple of the characteristic vector of $C^\perp$ with respect to the basis $\{\hat{\varepsilon} : \varepsilon \in X^*\}$. We now observe that

$$\langle \hat{C}, E_i\hat{C} \rangle = |X|^{-1}|C|^2 \cdot |X_i^* \cap C^\perp| \quad (0 \leqslant i \leqslant s). \tag{2.10}$$

In particular, combining this with (2.6), we have

$$|X_i^* \cap C^\perp| = |C|^{-1}(aQ)_i \quad (0 \leqslant i \leqslant s),$$

so that $|C|^{-1}(aQ)$ gives the weight distribution of $C^\perp$.

The group operation on X^* is multiplicative. In many cases (cf. Section 4.4), we fix a (non-canonical) isomorphism $X \to X^*$ ($x \mapsto \varepsilon_x$) such that

$$\varepsilon_x(y) = \varepsilon_y(x) \quad (x, y \in X). \tag{2.11}$$

Then the dual code of an additive code in X becomes again an *additive* code in X.

For more information about translation association schemes, the reader may refer to [13, §2.10], [18, Chapter 6], and [42, §6].

2.4 Extensions of commutative association schemes and Hamming association schemes

For the rest of this chapter, we fix an integer $n \geq 2$. Delsarte [18, §2.5] gave a construction of a new commutative association scheme from $(X, \mathscr{R})$ with vertex set X^n as follows. For a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathbb{N}^s$, let $|\alpha| = \sum_{i=1}^{s} \alpha_i$. For any two vertices $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), \boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in X^n$, define the *composition* of $\boldsymbol{x}, \boldsymbol{y}$ to be the vector $c(\boldsymbol{x}, \boldsymbol{y}) = (c_1, c_2, \ldots, c_s) \in \mathbb{N}^s$, where

$$c_i = |\{\ell : (x_\ell, y_\ell) \in R_i\}| \quad (1 \leqslant i \leqslant s).$$

It is clear that $|c(\boldsymbol{x}, \boldsymbol{y})| \leqslant n$. For every $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant n$, define the binary relation $\boldsymbol{R}_\alpha$ on X^n by

$$\boldsymbol{R}_\alpha = \{(\boldsymbol{x}, \boldsymbol{y}) \in X^n \times X^n : c(\boldsymbol{x}, \boldsymbol{y}) = \alpha\}.$$

Let

$$\mathrm{Sym}^n(\mathscr{R}) = \{\boldsymbol{R}_\alpha : \alpha \in \mathbb{N}^s,\ |\alpha| \leqslant n\}.$$

Then it follows that the pair $(X^n, \mathrm{Sym}^n(\mathscr{R}))$ is a commutative association scheme, called the *extension* of $(X, \mathscr{R})$ of length n. We identify its standard module with vector space $V^{\otimes n}$ so that $\hat{\boldsymbol{x}} := \hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_n$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in X^n$. For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathbb{N}^s$ with $|\alpha| \leqslant n$, the 0-1 adjacency matrix $\boldsymbol{A}_\alpha \in \mathrm{End}(V^{\otimes n})$ of the graph $(X^n, \boldsymbol{R}_\alpha)$ is then given by

$$\boldsymbol{A}_\alpha = \sum_{i_1, i_2, \ldots, i_n} A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n}, \tag{2.12}$$

where the sum is over $i_1, i_2, \ldots, i_n \in \mathbb{N}$ such that

$$\{i_1, i_2, \ldots, i_n\} = \{0^{n-|\alpha|}, 1^{\alpha_1}, 2^{\alpha_2}, \ldots, s^{\alpha_s}\}$$

as multisets. In particular, the Bose–Mesner algebra $\boldsymbol{M}$ of $(X^n, \mathrm{Sym}^n(\mathscr{R}))$ coincides with the n^{th} symmetric tensor space of M. Similar expressions hold for the primitive idempotents, dual idempotents and the dual adjacency matrices of $(X^n, \mathrm{Sym}^n(\mathscr{R}))$, denoted henceforth by the $\boldsymbol{E}_\alpha$, the $\boldsymbol{E}^*_\alpha$ and the $\boldsymbol{A}^*_\alpha$, respectively. For simplicity, we will always choose $\boldsymbol{x}_0 := (x_0, x_0, \ldots, x_0) \in X^n$ as the base vertex. We denote the corresponding dual Bose–Mesner algebra and the Terwilliger algebra by $\boldsymbol{M}^*$ and $\boldsymbol{T}$, respectively. We also consider the partition

$$X^n = \bigsqcup_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha| \leqslant n}} (X^n)_\alpha$$

corresponding to (2.5), i.e.,

$$(X^n)_\alpha = \{\boldsymbol{x} \in X^n : (\boldsymbol{x}_0, \boldsymbol{x}) \in \boldsymbol{R}_\alpha\}.$$

Let $\{\boldsymbol{e}_i : 1 \leqslant i \leqslant s\}$ be the standard basis of $\mathbb{R}^s$. Then in view of (2.3), we have

$$\boldsymbol{A}_{\boldsymbol{e}_i} = \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} \left(\sum_{j=0}^{s} \alpha_j P_{ji} \right) \boldsymbol{E}_\alpha, \quad \boldsymbol{A}^*_{\boldsymbol{e}_i} = \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} \left(\sum_{j=0}^{s} \alpha_j Q_{ji} \right) \boldsymbol{E}^*_\alpha, \tag{2.13}$$

where $\alpha_0 := n - |\alpha|$. More generally, Mizukawa and Tanaka [44] described the eigenmatrices of $(X^n, \mathrm{Sym}^n(\mathscr{R}))$ in terms of certain s-variable hypergeometric orthogonal polynomials which generalize the Krawtchouk polynomials (see also [36, 37]). Let $\boldsymbol{p}^\gamma_{\alpha\beta}$ (resp. $\boldsymbol{q}^\gamma_{\alpha\beta}$) denote the intersection numbers (resp. Krein parameters) of $(X^n, \mathrm{Sym}^n(\mathscr{R}))$. Then, for all $1 \leqslant i \leqslant s$ and $\beta, \gamma \in \mathbb{N}^s$ with $|\beta|, |\gamma| \leqslant n$, we have

$$\boldsymbol{p}^\gamma_{\boldsymbol{e}_i\beta} \neq 0 \iff \gamma \in \big\{\beta - \boldsymbol{e}_j + \boldsymbol{e}_k : p^k_{ij} \neq 0\big\}, \tag{2.14}$$

where we set $\boldsymbol{e}_0 := 0$. A similar result holds for the $\boldsymbol{q}^\gamma_{\boldsymbol{e}_i\beta}$.

Let $\xi = (\xi_0, \xi_1, \ldots, \xi_s)$ denote a sequence of $s+1$ mutually commuting indeterminates. For every $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant n$, we let

$$\xi^\alpha = \xi_0^{n-|\alpha|} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \xi_s^{\alpha_s}. \tag{2.15}$$

Then it follows from (2.12) that

$$\left(\sum_{i=0}^{s} \xi_i A_i \right)^{\otimes n} = \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} \xi^\alpha \boldsymbol{A}_\alpha,$$

and similarly for the $\boldsymbol{E}_\alpha$. Observe that

$$\sum_{i=0}^{s} \xi_i E_i = |X|^{-1} \sum_{i=0}^{s} (\xi Q^{\mathsf{T}})_i A_i.$$

Combining these comments, we have (cf. [24, 62])

$$\sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} \xi^\alpha \boldsymbol{E}_\alpha = |X|^{-n} \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} (\xi Q^{\mathsf{T}})^\alpha \boldsymbol{A}_\alpha. \tag{2.16}$$

(Here, we extend the notation (2.15) to the sequence ξQ^{T} as well.)

Now, let C denote a code in X^n with inner distribution $a = (a_\alpha)_{\alpha\in\mathbb{N}^s,\, |\alpha|\leqslant n}$. Consider the polynomial $w_C(\xi)$ in $\mathbb{R}[\xi] = \mathbb{R}[\xi_0, \xi_1, \ldots, \xi_s]$ defined by

$$w_C(\xi) = \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} a_\alpha \xi^\alpha.$$

Note that $w_C(\xi)$ is homogeneous of degree n. From (2.16) it follows that

$$|C|^{-1} \sum_{\substack{\alpha\in\mathbb{N}^s \\ |\alpha|\leqslant n}} \langle \hat{C}, \boldsymbol{E}_\alpha \hat{C} \rangle \xi^\alpha = |X|^{-n}\, w_C(\xi Q^{\mathsf{T}}). \tag{2.17}$$

Hence we can read which of the $\boldsymbol{E}_\alpha \hat{C}$ vanish from the expansion of $w_C(\xi Q^{\mathsf{T}})$.

Suppose for the moment that $(X, \mathscr{R})$ is a translation association scheme, and that C is an additive code in X^n. In this case, $w_C(\xi)$ is called the *weight enumerator* of C. It should be remarked that $(X^n, \mathrm{Sym}^n(\mathscr{R}))$ and $(X^{*n}, \mathrm{Sym}^n(\mathscr{R}^*))$ are dual to each other. By (2.10) and (2.17) we have (cf. [24])

$$w_{C^{\perp}}(\xi) = |C|^{-1}\, w_C(\xi Q^{\mathsf{T}}).$$

This generalizes the well-known *MacWilliams identity*.

In proving our results, we need to consider extensions of length n of one-class association scheme $(X, \{R_0, (X\times X)\backslash R_0\})$. This special fusion of $(X^n, \mathrm{Sym}^n(\mathscr{R}))$ is called the *Hamming association scheme* $H(n, |X|)$. Observe that $H(n, |X|)$ has n classes, and that the associated matrices as well as the partition of the vertex set X^n are parametrized by the integers $0, 1, \ldots, n$, i.e., $\boldsymbol{A}_i, \boldsymbol{E}_i, \boldsymbol{E}_i^*, \boldsymbol{A}_i^*$, and also $(X^n)_i$ ($0 \leqslant i \leqslant n$). We denote the corresponding Bose–Mesner algebra, the dual Bose–Mesner algebra, and the Terwilliger algebra by $\boldsymbol{M}_H, \boldsymbol{M}_H^*$, and $\boldsymbol{T}_H$, respectively. Note that

$$\boldsymbol{A}_1 = \sum_{i=0}^{n} \theta_i \boldsymbol{E}_i, \quad \boldsymbol{A}_1^* = \sum_{i=0}^{n} \theta_i^* \boldsymbol{E}_i^*, \tag{2.18}$$

where

$$\theta_i = \theta_i^* = n(|X|-1) - |X|i \quad (0 \leqslant i \leqslant n).$$

Below we collect important facts about the irreducible $\boldsymbol{T}_H$-modules, most of which can be found in Terwilliger's lecture notes [67]. See also [61, §5.1]. (Some of the results hold in the wider class of *metric* and *cometric* association schemes.)

Lemma 2.4.1. *Let W be an irreducible $\boldsymbol{T}_H$-module.*

(i) $\boldsymbol{A}_1\boldsymbol{E}_i^*W \subset \boldsymbol{E}_{i-1}^*W + \boldsymbol{E}_i^*W + \boldsymbol{E}_{i+1}^*W$ $(0 \leqslant i \leqslant n)$, *where* $\boldsymbol{E}_{-1}^* = \boldsymbol{E}_{n+1}^* = 0$.

(ii) $\boldsymbol{A}_1^*\boldsymbol{E}_iW \subset \boldsymbol{E}_{i-1}W + \boldsymbol{E}_iW + \boldsymbol{E}_{i+1}W$ $(0 \leqslant i \leqslant n)$, *where* $\boldsymbol{E}_{-1} = \boldsymbol{E}_{n+1} = 0$.

(iii) There are non-negative integers r *and* d *such that*

$$n-2r \leqslant d \leqslant n-r, \tag{2.19}$$

and

$$\dim \boldsymbol{E}_i^* W = \dim \boldsymbol{E}_i W = \begin{cases} 1 & \textit{if } r \leqslant i \leqslant r+d, \\ 0 & \textit{otherwise}, \end{cases} \qquad (0 \leqslant i \leqslant n).$$

(iv) $\boldsymbol{E}_i^* \boldsymbol{A}_1 \boldsymbol{E}_j^* W \neq 0$ *if* $|i-j|=1$ $(r \leqslant i, j \leqslant r+d)$.

(v) $\boldsymbol{E}_i \boldsymbol{A}_1^* \boldsymbol{E}_j W \neq 0$ *if* $|i-j|=1$ $(r \leqslant i, j \leqslant r+d)$.

The integers r and d in (iii) above are called the *endpoint* and the *diameter* of W, respectively. The integer $2r+d-n$ is called the *displacement* of W (see [66]). From (2.19) it follows that

$$0 \leqslant 2r+d-n \leqslant n.$$

For every $0 \leqslant c \leqslant n$, let U_c be the span of the irreducible $\boldsymbol{T}_H$-modules in $V^{\otimes n}$ with displacement c. Then we have

$$V^{\otimes n} = \bigoplus_{c=0}^{n} U_c.$$

and this decomposition is called the *displacement decomposition* of $V^{\otimes n}$. In [67], Terwilliger showed that

$$U_0 = (\mathbb{C}\hat{x}_0 + \mathbb{C}\hat{X})^{\otimes n}. \tag{2.20}$$

Chapter 3

On Lee association schemes over $\mathbb{Z}_4$ and their Terwilliger algebra

Codes over $\mathbb{Z}_4$ are an active area of research. Hammons, Kumar, Calderbank, Sloane, and Solé [28] studied $\mathbb{Z}_4$-linear codes to understand via the Gray map the 'duality' of several families of nonlinear binary codes such as the Kerdock codes and the Preparata codes. Certain $\mathbb{Z}_4$-codes are also relevant to the study of vertex operator algebras (see [30] for example). The aim in this chapter is to explore the algebraic structure of the space $\mathbb{Z}_4^n$ underlying the $\mathbb{Z}_4$-codes of length n.

When dealing with $\mathbb{Z}_4$-codes of length n and their symmetrized weight enumerators (see Section 4.4.2), we consider the so-called *Lee association scheme* $L(n)$ with vertex set $\mathbb{Z}_4^n$. The structure of the Bose–Mesner algebra of $L(n)$ is known as discussed in [24] or [44]. In this chapter, we focus on the Terwilliger algebra of $L(n)$, and determine all of its irreducible modules. We show that there is a homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_3(\mathbb{C}))$ to the Terwilliger algebra of $L(n)$, and that the latter is generated by this image together with the center. It follows that in this case every irreducible module of the Terwilliger algebra has the structure of an irreducible $\mathfrak{sl}_3(\mathbb{C})$-module. Our main results in this chapter are Theorems 3.3.12, 3.3.13, and 3.3.14. The situation here turns out to be much more complicated than in the case of the binary Hamming schemes (see Go [23]), and in proving our theorems we invoke several facts from the representation theory of the symmetric groups and the Lie algebras $\mathfrak{sl}_m(\mathbb{C})$. We plan to discuss applications of the Terwilliger algebra to $\mathbb{Z}_4$-codes in a future paper.

This chapter is organized as follows: In Section 3.1, we give a brief background on the representation theory of the symmetric groups and recall some known properties of the *Specht modules*. In Section 3.2, we recall some important results in representation theory particularly on the connection between Specht modules and irreducible $\mathfrak{sl}_m(\mathbb{C})$-modules. Also, we describe the irreducible $\mathfrak{sl}_m(\mathbb{C})$-modules from the points of view of highest weight theory and of Weyl modules. Finally, we prove our main results in Section 3.3. The entire chapter is based on [45].

3.1 Specht modules

In this section, we discuss irreducible modules of the symmetric groups which are called Specht modules. There are a lot of available references for this topic, for instance see [20, 25, 50, 56]

Throughout this section, let k denote a positive integer. A *partition* λ of k is a sequence $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ of nonnegative integers satisfying the conditions

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} \text{ and } \sum_{i=0}^{n-1} \lambda_i = k.$$

In symbols, we write $\lambda \vdash k$. Naturally, we identify λ with $(\lambda_0, \dots, \lambda_{n-1}, 0, \dots, 0)$ so that length is immaterial. We say λ has l *parts* and we write $\mathrm{part}(\lambda) = l$ if l is the largest integer such that $\lambda_l \neq 0$. For each positive integer l, we define

$$P(k,l) = \{\lambda \vdash k \ : \ \mathrm{part}(\lambda) \leqslant l\}$$

and we set $P(0,l) := \{(0,\dots,0)\}$ for convenience. Let λ and ε be partitions of k. We say that λ *dominates* ε and we write $\lambda \geq \varepsilon$ whenever

$$\lambda_0 + \cdots + \lambda_i \geqslant \varepsilon_0 + \cdots + \varepsilon_i \qquad (\forall i \in \mathbb{N}).$$

Observe that $\geq$ is a partial order on the set of all partitions of k.

To each partition $\lambda \vdash k$ we associate a *Ferrers diagram*, an array of k boxes arranged in rows and columns such that the ith row has λ_i boxes for every $i \in \mathbb{N}$. On the other hand, a λ-*tableau* (also known as a *Young tableau of shape* λ) is an array $\mathtt{t}$ obtained by filling the boxes of the Ferrers diagram associated to λ with integers from 1 to k without repetitions. A λ-tableau $\mathtt{t}$ is said to be *standard* if its entries are strictly increasing from left to right along each row and from top to bottom along each column. We denote by $\mathsf{STab}(\lambda)$ the set of all standard λ-tableaux.

Two λ-tableaux $\mathtt{t}, \mathtt{t}'$ are said to be *row equivalent* (resp. *column equivalent*) if the entries in the corresponding rows (resp. columns) of $\mathtt{t}$ and $\mathtt{t}'$ are the same up to permutation. Let $\mathfrak{S}_k$ denote the symmetric group on k objects. We observe that $\mathfrak{S}_k$ acts transitively on the set of all λ-tableaux by applying $\sigma \in \mathfrak{S}_k$ to the entries in the boxes. The result of the action of σ on the λ-tableau $\mathtt{t}$ will be denoted by $\sigma\mathtt{t}$. If $\mathtt{t}$ and $\sigma\mathtt{t}$ are row equivalent (resp. column equivalent), we say σ is a *row stabilizer (resp. column stabilizer)* of $\mathtt{t}$. Observe that the set of all row stabilizers (resp. column stabilizers) of $\mathtt{t}$ forms a subgroup of $\mathfrak{S}_k$.

Let $\lambda \vdash k$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Denote by $R_{\mathtt{t}}$ and $C_{\mathtt{t}}$ the subgroups of $\mathfrak{S}_k$ consisting of row and column stabilizers of $\mathtt{t}$, respectively. Define an element $s_{\mathtt{t}}$ in the group algebra $\mathbb{C}[\mathfrak{S}_k]$ such that

$$s_{\mathtt{t}} = \sum_{\sigma \in C_{\mathtt{t}}} sgn(\sigma)\sigma \sum_{\psi \in R_{\mathtt{t}}} \psi$$

where sgn denotes the sign character of $\mathfrak{S}_k$. We mention that $s_{\mathtt{t}}$ is proportional to an idempotent $g_{\mathtt{t}}$ (see [20, Lemma 5.13.3] or [25, Lemma 9.3.8]). We call $g_{\mathtt{t}}$ the *normalized Young symmetrizer* associated to $\mathtt{t}$.

Theorem 3.1.1. *Let $\lambda \vdash k$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the subspace $V_\lambda = \mathbb{C}[\mathfrak{S}_k]g_{\mathtt{t}}$ of $\mathbb{C}[\mathfrak{S}_k]$ is an irreducible module of $\mathbb{C}[\mathfrak{S}_k]$ under left multiplication and is independent of $\mathtt{t}$ up to isomorphism. Moreover, every irreducible module of $\mathbb{C}[\mathfrak{S}_k]$ is isomorphic to V_λ for a unique λ.*

Proof. See Theorem 5.12.2 and Section 5.13 of [20]. □

The spaces V_λ are called the *Specht modules* and the collection $\{V_\lambda \mid \lambda \vdash k\}$

forms a complete set of mutually non-isomorphic irreducible modules for $\mathbb{C}[\mathfrak{S}_k]$.

Remark 3.1.2. In other references such as [50] and [56], the Specht modules are defined in terms of polytabloids. In this setup, Theorem 3.1.1 is proven using the so-called submodule theorem (see [50, Theorem 2.4.4] or [56, Theorem 10.2.13]). In addition, the set of all polytabloids associated to a standard λ-tableau forms a basis for the Specht module V_λ (see [50, Theorem 2.5.2]). Thus $\dim(V_\lambda)$ is equal to the number of standard λ-tableaux. This quantity can be computed using known formulas which are as follows:

Consider the Ferrers diagram of $\lambda \vdash k$ with $\mathrm{part}(\lambda) = l$. To every box, we associate an ordered pair (i, j) of nonnegative integers such that $0 \leqslant i \leqslant l-1$ is its row position and $1 \leqslant j \leqslant \lambda_i$ is its column position. By the *hook-length* $h_\lambda(i, j)$ we mean the number of boxes (a, b) such that either $a = i$ and $b \geqslant j$ or $a \geqslant i$ and $b = j$. Then the dimension of V_λ is given by

$$\dim(V_\lambda) = \frac{k!}{\prod h_\lambda(i, j)}$$

where the product ranges to all boxes (i, j) of the Ferrers diagram of λ. This formula is called the *hook-length formula* and is proven by Frame, Robinson and Thrall in 1954 (see [50, Section 3.10]). The second one is due to Frobenius and Young, and is called the *determinantal formula*. To state the formula, we set $1/c! = 0$ whenever $c < 0$. Then the dimension of V_λ is given by

$$\dim(V_\lambda) = k! \det \left[\frac{1}{(\lambda_i - i + j)!} \right]_{i,j=0}^{l-1}.$$

This formula is much older than the hook-length formula (see [50, Section 3.11]).

3.2 $\mathfrak{sl}(V)$-modules

Throughout this section, let V denote an n-dimensional vector space over $\mathbb{C}$. The Lie algebra $\mathfrak{sl}(V)$ is the vector space over $\mathbb{C}$ of traceless linear operators on V together with the Lie bracket $[x, y] = xy - yx$ for all $x, y \in \mathfrak{sl}(V)$. Fixing an ordered basis for V means we may identify the linear operators on V with $n \times n$ matrices and write $\mathfrak{sl}(V) = \mathfrak{sl}_n(\mathbb{C})$. In this section we discuss the irreducible modules for $\mathfrak{sl}(V)$ from two points of view. The first is described as follows: For every $\sigma \in \mathfrak{S}_k$ and for vectors $v_1, \ldots, v_k \in V$, define $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ so that $V^{\otimes k}$ becomes a module for $\mathbb{C}[\mathfrak{S}_k]$. Then for every $\lambda \in P(k, n)$ and for every $\mathtt{t} \in \mathsf{STab}(\lambda)$, the space $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}(V)$ and is called a *Weyl module*. The other one is by means of a theorem of the highest weight. This states that every irreducible $\mathfrak{sl}(V)$-module has a unique highest weight and two irreducible modules with the same highest weight are isomorphic. We establish the connection between these two points of view in the latter part of the section. The reader may refer to [20, 25, 27, 35] for more background information.

Let $\mathfrak{gl}(V)$ be the Lie algebra of linear operators on V with the usual Lie bracket. Similarly, we write $\mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$ if an ordered basis for V is fixed. Let I denote the identity operator in $\mathfrak{gl}(V)$. For $\sigma \in \mathfrak{S}_k$ and for operators $M_1, \ldots, M_k \in \mathfrak{gl}(V)$, define $\sigma(M_1 \otimes \cdots \otimes M_k) = M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(k)}$ so that $\mathfrak{gl}(V)$ acts on $V^{\otimes k}$ by

$$F(v_1 \otimes \cdots \otimes v_k) = \frac{1}{(k-1)!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(F \otimes I \otimes \cdots \otimes I)(v_1 \otimes \cdots \otimes v_k) \qquad (3.1)$$

for every $F \in \mathfrak{gl}(V)$. We see that the space $V^{\otimes k}$ supports a module structure for both the group algebra $\mathbb{C}[\mathfrak{S}_k]$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(V))$.

Theorem 3.2.1. *(Schur-Weyl duality) Let $\mathcal{A}$ and $\mathcal{B}$ denote the homomorphic images of $\mathbb{C}[\mathfrak{S}_k]$ and $\mathcal{U}(\mathfrak{gl}(V))$ in End$(V^{\otimes k})$, respectively. Then each of the following statements holds:*

i) $\mathcal{A}$ and $\mathcal{B}$ are the centralizers of each other.

ii) $\mathcal{A}$ and $\mathcal{B}$ are semisimple and in particular $V^{\otimes k}$ is a semisimple $\mathfrak{gl}(V)$-module.

iii) $V^{\otimes k} = \bigoplus_{\lambda \vdash k} V_\lambda \otimes L_\lambda$ is a direct sum decomposition into $\mathcal{A}\otimes\mathcal{B}$-modules where $\{V_\lambda\}$ are Specht modules and $\{L_\lambda\}$ are some non-isomorphic irreducible modules for $\mathfrak{gl}(V)$ or zero.

Proof. This is found in [20, Theorem 5.18.4] which is in fact a consequence of the double centralizer theorem [20, Theorem 5.18.1]. □

According to the Weyl character formula [20, Theorem 5.22.1], dim(L_λ) is zero if and only if part$(\lambda) > n$. Thus $V^{\otimes k}$ decomposes into irreducible $\mathfrak{gl}(V)$-modules

$$V^{\otimes k} = \bigoplus_{\lambda \in P(k,n)} V_\lambda \otimes L_\lambda \cong \bigoplus_{\lambda \in P(k,n)} \dim(V_\lambda)\, L_\lambda. \tag{3.2}$$

In other words, a complete set of mutually non-isomorphic irreducible $\mathfrak{gl}(V)$-modules on $V^{\otimes k}$ is in bijection with the set $P(k,n)$.

Lemma 3.2.2. *On the space $V^{\otimes k}$, a complete set of mutually non-isomorphic irreducible modules for $\mathfrak{gl}(V)$ is also a complete set of mutually non-isomorphic irreducible modules for $\mathfrak{sl}(V)$.*

Proof. Let W denote a subspace of $V^{\otimes k}$. Note that every operator $F \in \mathfrak{gl}(V)$ can be written as $F = S + cI$ where $S \in \mathfrak{sl}(V)$ and $c \in \mathbb{C}$. From this and (3.1), we find W is an irreducible $\mathfrak{sl}(V)$-module if and only if W is an irreducible $\mathfrak{gl}(V)$-module. Suppose W is a $\mathfrak{gl}(V)$-module. Let W' denote a $\mathfrak{gl}(V)$-module on $V^{\otimes k}$ and let $f : W \to W'$ be a vector space isomorphism. Then for all $w \in W$ we have

$$\begin{aligned} f(Mw) - Mf(w) &= f((S+cI)w) - (S+cI)f(w) \\ &= f(Sw + ckw) - (Sf(w) + ckf(w)) \\ &= f(Sw) - Sf(w). \end{aligned}$$

Hence f is an isomorphism of $\mathfrak{gl}(V)$-modules if and only if f is an isomorphism of $\mathfrak{sl}(V)$-modules. □

Lemma 3.2.3. *Let $\lambda \in P(k,n)$. Then for every standard λ-tableau $\mathtt{t}$, the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}(V)$ isomorphic to L_λ.*

Proof. Recall that $g_{\mathtt{t}}$ is an idempotent in $\mathbb{C}[\mathfrak{S}_k]$ and V_λ is isomorphic to $\mathbb{C}[\mathfrak{S}_k]g_{\mathtt{t}}$. Then $\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_k]}(V_\lambda, V^{\otimes k}) \cong g_{\mathtt{t}}(V^{\otimes k})$ by [20, Lemma 5.13.4]. On the other hand, we obtain $L_\lambda \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_k]}(V_\lambda, V^{\otimes k})$ by the double centralizer theorem. □

Remark 3.2.4. Fix an ordered basis $\{v_0, v_1, \ldots, v_{n-1}\}$ for V and identify $\mathfrak{sl}(V)$ and $\mathfrak{gl}(V)$ with $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$, respectively. Let $\mathfrak{h}$ denote the set of all complex diagonal matrices of the form $H = \mathrm{diag}(a_0, a_1, \ldots, a_{n-1})$ such that $\sum_{j=0}^{n-1} a_j = 0$. Recall that $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{sl}_n(\mathbb{C})$. For $H = \mathrm{diag}(a_0, \ldots, a_{n-1})$ and $H' = \mathrm{diag}(b_0, \ldots, b_{n-1})$, define an inner product on $\mathfrak{h}$ such that

$$\langle H, H' \rangle = \sum_{j=0}^{n-1} \overline{a_j} b_j.$$

For integers $0 \leqslant r, s \leqslant n-1$, let E_{rs} denote the matrix in $\mathfrak{gl}_n(\mathbb{C})$ that has a 1 on the (r,s)-entry and 0 on all other entries. Define the linear functional $\alpha_{rs} : \mathfrak{h} \to \mathbb{C}$ that sends $H \mapsto (a_r - a_s)$ for all $H \in \mathfrak{h}$. Then for integers $0 \leqslant r, s \leqslant n-1$ such that $r \neq s$ we find $E_{rs} \in \mathfrak{sl}_n(\mathbb{C})$ and

$$[H, E_{rs}] = (a_r - a_s)E_{rs} = \alpha_{rs}(H)E_{rs} \quad \forall H \in \mathfrak{h}.$$

We call α_{rs} a *root* of $\mathfrak{sl}_n(\mathbb{C})$ relative to the Cartan subalgebra $\mathfrak{h}$ with corresponding *root vector* E_{rs}. Clearly, $\alpha_{rs}(H) = \langle E_{rr} - E_{ss}, H \rangle$ for all $H \in \mathfrak{h}$ and thus we can transfer the roots to $\mathfrak{h}$ via the map $\alpha_{rs} \mapsto (E_{rr} - E_{ss})$. Let R denote the set of all roots and let E denote the $\mathbb{R}$-linear span of R. Then R forms a *root system* that is conventionally called A_{n-1}. We abbreviate $H_j = E_{jj} - E_{j+1,j+1}$ for each integer $0 \leqslant j \leqslant n-2$ so that $\{H_0, \ldots, H_{n-2}\}$ is a *base* for E. We comment about finite-dimensional modules for $\mathfrak{sl}_n(\mathbb{C})$. If W is a finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module, then W has a basis consisting of simultaneous eigenvectors for $\mathfrak{h}$. In fact, for each basis vector v there exists $\mu \in \mathfrak{h}$ such that

$$Hv = \langle \mu, H \rangle v \qquad \forall H \in \mathfrak{h}.$$

We call μ a *weight* in W with corresponding *weight vector* v. Recall that $\langle \mu, H_j \rangle$ is an integer for every $0 \leqslant j \leqslant n-2$. There exist elements $\omega_0, \omega_1, \ldots, \omega_{n-2} \in \mathfrak{h}$ called *fundamental weights* where $\langle \omega_i, H_j \rangle = \delta_{ij}$ for all integers $0 \leqslant i, j \leqslant n-2$. Consequently, the weight μ is written as $\mu = \sum_{j=0}^{n-2} \langle \mu, H_j \rangle \omega_j$. We say that a weight μ is *dominant* if $\langle \mu, H_j \rangle$ is nonnegative for all $0 \leqslant j \leqslant n-2$. Suppose μ, μ' are weights in W. We say that μ is *higher than* μ' if there exists nonnegative real numbers $c_0, c_1, \ldots, c_{n-2}$ such that

$$\mu - \mu' = \sum_{j=0}^{n-2} c_j H_j.$$

A weight λ occuring in W is said to be the *highest weight* if λ is higher than any other weight in W. The highest weight theory of irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules states that every irreducible module has a unique highest weight and that two irreducible modules are isomorphic if and only if they have the same highest weight. Suppose W is an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module with highest weight λ. The *multiplicity* of the weight μ in W is the dimension of the μ-weight space in W and this quantity is determined using two known formulas: Freudenthal's (see [35, Section 22]) and Kostant's formulas (see [35, Section 24]).

We resume our discussion on Weyl modules. Recall that $\{v_0, v_1, \ldots, v_{n-1}\}$ is a fixed ordered basis for V. For convenience, every vector of the form

$$\beta = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k} \qquad (j_1, j_2, \ldots, j_k \in \{0, 1, \ldots, n-1\})$$

will be called a *simple tensor*. We shall identify the set of all simple tensors with the set of all ordered k-tuples on $\{0, 1, \ldots, n-1\}$. Let $\beta = (j_1, j_2, \ldots, j_k)$ denote a simple tensor and define the ordered tuple $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1})$ such that

$$\varepsilon_r = |\{1 \leqslant s \leqslant k \mid j_s = r\}| \qquad (0 \leqslant r \leqslant n-1).$$

We call ε the *content* of β and write $\mathsf{cont}(\beta) = \varepsilon$. Let $\mathsf{span}(\varepsilon)$ denote the subspace of $V^{\otimes k}$ spanned by simple tensors β with $\mathsf{cont}(\beta)$.

Let $\lambda \in P(k,n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. To each simple tensor $\beta = (j_1, j_2, \ldots, j_k)$ we associate a substitution $\beta_{\mathtt{t}}$ such that for every integer $1 \leqslant s \leqslant k$ we write j_s in the box labeled s in $\mathtt{t}$. We say β is a $(\lambda, \mathtt{t})$-*semistandard simple tensor* if the corresponding $\beta_{\mathtt{t}}$ satisfies each of the following conditions:

(SST1) the numbers are weakly increasing from left to right along each row, and

(SST2) the numbers are strictly increasing from top to bottom along each column.

Lemma 3.2.5. *Let $\lambda \in P(k,n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the set of all vectors $g_{\mathtt{t}}(\beta)$ where β is a $(\lambda, \mathtt{t})$-semistandard simple tensor forms a basis for $g_{\mathtt{t}}(V^{\otimes k})$. Furthermore, we have*

$$\mathit{dim}(g_{\mathtt{t}}(V^{\otimes k}) \cap \mathsf{span}(\varepsilon)) = K_{\lambda,\varepsilon}$$

where the scalar $K_{\lambda,\varepsilon}$ (called the Kostka number) is equal to the number of distinct $(\lambda, \mathtt{t})$-semistandard simple tensors with content ε.

Proof. See [8, Theorem 8.11]. □

In view of Lemmas 3.2.3 and 3.2.5, the space $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}_n(\mathbb{C})$ with basis consisting of vectors $g_{\mathtt{t}}(\beta)$ such that β is a $(\lambda, \mathtt{t})$-semistandard simple tensor. Pick an arbitrary basis vector $g_{\mathtt{t}}(\beta)$ and suppose $\mathsf{cont}(\beta) = \varepsilon$ where $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1})$. Let $H_0, H_1, \ldots, H_{n-2}$ and $\omega_0, \omega_1, \ldots, \omega_{n-2}$ denote elements of $\mathfrak{h}$ as described in Remark 3.2.4. Observe that

$$H_j g_{\mathtt{t}}(\beta) = (\varepsilon_j - \varepsilon_{j+1}) g_{\mathtt{t}}(\beta) \quad \text{for each } j \in \{0, 1, \ldots, n-2\}$$

and thus, ε is viewed as a weight via the map ω that sends $\varepsilon \mapsto \sum_{i=0}^{n-2}(\varepsilon_i - \varepsilon_{i+1})\omega_i$.

Theorem 3.2.6. *Let $\lambda \in P(k,n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\omega(\lambda)$.*

Proof. The Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is a highest weight cyclic representation of $\mathfrak{sl}_n(\mathbb{C})$ with weight $\omega(\lambda)$ in view of [27, Definition 7.16]. The theorem then follows immediately from [27, Proposition 7.17]. □

Corollary 3.2.7. *The set of all dominant weights occuring in the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is $\{\omega(\varepsilon) \mid \varepsilon \in P(k,n)$ and $\lambda \geq \varepsilon\}$.*

Proof. See [8, Lemma 4.5]. □

Remark 3.2.8. The set of all weights occurring in the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ together with their corresponding multiplicities can be described as follows: Let $\varepsilon \in P(k,n)$ such that $\lambda \geq \varepsilon$. Thus, ε is a content occurring in $g_{\mathtt{t}}(V^{\otimes k})$ and in particular $\omega(\varepsilon)$ is a dominant weight. Suppose ε' is an n-tuple obtained by permuting the entries of ε. By applying the *Bender-Knuth involution* (see [7]) repeatedly, we see that there exists a bijection between the set of all $(\lambda, \mathtt{t})$-semistandard simple tensors with content ε and that of content ε'. Hence, $\omega(\varepsilon')$ is a weight occurring in $g_{\mathtt{t}}(V^{\otimes k})$ and the multiplicity of $\omega(\varepsilon')$ is equal to $K_{\lambda,\varepsilon}$.

3.3 Terwilliger algebras of Lee association schemes over $\mathbb{Z}_4$ and irreducible modules

Throughout the section, $(X, \mathscr{R})$ denotes the commutative association scheme with vertex set $X = \{0, 1, 2, 3\}$ and associate classes $\mathscr{R} = \{R_0, R_1, R_2\}$ such that

$$R_i = \{(x, y) \in X \times X \;:\; x - y \equiv \pm i \pmod 4\} \quad (0 \leqslant i \leqslant 2).$$

Note that X is an abelian group with respect to addition reduced modulo 4. Hence, $(X, \mathscr{R})$ is a translation association scheme. Let A_0, A_1, A_2 and E_0, E_1, E_2 denote the associate matrices and the primitive idempotents, respectively. Write the dual matrices as $A_i^* = A_i^*(0)$ and $E_i^* = E_i^*(0)$ for each integer $0 \leqslant i \leqslant 2$. Let V denote the standard module for $(X, \mathscr{R})$ and let $T = T(0)$ denote the Terwilliger algebra of $(X, \mathscr{R})$ with respect to vertex 0. Let W_0 denote the primary T-module with bases $\{v_0, v_1, v_2\}$ and $\{v_0^*, v_1^*, v_2^*\}$ (as described in Section 2.2) and let W_1 denote the orthogonal completement of W_0 in V. Finally, let T_L denote the Lie algebra over $\mathbb{C}$ obtained by endowing T with the usual Lie bracket.

From here on, we fix an integer $n \geqslant 2$ and let $L(n)$ denote the commutative association scheme $(X^n, \mathrm{Sym}^n(\mathscr{R}))$. Observe that $L(n)$ is a translation association scheme and the identity element is the zero codeword. We refer to $L(n)$ as the *Lee association scheme over* $\mathbb{Z}_4$. We recall the Bose–Mesner algebra $\boldsymbol{M}$ of $L(n)$. For every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\alpha| \leqslant n$, we have

$$\boldsymbol{A}_\alpha = \frac{1}{(n-|\alpha|)!\alpha_1!\alpha_2!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(A_0^{\otimes(n-|\alpha|)} \otimes A_1^{\otimes \alpha_1} \otimes A_2^{\otimes \alpha_2}\right),$$
$$\boldsymbol{E}_\alpha = \frac{1}{(n-|\alpha|)!\alpha_1!\alpha_2!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(E_0^{\otimes(n-|\alpha|)} \otimes E_1^{\otimes \alpha_1} \otimes E_2^{\otimes \alpha_2}\right).$$

Then the $\boldsymbol{A}_\alpha$ are the adjacency matrices and the $\boldsymbol{E}_\alpha$ are the primitive idempotents. Now consider the dual Bose–Mesner algebra $\boldsymbol{M}^*$ of $L(n)$ with respect to the zero codeword. Then similar expressions hold for the dual primitive idempotents $\boldsymbol{E}_\alpha^*$ and the dual adjacency matrices $\boldsymbol{A}_\alpha^*$.

Let $\boldsymbol{T}$ denote the Terwilliger algebra of $L(n)$ with respect to the zero codeword of length n. Recall that $\boldsymbol{T}$ is the subalgebra of $\mathrm{End}(V^{\otimes n})$ generated by the associate and the dual associate matrices. In this section, we determine and describe the irreducible modules for the Terwilliger algebra $\boldsymbol{T}$. Our methods and techniques are inspired by [67].

Lemma 3.3.1. *With above notation, each of the following relations holds:*

$$\begin{aligned} 4E_1 &= 2A_0 - 2A_2, \\ 4E_2 &= A_0 - A_1 + A_2, \\ A_1^* &= 2E_0^* - 2E_2^*, \\ A_2^* &= E_0^* - E_1^* + E_2^*. \end{aligned}$$

Proof. Label the coordinates of the vectors in V, and the rows and columns of the matrices in $\mathrm{End}(V)$ with the natural ordering of the vertices in X. Let V_1 (resp. V_2) denote the eigenspace of A_1 corresponding to the eigenvalue $\tau_1 = 0$ (resp.

$\tau_2 = -2$). Then V_1 has an orthonormal basis $\{x_1, x_2\}$ and V_2 has an orthonormal basis $\{x_3\}$ where

$$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \; x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \text{ and } x_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Consequently,

$$\begin{aligned} E_1 &= x_1\bar{x}_1^t + x_2\bar{x}_2^t &= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \\ E_2 &= x_3\bar{x}_3^t &= \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

These prove the first two equations of the lemma. The remaining equations follow immediately from the first two and the definition of dual matrices. □

Lemma 3.3.2. *With above notation, for* $F \in \{A_1, A_2, E_0^*, E_1^*, E_2^*\}$ *the matrix representing* $F|_{W_0}$ *with respect to the ordered basis* $\{v_0, v_1, v_2\}$ *and the matrix repre-*

senting $F|_{W_1}$ are given by

F	A_1	A_2	E_0^*	E_1^*	E_2^*
$F\|_{W_0}$	$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
$F\|_{W_1}$	$[0]$	$[-1]$	$[0]$	$[1]$	$[0]$

Proof. Routine. □

Lemma 3.3.3. *With above notation, for $F \in \{A_1^*, A_2^*, E_1, E_2\}$ the matrix representing $F|_{W_0}$ with respect to the ordered basis $\{v_0, v_1, v_2\}$ and the matrix representing $F|_{W_1}$ are given by*

F	A_1^*	A_2^*	E_1	E_2
$F\|_{W_0}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$.
$F\|_{W_1}$	$[0]$	$[-1]$	$[1]$	$[0]$

Proof. Follows immediately from Lemma 3.3.1 and Lemma 3.3.2. □

Lemma 3.3.4. *With above notation, the matrices*

$$E_0^*,\ E_1E_1^*,\ E_2^*,\ E_1^*A_2,\ E_0^*A_2,\ A_2E_0^*,\ E_0^*A_1,\ A_1E_0^*,\ E_1^*A_1E_2^*,\ E_2^*A_1E_1^*$$

form a basis for T.

Proof. Use Lemmas 3.3.2 and 3.3.3. □

Lemma 3.3.5. *With above notation, each of the following relations holds:*

$$\begin{aligned}
E_1^*A_2 &= \tfrac{1}{2}A_2 + \tfrac{1}{2}A_2^* - \tfrac{1}{16}[A_1,[A_1,A_2^*]],\\
E_0^*A_2 &= \tfrac{1}{4}A_2 - \tfrac{1}{4}A_2^* + \tfrac{1}{8}[A_1^*,A_2] + \tfrac{1}{32}[A_1,[A_1,A_2^*]],\\
E_0^*A_1 &= \tfrac{1}{4}A_1 - \tfrac{1}{8}[A_1,A_1^*] - \tfrac{1}{8}[A_1,A_2^*] + \tfrac{1}{16}[A_2,[A_1,A_1^*]],\\
E_1^*A_1E_2^* &= \tfrac{1}{4}A_1 - \tfrac{1}{8}[A_1,A_1^*] + \tfrac{1}{8}[A_1,A_2^*] - \tfrac{1}{16}[A_2,[A_1,A_1^*]],\\
E_0^* - E_1E_1^* &= \tfrac{3}{4}A_2^* + \tfrac{1}{4}A_2 + \tfrac{1}{4}A_1^* - \tfrac{1}{32}[A_1,[A_1,A_2^*]],\\
E_2^* - E_1E_1^* &= \tfrac{3}{4}A_2^* + \tfrac{1}{4}A_2 - \tfrac{1}{4}A_1^* - \tfrac{1}{32}[A_1,[A_1,A_2^*]].
\end{aligned}$$

Proof. Use Lemmas 3.3.2 and 3.3.3. □

Let I denote the identity matrix in T and let $\mathfrak{g}$ denote the Lie subalgebra of T_L consisting of matrices with trace 0. We define the unique element $\Phi \in \mathfrak{g}$ for which $\Phi|_{W_0}$ acts as identity on W_0 and $\Phi|_{W_1}$ acts as the scalar -3 on W_1.

Lemma 3.3.6. *With above notation, each of the following statements holds:*

i) The Terwilliger algebra T is a direct sum of $\mathfrak{g}$ and $\mathbb{C}I$.

ii) The matrices

$$E_0^*-E_1E_1^*,\ E_2^*-E_1E_1^*,\ E_1^*A_2,\ E_0^*A_2,\ A_2E_0^*,\ E_0^*A_1,\ A_1E_0^*,\ E_1^*A_1E_2^*,\ E_2^*A_1E_1^*$$

form a basis for $\mathfrak{g}$.

iii) The Lie algebra $\mathfrak{g}$ is precisely the Lie subalgebra of T_L that is generated by A_1, A_2, A_1^* and A_2^*.

Proof. If $F \in \mathfrak{g}$, then $F|_{W_1}$ acts as $-\operatorname{trace}(F|_{W_0})$ on W_1. Thus, dim$(\mathfrak{g}) = 9$ and (i) holds since $I \notin \mathfrak{g}$. Statement (ii) holds since each matrix has trace 0 and are linearly independent by Lemma 3.3.4. Statement (iii) follows immediately from (ii) and Lemma 3.3.5. □

Proposition 3.3.7. *With above notation, each of the following statements holds:*

i) The map $\mathfrak{g} \to \mathfrak{gl}(W_0)$ that sends every matrix $F \in \mathfrak{g}$ to the restriction $F|_{W_0}$ is an isomorphism of Lie algebras.

ii) The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is isomorphic to $\mathfrak{sl}(W_0)$.

Proof. Let τ denote the map in (i). Note that τ is a Lie algebra homomorphism and the spaces $\mathfrak{g}$ and $\mathfrak{gl}(W_0)$ are of equal dimension. Thus, it suffices to show that Ker $\tau = \{0\}$. Suppose $F \in$ Ker τ. Then both F and $F|_{W_0}$ must have trace 0 and so must $F|_{W_1}$. Since $\dim(W_1) = 1$, the matrix F acts as 0 on W_1. This implies that F is the zero matrix. Statement (ii) follows immediately from (i). □

Lemma 3.3.8. *With above notation, each of the following statements holds:*

i) The Lie algebra $\mathfrak{g}$ is a direct sum of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathbb{C}\Phi$.

ii) The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is precisely the Lie subalgebra of T_L consisting of matrices F such that both $F|_{W_0}$ and $F|_{W_1}$ have trace 0.

Proof. Let F and G denote matrices in $\mathfrak{g}$ and observe that both $[F, G]|_{W_0}$ and $[F, G]|_{W_1}$ have trace 0. Hence, $\Phi \notin [\mathfrak{g}, \mathfrak{g}]$ and (i) holds by Proposition 3.3.7. Statement (ii) follows immediately from (i). □

Lemma 3.3.9. *With above notation, the Terwilliger algebra T is a direct sum of $[\mathfrak{g}, \mathfrak{g}]$ and the center $Z(T)$ of T. In particular, $Z(T)$ is spanned by Φ and I.*

Proof. Observe that span$\{\Phi, I\} \subseteq Z(T)$. By Lemmas 3.3.6 (i) and 3.3.8 (i), we obtain the direct sum $T = [\mathfrak{g}, \mathfrak{g}] \oplus \text{span}\{\Phi, I\}$. Suppose F is a matrix contained in $Z(T) \cap [\mathfrak{g}, \mathfrak{g}]$. Then $F|_{W_0}$ has trace 0 and is a scalar multiple of the identity map in $\mathfrak{gl}(W_0)$. Hence, $F|_{W_0}$ is the zero map in $\mathfrak{gl}(W_0)$. Consequently, F is the zero matrix by the isomorphism in Proposition 3.3.7 (i). □

Lemma 3.3.10. *Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ denote the standard basis for $\mathbb{R}^2$. With above notation, the Terwilliger algebra $\boldsymbol{T}$ has four generators namely $\boldsymbol{A}_{\boldsymbol{e}_1}, \boldsymbol{A}_{\boldsymbol{e}_2}, \boldsymbol{A}^*_{\boldsymbol{e}_1}$ and $\boldsymbol{A}^*_{\boldsymbol{e}_2}$.*

Proof. Let $\boldsymbol{M}'$ be the subalgebra of the Bose–Mesner algebra $\boldsymbol{M}$ generated by $\boldsymbol{A}_{\boldsymbol{e}_1}$ and $\boldsymbol{A}_{\boldsymbol{e}_2}$. For each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\alpha| \leqslant n$, observe that the matrix $\boldsymbol{A}_\alpha$ is in the expansion of $[\boldsymbol{A}_{\boldsymbol{e}_1}]^{\alpha_1} [\boldsymbol{A}_{\boldsymbol{e}_2}]^{\alpha_2}$ and hence, $\boldsymbol{A}_\alpha \in \boldsymbol{M}'$ by induction on $|\alpha|$. Hence $\boldsymbol{M}'$ contains all the adjacency matrices and is in fact the Bose–Mesner algebra $\boldsymbol{M}$. Similarly, we can prove that the dual Bose–Mesner algebra $\boldsymbol{M}^*$ is generated by $\boldsymbol{A}^*_{\boldsymbol{e}_1}$ and $\boldsymbol{A}^*_{\boldsymbol{e}_2}$. □

Define the unique matrix $\Delta(P)$ in $\mathrm{End}(V^{\otimes n})$ for every matrix $P \in T$ such that $\Delta(P)$ is given by

$$\Delta(P) = \frac{1}{(k-1)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(P \otimes I \otimes \cdots \otimes I).$$

Endow $V^{\otimes n}$ with a T_L-module structure such that $P \in T$ acts as $\Delta(P)$ under left multiplication. Consequently, $V^{\otimes n}$ becomes a module for Lie algebras $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 3.3.11. *With above notation, suppose W is a non-zero subspace of $V^{\otimes n}$ such that Φ acts as a scalar on W. Then the following are equivalent:*

i) W is an irreducible $\boldsymbol{T}$-module, *ii)* W *is an irreducible* $\mathfrak{g}$*-module,*

iii) W *is an irreducible* $[\mathfrak{g}, \mathfrak{g}]$*-module.*

Proof. Immediate from Lemma 3.3.6 (iii), Lemma 3.3.8 (i) and Lemma 3.3.10. □

Theorem 3.3.12. *With above notation, there exists a unital algebra homomorphism from the universal enveloping algebra* $\mathcal{U}(\mathfrak{sl}(W_0))$ *to* $\boldsymbol{T}$*. Furthermore,* $\boldsymbol{T}$ *is generated by the image of* $\mathcal{U}(\mathfrak{sl}(W_0))$ *and the center* $Z(\boldsymbol{T})$*.*

Proof. View $\boldsymbol{T}$ as a Lie algebra with respect to the usual Lie bracket and consider the map $\mathfrak{g} \to \boldsymbol{T}$ such that $P \mapsto \Delta(P)$. Under this mapping, observe that

$$A_1 \mapsto \boldsymbol{A}_{\boldsymbol{e}_1}, \qquad A_1^* \mapsto \boldsymbol{A}^*_{\boldsymbol{e}_1},$$
$$A_2 \mapsto \boldsymbol{A}_{\boldsymbol{e}_2}, \qquad A_2^* \mapsto \boldsymbol{A}^*_{\boldsymbol{e}_2}.$$

So this map is a well-defined Lie algebra homomorphism (see Lemma 3.3.6 (iii) and Lemma 3.3.10). Thus there exists a unique unital algebra homomorphism ρ from the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ to $\boldsymbol{T}$ and in fact, ρ is an epimorphism. The theorem follows from Proposition 3.3.7 (ii) and Lemma 3.3.9. □

Theorem 3.3.13. *With above notation,*

$$V^{\otimes n} \cong \bigoplus_{k=0}^{n} \bigoplus_{\lambda \in P(k,3)} \binom{n}{k} dim(V_\lambda)\ L_\lambda \otimes (W_1)^{\otimes(n-k)} \tag{3.3}$$

is a decomposition into irreducible $\boldsymbol{T}$*-modules where* $\{V_\lambda\}$ *are the Specht modules, and* $\{L_\lambda\}$ *are irreducible* $\mathfrak{sl}(W_0)$*-modules. Moreover,* $(k, \lambda) = (k', \lambda')$ *if and only if the summands* $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ *and* $L_{\lambda'} \otimes (W_1)^{\otimes(n-k')}$ *are isomorphic* $\boldsymbol{T}$*-modules.*

Proof. First we prove the isomorphism (3.3). Recall that $V = W_0 \oplus W_1$ so we obtain the isomorphism

$$V^{\otimes n} \cong \bigoplus_{k=0}^{n} \binom{n}{k} (W_0)^{\otimes k} \otimes (W_1)^{\otimes (n-k)}.$$

We obtain (3.3) by applying (3.2) to $(W_0)^{\otimes k}$. Fix an ordered pair (k, λ) where $k \leqslant n$ is a nonnegative integer and $\lambda \in P(k, 3)$. That the summand $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ is an irreducible $[\mathfrak{g}, \mathfrak{g}]$-module follows from Proposition 3.3.7 (ii) and Lemma 3.3.8 (ii). Moreover, Φ acts on $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ as a scalar multiplication by $4k - 3n$. Hence, $L_\lambda \otimes (W_1)^{\otimes k}$ is an irreducible $\boldsymbol{T}$-module by Proposition 3.3.11. The last statement follows from the action of $\Delta(\Phi)$ and Theorem 3.3.12. □

Observe that a complete set of mutually non-isomorphic irreducible $\boldsymbol{T}$-modules on $V^{\otimes n}$ is in bijection with the set of all ordered pairs (k, λ) where k is a nonnegative integer such that $k \leqslant n$ and $\lambda \in P(k, 3)$. Recall that the support W_s (resp. dual support W_s^*) of an irreducible $\boldsymbol{T}$-module W is the set of all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\alpha| \leqslant n$ such that $\boldsymbol{E}^*_\alpha W \neq 0$ (resp. $\boldsymbol{E}_\alpha W \neq 0$).

Theorem 3.3.14. *With above notation, abbreviate $W = W_{(k,\lambda)} := L_\lambda \otimes (W_1)^{\otimes(n-k)}$ for a fixed integer $0 \leqslant k \leqslant n$ and a fixed partition $\lambda \in P(k, 3)$. Let $P(\lambda)$ denote the set of all partitions in $P(k, 3)$ that are dominated by λ. Then each of W_s and W_s^* is equal to*

$$\{(\mu_0, \mu_1 + n - k, \mu_2) \mid (\mu_0, \mu_1, \mu_2) \textit{ is a permutation of some } \varepsilon \in P(\lambda)\}. \quad (3.4)$$

Moreover if $\mu = (\mu_0, \mu_1, \mu_2)$ is a permutation of some $\varepsilon \in P(\lambda)$, then each of $\boldsymbol{E}^*_\alpha W$ and $\boldsymbol{E}_\alpha W$ has dimension $K_{\lambda,\varepsilon}$ whenever $\alpha_1 = \mu_1 + n - k$ and $\alpha_2 = \mu_2$.

Proof. Identify W with $g_{\mathtt{t}}(W_0^{\otimes k}) \otimes (W_1)^{\otimes(n-k)}$ for a fixed standard λ-tableau $\mathtt{t}$ in view of Lemma 3.2.3. Now we give a basis for W consisting of common eigenvectors for the dual primitive idempotents. Fix the ordered basis $\{v_0, v_1, v_2\}$ for W_0 and the basis $\{v\}$ for W_1. Then by Lemma 3.2.5, the set of all vectors $g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)}$ forms a basis for W where β is $(\lambda, \mathtt{t})$-semistandard simple tensor. Pick a $(\lambda, \mathtt{t})$-semistandard simple tensor β and suppose $\mathsf{cont}(\beta) = (\mu_0, \mu_1, \mu_2)$. Then for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\alpha| \leqslant n$, we obtain

$$\boldsymbol{E}^*_\alpha \left(g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)}\right) = \begin{cases} g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)} & \text{if } \alpha_1 = \mu_1 + n - k \text{ and } \alpha_2 = \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

We see that W_s is equal to the set (3.4) by Corollary 3.2.7 and Remark 3.2.8. Similarly we get W_s^* by using the ordered basis $\{v_0^*, v_1^*, v_2^*\}$ for W_0 and the basis $\{v\}$ for W_1. The last statement follows from Lemma 3.2.5 and Remark 3.2.8. □

Remark 3.3.15. In [1], there is a method that describes how the multiplicities of the weights in the root system A_2 can be obtained. The set of all weights in an irreducible module for $\mathfrak{sl}_3(\mathbb{C})$ is then partitioned into layers such that weights lying on the same layer have the same multiplicities. In particular, the Kostka number $K_{\lambda,\varepsilon}$ mentioned in Theorem 3.3.14 only depends on which layer does ε belong. To further explain this, write $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ and let $P(\lambda)$ denote the set of all partitions in $P(k, 3)$ dominated by λ. Let r denote the largest nonnegative integer for which $\lambda^r := (\lambda_0 - r, \lambda_1, \lambda_2 + r) \in P(\lambda)$. Suppose $r > 0$. Then the sequence $\lambda^0, \lambda^1, \ldots, \lambda^r$ determines the layers and the Kostka number $K_{\lambda,\varepsilon}$ for each $\varepsilon \in P(\lambda)$

is given by

$$K_{\lambda,\varepsilon} = \begin{cases} s & \text{if } \lambda^{s-1} \geq \varepsilon \text{ and } \lambda^{s} \not\geq \varepsilon \text{ for some integer } 1 \leqslant s \leqslant r, \\ r+1 & \text{if } \lambda^{r} \geq \varepsilon. \end{cases}$$

Suppose $r = 0$. In this case, there is exactly one layer and $K_{\lambda,\varepsilon} = 1$ for each $\varepsilon \in P(\lambda)$. Hence, the corresponding irreducible $\mathcal{T}$-module is both thin and dual thin.

Remark 3.3.16. Observe that the irreducible $\boldsymbol{T}$-module $W_{(k,\lambda)} = L_{\lambda} \otimes (W_1)^{\otimes(n-k)}$ for a positive integer $k \leqslant n$ and $\lambda \in P(k,1)$ is isomorphic as an $\mathfrak{sl}_3(\mathbb{C})$-module to the vector space over $\mathbb{C}$ of homogeneous polynomials in mutually commuting indeterminates x, y, z with total degree k. On such an irreducible $\boldsymbol{T}$-module, we can show the relationship among the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, Rahman polynomials and rank two extension of Leonard pair as discussed in [37].

Chapter 4

An Assmus-Mattson theorem for codes over commutative association schemes

We begin by recalling the famous *Assmus–Mattson theorem* which relates linear codes and combinatorial designs:

Theorem 4.0.1 (Assmus and Mattson [2, Theorem 4.2])**.** *Let C be a linear code of length n over $\mathbb{F}_q$ with minimum weight δ. Let $C^{\perp}$ be the dual code of C, with minimum weight δ^*. Suppose that an integer t $(1 \leqslant t \leqslant n)$ is such that there are at most $\delta - t$ weights of $C^{\perp}$ in $\{1, 2, \ldots, n-t\}$, or such that there are at most $\delta^* - t$ weights of C in $\{1, 2, \ldots, n-t\}$. Then the supports of the words of any fixed weight in C form a t-design (with possibly repeated blocks).*

We remark that [2, Theorem 4.2] also includes a criterion for obtaining simple t-designs (see Section 4.1 for definition), but we will not pay much attention in this chapter to the simplicity of the resulting designs. There are several proofs and strengthenings of Theorem 4.0.1; see, e.g., [3, 15, 16, 38, 53, 58, 60]. The purpose of this chapter is to establish a theorem which unifies many of the known generalizations and extensions of Theorem 4.0.1.

Constructing t-designs from codes received renewed interest when Gulliver and Harada [26] and Harada [29] found new 5-designs by computer from the lifted Golay code of length 24 over $\mathbb{Z}_4$ (among others). Their constructions were later explained and generalized further by Bonnecaze, Rains, and Solé [9]. Motivated by these results, Tanabe [57] obtained an Assmus–Mattson-type theorem for $\mathbb{Z}_4$-linear codes with respect to the symmetrized weight enumerator. Tanabe's theorem can indeed capture the 5-designs from the lifted Golay code over $\mathbb{Z}_4$, but the conditions in his theorem involve finding the ranks of matrices having quite complicated entries, so that it is hard to verify the conditions without the help of a computer. Tanabe [59] then presented a simpler version of his theorem, and we can easily check its conditions by hand for the lifted Golay code over $\mathbb{Z}_4$.

To be somewhat concrete, by an *Assmus–Mattson-type* theorem, we mean in this chapter a theorem which enables us to find t-designs by just looking at some kind of weight enumerator of a code (plus a bit of extra information in some cases, e.g., linearity). Such a theorem is not always the best way to estimate the parameter t of the resulting designs as it does not take into account the structure of the code at all (cf. Remark 4.4.5), but instead it has a great advantage in its wide range of applicability.

As stated in the introduction, we consider the Hamming association schemes when we are dealing with codes and their Hamming weight enumerator (see Section 4.4.1 for definition) as in Theorem 4.0.1. Hamming association schemes are examples of metric and cometric association schemes, and Theorem 4.0.1 can be interpreted and generalized from this point of view; cf. [60]. On the other hand, in situations where we focus on a more complicated type of weight enumerator of a block code, we think of the code in question (say, of length n) as lying in a structure much finer than a Hamming association scheme; that is to say, the alphabet itself naturally becomes the vertex set of a commutative association scheme with s classes where $s \geqslant 2$, and we consider its extension of length n. Hamming association schemes are the same thing as extensions of 1-class (i.e., trivial) association schemes, but if $s \geqslant 2$ then its extensions are no longer metric nor cometric.

In this chapter, we prove a general Assmus–Mattson-type theorem for codes in extensions of arbitrary commutative association schemes. Our main results are Theorem 4.1.1 and Supplements 4.1.2–4.1.5. In general, the weights of a code take the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$, where the α_i are non-negative integers such that $\sum_{i=1}^{s} \alpha_i \leqslant n$. We count the *number* of weights in a given interval when $s = 1$ as in Theorem 4.0.1, but if $s \geqslant 2$ then instead we speak of the *minimal degree* of subspaces of the polynomial ring $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$ which allow unique Lagrange interpolation with respect to those weights (which are lattice points in $\mathbb{R}^s$) contained in a given region. When specialized to the case of $\mathbb{Z}_4$-linear codes with the symmetrized weight enumerator as in [57, 59], the association scheme on the alphabet $\mathbb{Z}_4$ has two classes R_1 and R_2, together with the identity class R_0, defined by

$$(x, y) \in R_i \iff y - x = \pm i \pmod{4} \quad (x, y \in \mathbb{Z}_4)$$

for $i \in \{0, 1, 2\}$, and our results give a slight extension of Tanabe's theorem in [59]. The Assmus–Mattson-type theorem for $\mathbb{Z}_4$-linear codes with the Hamming weight enumerator due to Shin, Kumar, and Helleseth [52] can also be recovered. To prove our results, we make heavy use of the representation theory of the Terwilliger algebra [63, 64, 65], which is a non-commutative semisimple matrix $\mathbb{C}$-algebra attached to each vertex of an association scheme. See, e.g., [51, 22, 60, 4] for more applications of the Terwilliger algebra to coding theory and design theory.

The layout of this chapter is as follows. In Section 4.1, we recall important concepts from polynomial interpolation and state our main results. Section 4.2 is devoted to their proofs. Finally, we discuss a number of examples in Section 4.4. The entire chapter is based on [47].

4.1 Statement of main results

We recall some concepts from polynomial interpolation (see [21]). Throughout the section, let S denote a finite set of points in $\mathbb{R}^s$. A linear subspace $\mathscr{L}$ of the polynomial ring $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$ is called an *interpolation space* with respect to S if, for every $f \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$, there is a unique $g \in \mathscr{L}$ such that $f(z) = g(z)$ for all $z = (z_1, \ldots, z_s) \in S$. We call $\mathscr{L}$ a *minimal degree interpolation space* with respect to S if, moreover, the *interpolant* g always satisfies $\deg f \geqslant \deg g$.

Let $\mathscr{M}(S)$ be a minimal degree interpolation space with respect to S and define

$$\mu(S) = \max\{\deg f : f \in \mathscr{M}(S)\}.$$

We note that $\mathscr{M}(S)$ exists; see Theorem 4.1.6 below. Observe also that the scalar $\mu(S)$ is well-defined, that is, independent of the choice of $\mathscr{M}(S)$.

From here on, we assume that $(X, \mathscr{R})$ is a commutative association scheme with fixed base vertex x_0 and we shall adopt the notation in Chapter 2. For each vertex $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in X^n$, define

$$\mathrm{supp}(\boldsymbol{x}) = \{\ell : x_\ell \neq x_0\} \subset \{1, 2, \ldots, n\}.$$

We call $\mathrm{supp}(\boldsymbol{x})$ the *support* of $\boldsymbol{x}$ (with respect to $\boldsymbol{x}_0$). Recall that a t-(n,k,λ) *design* (or simply a t-*design*) is an incidence structure of points and blocks such that the following conditions hold:

(TD1) there are exactly n points,

(TD2) each block contains exactly k points,

(TD3) for any t points, there are exactly λ blocks containing these points.

In this section, we state our main results and we discuss proofs of our main results in the next section.

Theorem 4.1.1. *Let C be a code in X^n. Let*

$$S_r = \{\alpha \in \mathbb{N}^s : r \leqslant |\alpha| \leqslant n - r,\, \boldsymbol{E}^*_\alpha \hat{C} \neq 0\} \quad (1 \leqslant r \leqslant \lfloor n/2 \rfloor),$$

and let $\delta^* = \min\{i \neq 0 : \boldsymbol{E}_i \hat{C} \neq 0\}$. Suppose there exists an integer t $(1 \leqslant t \leqslant n)$

such that

$$\mu(S_r) < \delta^* - r \quad (1 \leqslant r \leqslant t). \tag{4.1}$$

Then the multiset

$$\{\mathrm{supp}(\boldsymbol{x}) : \boldsymbol{x} \in (X^n)_\alpha \cap C\} \tag{4.2}$$

is a t-design (with block size $|\alpha|$) for every $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant n$.

We use Theorem 4.1.1 together with the following "supplements".

Supplement 4.1.2. *Let C be a code in X^n. Assume that we are given in advance a set $K \subset \mathbb{N}^s$ such that the multiset* (4.2) *is a t-design for every $\alpha \in K$. Then the condition* (4.1) *in Theorem 4.1.1 may be replaced by*

$$\mu(S_r \backslash K) < \delta^* - r \quad (1 \leqslant r \leqslant t).$$

We call a subset C of X^n a *weakly t-balanced array*[1] over $(X, \mathscr{R})$ (with respect to $\boldsymbol{x}_0$) if, for any $\Lambda \subset \{1, 2, \ldots, n\}$ and $\gamma \in \mathbb{N}^s$ such that $|\gamma| \leqslant |\Lambda| \leqslant t$, the number

$$\left|\left\{\boldsymbol{x} \in C : (x_i)_{i \in \Lambda} \in (X^{|\Lambda|})_\gamma\right\}\right|$$

depends only on $|\Lambda|$ and γ.

Recall that, when considering translation association scheme, we always choose the identity as the base vertex.

Supplement 4.1.3. *Suppose that $(X, \mathscr{R})$ is a translation association scheme, and that C is an additive code in X^n. Assume that we are given in advance a set*

[1]This term is meant as only provisional (see [55]).

$L \subset \mathbb{N}^s$ such that, for every $\alpha \in L$, $(X^{*n})_\alpha \cap C^\perp$ is a weakly t-balanced array over $(X^*, \mathscr{R}^*)$. Then the scalar δ^* in Theorem 4.1.1 may be replaced by

$$\min\{|\alpha| : 0 \neq \alpha \in \mathbb{N}^s \backslash L,\ \boldsymbol{E}_\alpha \hat{C} \neq 0\}. \tag{4.3}$$

Remark 4.1.4. We discuss a special case of weakly t-balanced array over $(X^*, \mathscr{R}^*)$. Recall that $\{\boldsymbol{e}_i\}_{i=1}^s$ is the standard basis for $\mathbb{R}^s$. Let $\alpha = h\boldsymbol{e}_i$ for some $0 \leq i \leq s$ and for some $h > 0$. Observe that the condition $(X^{*n})_\alpha \cap C^\perp$ is a weakly-t-balanced array over $(X^*, \mathscr{R}^*)$ is equivalent to saying that the multiset

$$\{\mathrm{supp}(\boldsymbol{x}) : \boldsymbol{x} \in (X^{*n})_\alpha \cap C^\perp\}$$

is a t-design.

Supplement 4.1.5 below was inspired by [59, Theorem 2] and allows us to estimate $\mu(S)$, and hence t, by geometrical considerations; see Section 4.4. It is a general result about minimal degree interpolation spaces, so that we give a proof right after the statement.

Supplement 4.1.5. *Let S be a finite set of points in $\mathbb{R}^s$. Suppose that there are real scalars $z_{i\ell}$ $(1 \leqslant i \leqslant s,\ \ell \in \mathbb{N})$, a positive integer m, and a linear automorphism $\sigma \in \mathrm{GL}(\mathbb{R}^s)$ such that $z_{ik} \neq z_{i\ell}$ whenever $k \neq \ell$, and that*

$$\sigma(S) \subset \{(z_{1\alpha_1}, z_{2\alpha_2}, \ldots, z_{s\alpha_s}) \in \mathbb{R}^s : \alpha \in \mathbb{N}^s,\ |\alpha| \leqslant m\}. \tag{4.4}$$

Then $\mu(S) \leqslant m$.

Proof. We abbreviate $z_\alpha := (z_{1\alpha_1}, z_{2\alpha_2}, \ldots, z_{s\alpha_s})$. Let Σ denote the RHS in (4.4).

It suffices to show that $\mu(\Sigma) \leqslant m$. To this end, we construct an interpolation space with respect to Σ with maximum degree at most m as follows. Let $\alpha \in \mathbb{N}^s$ be given with $|\alpha| \leqslant m$, and assume that we have constructed polynomials

$$f_\beta \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s] \qquad (\beta \in \mathbb{N}^s,\ |\alpha| < |\beta| \leqslant m)$$

such that $\deg f_\beta \leqslant m$ and

$$f_\beta(z_\gamma) = \delta_{\beta\gamma} \quad (\gamma \in \mathbb{N}^s,\ |\gamma| \leqslant m).$$

Define $g_\alpha \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$ by

$$g_\alpha = \prod_{i=1}^{s} \prod_{\ell=0}^{\alpha_i - 1} \frac{\xi_i - z_{i\ell}}{z_{i\alpha_i} - z_{i\ell}},$$

and let

$$f_\alpha = g_\alpha - \sum_{\substack{\beta \in \mathbb{N}^s \\ |\alpha| < |\beta| \leqslant m}} g_\alpha(z_\beta) f_\beta.$$

Then $\deg f_\alpha \leqslant m$, and it is easy to see that

$$f_\alpha(z_\gamma) = \delta_{\alpha\gamma} \quad (\gamma \in \mathbb{N}^s,\ |\gamma| \leqslant m). \tag{4.5}$$

Thus, by induction we obtain polynomials f_α with $\deg f_\alpha \leqslant m$ satisfying (4.5) for all $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant m$. It is clear that the subspace

$$\sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha| \leqslant m}} \mathbb{R} f_\alpha \subset \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$$

is an interpolation space with respect to Σ and hence $\mu(\Sigma) \leq m$. $\square$

We end this section by recalling a construction of a minimal degree interpolation space due to de Boor and Ron (see [10, 11]). See also [21, §3]. For every non-zero element $f = \sum_{i=0}^{\infty} f_i$ in the ring of formal power series $\mathbb{R}[[\xi_1, \xi_2, \ldots, \xi_s]]$ where f_i is homogeneous of degree i, let

$$f_{\downarrow} = f_{i_0},$$

where $i_0 = \min\{i : f_i \neq 0\}$. We conventionally set $0_{\downarrow} := 0$.

Theorem 4.1.6 ([10, 11])**.** *Let S be a finite set of points in $\mathbb{R}^s$. Let $\mathscr{E}$ be the subspace of $\mathbb{R}[[\xi_1, \xi_2, \ldots, \xi_s]]$ spanned by the exponential functions*

$$\exp\left(\sum_{i=1}^{s} z_i \xi_i\right) \quad ((z_1, z_2, \ldots, z_s) \in S).$$

Then the subspace

$$\sum_{f \in \mathscr{E}} \mathbb{R} f_{\downarrow} \subset \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_s]$$

is a minimal degree interpolation space with respect to S.

Theorem 4.1.6 immediately leads to the following formula for $\mu(S)$ which is well suited for computer calculations:

Supplement 4.1.7. *For every finite set S of points in $\mathbb{R}^s$, the scalar $\mu(S)$ equals the smallest $m \in \mathbb{N}$ for which the polynomials*

$$\sum_{k=0}^{m} \left(\sum_{i=1}^{s} z_i \xi_i\right)^k \quad ((z_1, z_2, \ldots, z_s) \in S)$$

are linearly independent.

(Note that we just discarded the irrelevant factors $1/(k!)$ in the Taylor polynomials of these exponential functions.)

4.2 Preliminary lemmas

We being by proving a few preliminary lemmas that are necessary in verifying our main results. Recall the space U_0 spanned by the irreducible $\boldsymbol{T}_H$-modules in $V^{\otimes n}$ with displacement 0 (see latter part of Section 2.4). Let

$$\pi_{U_0} : V^{\otimes n} \to U_0$$

denote the orthogonal projection onto U_0. Note that π_{U_0} is a $\boldsymbol{T}_H$-homomorphism.

Lemma 4.2.1. *The primary $\boldsymbol{T}$-module $\boldsymbol{M}\hat{\boldsymbol{x}}_0$ is orthogonal to every non-primary irreducible $\boldsymbol{T}_H$-module in U_0.*

Proof. Let $u_0 = \hat{x}_0$ and $u_1 = \hat{X} - \hat{x}_0$. For every $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \{0,1\}^n$, let

$$\boldsymbol{u}_\tau = u_{\tau_1} \otimes u_{\tau_2} \otimes \cdots \otimes u_{\tau_n} \in \boldsymbol{E}^*_{|\tau|} U_0,$$

where $|\tau| = \sum_{\ell=1}^{n} \tau_\ell$ denotes the weight of τ. The $\boldsymbol{u}_\tau$ form an orthogonal basis of U_0 by (2.20), and we have

$$||\boldsymbol{u}_\tau||^2 = (|X| - 1)^{|\tau|}. \tag{4.6}$$

For every $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant n$, we have

$$\begin{aligned}\langle \boldsymbol{A}_\alpha \hat{\boldsymbol{x}}_0, \boldsymbol{u}_\tau \rangle &= \delta_{|\tau|,|\alpha|} \sum_{(\boldsymbol{x}_0, \boldsymbol{y}) \in \boldsymbol{R}_\alpha} \langle \hat{\boldsymbol{y}}, \boldsymbol{u}_\tau \rangle \\ &= \delta_{|\tau|,|\alpha|} \binom{|\alpha|}{\alpha_1, \alpha_2, \dots, \alpha_s} \prod_{i=1}^{s} (P_{0i})^{\alpha_i}, \end{aligned} \tag{4.7}$$

where we recall that P_{0i} is the degree of graph (X, R_i). Since

$$\pi_{U_0} \boldsymbol{A}_\alpha \hat{\boldsymbol{x}}_0 = \sum_{\tau \in \{0,1\}^n} ||\boldsymbol{u}_\tau||^{-2} \langle \boldsymbol{A}_\alpha \hat{\boldsymbol{x}}_0, \boldsymbol{u}_\tau \rangle \boldsymbol{u}_\tau,$$

it follows from (4.6) and (4.7) that $\pi_{U_0} \boldsymbol{A}_\alpha \hat{\boldsymbol{x}}_0$ is a scalar multiple of

$$\sum_{\substack{\tau \in \{0,1\}^n \\ |\tau| = |\alpha|}} \boldsymbol{u}_\tau = \boldsymbol{A}_{|\alpha|} \hat{\boldsymbol{x}}_0 \in \boldsymbol{M}_H \hat{\boldsymbol{x}}_0.$$

It follows that $\pi_{U_0} \boldsymbol{M} \hat{\boldsymbol{x}}_0 = \boldsymbol{M}_H \hat{\boldsymbol{x}}_0$, as desired. □

Lemma 4.2.2. *Let C be a non-empty subset of $(X^n)_k$ for some $0 \leqslant k \leqslant n$. Then the following are equivalent:*

(i) The multiset $\{\mathrm{supp}(\boldsymbol{x}) : \boldsymbol{x} \in C\}$ is a t-design.

(ii) $\boldsymbol{E}_i \pi_{U_0} \hat{C}$ is a scalar multiple of $\boldsymbol{E}_i \hat{\boldsymbol{x}}_0$ for every $0 \leqslant i \leqslant t$.

(iii) $\hat{C}$ is orthogonal to every non-primary irreducible $\boldsymbol{T}_H$-module in U_0 with endpoint at most t.

Proof. First, we show the equivalence of (i) and (ii). To this end, we introduce another orthogonal basis of U_0 as follows. Define $v_0, v_1 \in V$ by

$$v_0 = E_0\hat{x}_0 = |X|^{-1}\hat{X}, \quad v_1 = (I - E_0)\hat{x}_0 = \hat{x}_0 - |X|^{-1}\hat{X}. \tag{4.8}$$

Note that

$$||v_0||^2 = |X|^{-1}, \quad ||v_1||^2 = 1 - |X|^{-1}, \quad \langle v_0, v_1\rangle = 0.$$

For every $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \{0,1\}^n$, let

$$\boldsymbol{v}_\tau = v_{\tau_1} \otimes v_{\tau_2} \otimes \cdots \otimes v_{\tau_n} \in \boldsymbol{E}_{|\tau|}U_0,$$

where $|\tau| = \sum_{\ell=1}^{n} \tau_\ell$. The $\boldsymbol{v}_\tau$ form an orthogonal basis of U_0 by (2.20), and we have

$$||\boldsymbol{v}_\tau||^2 = |X|^{-n}\,(|X| - 1)^{|\tau|}$$

Moreover, observe that

$$\sum_{\substack{\tau\in\{0,1\}^n \\ |\tau|=i}} \boldsymbol{v}_\tau = \boldsymbol{E}_i\hat{\boldsymbol{x}}_0.$$

By these comments and since

$$\boldsymbol{E}_i\pi_{U_0}\hat{C} = \sum_{\substack{\tau\in\{0,1\}^n \\ |\tau|=i}} ||\boldsymbol{v}_\tau||^{-2}\langle\hat{C}, \boldsymbol{v}_\tau\rangle\, \boldsymbol{v}_\tau,$$

it follows that (ii) holds if and only if $\langle\hat{C}, \boldsymbol{v}_\tau\rangle$ depends only on $|\tau|$ whenever $|\tau| \leqslant t$.

Assume that (i) holds. Let $\tau \in \{0,1\}^n$ with $|\tau| \leqslant t$. From (4.8) it follows that

$$\langle \hat{C}, \boldsymbol{v}_\tau \rangle = |X|^{-n} \sum_{i=0}^{|\tau|} (-1)^i (|X|-1)^{|\tau|-i} \times \Big| \big\{ \boldsymbol{x} \in C : |\operatorname{supp}(\tau) \cap \operatorname{supp}(\boldsymbol{x})| = i \big\} \Big|,$$

which is indeed a constant depending only on $|\tau|$, and hence (ii) holds.

Conversely, assume that (ii) holds. Let $\tau \in \{0,1\}^n$ with $|\tau| = t$, and let

$$\boldsymbol{w}_\tau = w_{\tau_1} \otimes w_{\tau_2} \otimes \cdots \otimes w_{\tau_n},$$

where $w_0 = \hat{X} = |X| v_0$ and $w_1 = \hat{x}_0 = v_0 + v_1$. On the one hand, we have

$$\langle \hat{C}, \boldsymbol{w}_\tau \rangle = |\{ \boldsymbol{x} \in C : \operatorname{supp}(\tau) \cap \operatorname{supp}(\boldsymbol{x}) = \emptyset \}|. \tag{4.9}$$

On the other hand, observe that

$$\boldsymbol{w}_\tau = |X|^{n-t} \sum_{\rho} \boldsymbol{v}_\rho,$$

where the sum is over $\rho \in \{0,1\}^n$ with $\operatorname{supp}(\rho) \subset \operatorname{supp}(\tau)$. It follows that the common value in (4.9) is independent of the choice of τ, and hence (i) holds. We have now shown the equivalence of (i) and (ii).

Finally, we show the equivalence of (ii) and (iii). Observe that

$$\boldsymbol{E}_i U_0 = \mathbb{C}\boldsymbol{E}_i \hat{\boldsymbol{x}}_0 \perp \sum_{W} \boldsymbol{E}_i W \quad (0 \leqslant i \leqslant t), \tag{4.10}$$

where the sum is over the non-primary irreducible $\boldsymbol{T}_H$-modules W in U_0 with endpoint at most i. If (iii) holds, then the vectors $\boldsymbol{E}_i \hat{C}$ $(0 \leqslant i \leqslant t)$ are also orthogonal to every non-primary irreducible $\boldsymbol{T}_H$-module in U_0 with endpoint at most t, and hence the vector $\pi_{U_0} \boldsymbol{E}_i \hat{C} \in \boldsymbol{E}_i U_0$ vanishes on the second term of the RHS in (4.10) for every $0 \leqslant i \leqslant t$. In other words, (ii) holds.

Conversely, let W be a non-primary irreducible $\boldsymbol{T}_H$-module in U_0 with endpoint $r \leqslant t$, and assume that $\hat{C}$ is not orthogonal to W. Let $\pi_W : V^{\otimes n} \to W$ be the orthogonal projection onto W. Then we have $\pi_W \hat{C} \neq 0$. Let

$$\ell = \min\{i : \boldsymbol{E}_i \pi_W \hat{C} \neq 0\}.$$

By Lemma 2.4.1 (iii), $\boldsymbol{E}_\ell \pi_W \hat{C}$ spans $\boldsymbol{E}_\ell W$. In view of Lemma 2.4.1 (ii), (v), we have

$$\boldsymbol{E}_r (\boldsymbol{A}_1^*)^{\ell - r} \pi_W \hat{C} = \boldsymbol{E}_r (\boldsymbol{A}_1^*)^{\ell - r} \sum_{j=0}^{n} \boldsymbol{E}_j \pi_W \hat{C} = \boldsymbol{E}_r (\boldsymbol{A}_1^*)^{\ell - r} \boldsymbol{E}_\ell \pi_W \hat{C} \neq 0.$$

Since π_W is a $\boldsymbol{T}_H$-homomorphism and since $\hat{C} \in \boldsymbol{E}_k^* V^{\otimes n}$, it follows from (2.18) that

$$0 \neq \boldsymbol{E}_r \pi_W (\boldsymbol{A}_1^*)^{\ell - r} \hat{C} = (\theta_k^*)^{\ell - r} \boldsymbol{E}_r \pi_W \hat{C},$$

and hence we have $\ell = r$. It follows that $\boldsymbol{E}_i \pi_{U_0} \hat{C}$ does not vanish on the second term of the RHS in (4.10) when $i = r$, and hence (ii) fails to hold. We have now shown that (ii) and (iii) are equivalent. This completes the proof of the lemma. $\square$

Lemma 4.2.3. *Let C be a non-empty subset of $(X^n)_\alpha$ for some $\alpha \in \mathbb{N}^s$ satisfying $|\alpha| \leqslant n$. Suppose that C is a weakly t-balanced array over $(X, \mathscr{R})$. Then*

$$\boldsymbol{E}_i^* \pi_{U_0} \boldsymbol{M} \hat{C} = \mathbb{C} \boldsymbol{A}_i \hat{\boldsymbol{x}}_0 \quad (0 \leqslant i \leqslant t).$$

Proof. Fix an element $\beta \in \mathbb{N}^s$ with $|\beta| \leqslant n$, and consider the vector $\boldsymbol{A}_\beta \hat{C} \in \boldsymbol{M}\hat{C}$. We use the notation in the proof of Lemma 4.2.1. Let $\tau \in \{0,1\}^n$ with $|\tau| \leqslant t$. We will use $'$ and $''$ to denote objects associated with the extensions of $(X, \mathscr{R})$ of lengths $|\tau|$ and $n - |\tau|$, respectively; e.g., $\boldsymbol{A}'_\gamma$ ($\gamma \in \mathbb{N}^s$, $|\gamma| \leqslant |\tau|$), $\boldsymbol{A}'_i$ ($0 \leqslant i \leqslant |\tau|$), $\boldsymbol{x}'_0 \in X^{|\tau|}$ for the former. We understand that the coordinates of $X^{|\tau|}$ and $X^{n-|\tau|}$ are indexed by $\operatorname{supp}(\tau)$ and $\{1, 2, \ldots, n\} \setminus \operatorname{supp}(\tau)$, respectively. With this notation established, we have

$$\boldsymbol{A}_\beta = \sum_\nu \boldsymbol{A}'_\nu \otimes \boldsymbol{A}''_{\beta-\nu},$$

where the sum is over $\nu \in \mathbb{N}^s$ such that $\beta - \nu \in \mathbb{N}^s$, $|\nu| \leqslant |\tau|$, and $|\beta - \nu| \leqslant n - |\tau|$. Observe also that

$$\boldsymbol{u}_\tau = \boldsymbol{A}'_{|\tau|} \hat{\boldsymbol{x}}'_0 \otimes \hat{\boldsymbol{x}}''_0.$$

Hence we have

$$\begin{aligned} \langle \boldsymbol{A}_\beta \hat{C}, \boldsymbol{u}_\tau \rangle &= \sum_{\nu,\rho} g_{\nu\rho} \cdot \left\langle \hat{C}, (\boldsymbol{A}'_\rho)^\dagger \hat{\boldsymbol{x}}'_0 \otimes (\boldsymbol{A}''_{\beta-\nu})^\dagger \hat{\boldsymbol{x}}''_0 \right\rangle \\ &= \sum_{\nu,\rho} g_{\nu\rho} \cdot \left| \left\{ \boldsymbol{x} \in C : (x_i)_{i \in \operatorname{supp}(\tau)} \in (X^{|\tau|})_\rho \right\} \right|, \qquad (4.11) \end{aligned}$$

where the sums are over $\nu, \rho \in \mathbb{N}^s$ such that $|\nu|, |\rho| \leqslant |\tau|$, $\beta - \nu = \alpha - \rho \in \mathbb{N}^s$, and $|\beta - \nu| \leqslant n - |\tau|$, and where we write

$$\boldsymbol{A}'_\nu \boldsymbol{A}'_{|\tau|} = \boldsymbol{A}'_{|\tau|} \boldsymbol{A}'_\nu = \sum_{\substack{\rho \in \mathbb{N}^s \\ |\rho| \leqslant |\tau|}} g_{\nu\rho} \boldsymbol{A}'_\rho.$$

By the assumption, the RHS in (4.11) depends only on $|\tau| \leqslant t$, and hence it follows that $\boldsymbol{E}_i^* \pi_{U_0} \boldsymbol{A}_\beta \hat{C}$ is a scalar multiple of $\boldsymbol{A}_i \hat{\boldsymbol{x}}_0$ for every $0 \leqslant i \leqslant t$ as in the proof of Lemma 4.2.1. We have now shown that $\boldsymbol{E}_i^* \pi_{U_0} \boldsymbol{M} \hat{C}$ is a subspace of $\mathbb{C} \boldsymbol{A}_i \hat{\boldsymbol{x}}_0$ for $0 \leqslant i \leqslant t$. That it is non-zero and hence agrees with $\mathbb{C} \boldsymbol{A}_i \hat{\boldsymbol{x}}_0$ follows from

$$\boldsymbol{E}_i^* \pi_{U_0} J^{\otimes n} \hat{C} = |C| \boldsymbol{E}_i^* \pi_{U_0} \hat{X}^{\otimes n} = |C| \boldsymbol{A}_i \hat{\boldsymbol{x}}_0.$$

This completes the proof. □

4.3 Proofs of main results

In this section, we provide proofs of the main results mentioned in Section 4.1. For convenience, we break down the proofs into subsections.

4.3.1 Proof of Theorem 4.1.1

Define $\boldsymbol{D}_1^*, \boldsymbol{D}_2^*, \ldots, \boldsymbol{D}_s^* \in \boldsymbol{M}^*$ by

$$\boldsymbol{D}_i^* = \sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha| \leqslant n}} \alpha_i \boldsymbol{E}_\alpha^* \quad (1 \leqslant i \leqslant s).$$

Observe that the $\boldsymbol{D}_i^*$ generate $\boldsymbol{M}^*$. By (2.2), (2.3), and (2.13), for $1 \leqslant j \leqslant s$ we have

$$\begin{aligned}\sum_{i=1}^{s} P_{ij}\boldsymbol{A}^*_{\boldsymbol{e}_i} &= \sum_{\substack{\alpha\in\mathbb{N}^s\\|\alpha|\leqslant n}}\sum_{i=1}^{s} P_{ij}\left(\sum_{h=0}^{s}\alpha_h Q_{hi}\right)\boldsymbol{E}^*_\alpha\\ &= \sum_{\substack{\alpha\in\mathbb{N}^s\\|\alpha|\leqslant n}}\left(\sum_{h=0}^{s}\alpha_h\sum_{i=1}^{s} Q_{hi}P_{ij}\right)\boldsymbol{E}^*_\alpha\\ &= \sum_{\substack{\alpha\in\mathbb{N}^s\\|\alpha|\leqslant n}}\left(\sum_{h=0}^{s}\alpha_h\big(|X|\delta_{hj} - Q_{h0}P_{0j}\big)\right)\boldsymbol{E}^*_\alpha\\ &= |X|\boldsymbol{D}^*_j - nP_{0j}\, I^{\otimes n}, \end{aligned} \tag{4.12}$$

where we have used $\alpha_0 = n - |\alpha|$. In particular, the $\boldsymbol{A}^*_{\boldsymbol{e}_i}$ also generate $\boldsymbol{M}^*$.

Now, fix $\alpha \in \mathbb{N}^s$ with $|\alpha| \leqslant n$. We invoke Lemma 4.2.2 to show that the multiset (4.2) is a t-design. Let W be a non-primary irreducible $\boldsymbol{T}_H$-module in U_0 with endpoint $r \leqslant t$. Recall that W has diameter $n - 2r$. It suffices to show that $\boldsymbol{E}^*_\alpha \hat{C}$ is orthogonal to W. Let $\pi_W : V^{\otimes n} \to W$ denote the orthogonal projection onto W. First, we show that

$$\pi_W \boldsymbol{E}^*_\alpha \hat{C} \in \sum_{i=\delta^*-\mu_r}^{n-r} \boldsymbol{E}_i W, \tag{4.13}$$

where $\mu_r := \mu(S_r)$. To this end, let $f \in \mathscr{M}(S_r)$ be such that

$$f(\beta) = \delta_{\alpha\beta} \quad (\beta \in S_r).$$

Observe that

$$f(\boldsymbol{D}_1^*, \boldsymbol{D}_2^*, \ldots, \boldsymbol{D}_s^*) - \boldsymbol{E}_\alpha^* \in \sum_{\beta \notin S_r} \mathbb{R}\boldsymbol{E}_\beta^*.$$

Since

$$W \subset \sum_{i=r}^{n-r} \boldsymbol{E}_i^* V^{\otimes n}$$

by Lemma 2.4.1 (iii), we have

$$\pi_W \boldsymbol{E}_\beta^* \hat{C} = 0 \ \text{ unless } \ \beta \in S_r, \tag{4.14}$$

from which it follows that

$$\pi_W \boldsymbol{E}_\alpha^* \hat{C} = \pi_W f(\boldsymbol{D}_1^*, \boldsymbol{D}_2^*, \ldots, \boldsymbol{D}_s^*)\hat{C}. \tag{4.15}$$

Let U denote the orthogonal complement of the primary $\boldsymbol{T}$-module $\boldsymbol{M}\hat{\boldsymbol{x}}_0$ in $V^{\otimes n}$, and let $\pi_U : V^{\otimes n} \to U$ denote the orthogonal projection onto U. We note that $\pi_U \boldsymbol{E}_0 = \boldsymbol{E}_0 \pi_U = 0$ since $\boldsymbol{E}_0 V^{\otimes n} \subset \boldsymbol{M}\hat{\boldsymbol{x}}_0$, so that

$$\pi_U \hat{C} \in \sum_{i=\delta^*}^{n} \boldsymbol{E}_i V^{\otimes n}. \tag{4.16}$$

Moreover, since π_U is a $\boldsymbol{T}$-homomorphism and since $W \subset U$ by Lemma 4.2.1, we have

$$\pi_W \boldsymbol{B}^* \hat{C} = \pi_W \pi_U \boldsymbol{B}^* \hat{C} = \pi_W \boldsymbol{B}^* \pi_U \hat{C} \quad (\boldsymbol{B}^* \in \boldsymbol{M}^*). \tag{4.17}$$

By the definition of μ_r and (4.12), $f(\boldsymbol{D}_1^*, \boldsymbol{D}_2^*, \ldots, \boldsymbol{D}_s^*)$ is written as a polynomial in the $\boldsymbol{A}_{\boldsymbol{e}_i}^*$ with degree at most μ_r. For any $\beta, \gamma \in \mathbb{N}^s$ with $|\beta|, |\gamma| \leqslant n$, we also have

$$\boldsymbol{E}_\beta \boldsymbol{A}^*_{\boldsymbol{e}_i} \boldsymbol{E}_\gamma = 0 \quad \text{if } \big||\beta| - |\gamma|\big| > 1$$

by virtue of (2.4) and (the dual of) (2.14). Hence it follows from (4.15), (4.16), and (4.17) that

$$\begin{aligned}\pi_W \boldsymbol{E}^*_\alpha \hat{C} &\in \pi_W \sum_{i=\delta^*-\mu_r}^{n} \boldsymbol{E}_i V^{\otimes n} \\ &= \sum_{i=\delta^*-\mu_r}^{n-r} \boldsymbol{E}_i W.\end{aligned}$$

This proves (4.13).

Assume now that $\boldsymbol{E}^*_\alpha \hat{C}$ is not orthogonal to W, i.e., $\pi_W \boldsymbol{E}^*_\alpha \hat{C} \neq 0$. Let

$$\ell = \min\{i : \boldsymbol{E}_i \pi_W \boldsymbol{E}^*_\alpha \hat{C} \neq 0\}.$$

By Lemma 2.4.1 (iii), $\boldsymbol{E}_\ell \pi_W \boldsymbol{E}^*_\alpha \hat{C}$ spans $\boldsymbol{E}_\ell W$. In view of Lemma 2.4.1 (ii), (v), we have

$$\boldsymbol{E}_r (\boldsymbol{A}^*_1)^{\ell-r} \pi_W \boldsymbol{E}^*_\alpha \hat{C} = \boldsymbol{E}_r (\boldsymbol{A}^*_1)^{\ell-r} \boldsymbol{E}_\ell \pi_W \boldsymbol{E}^*_\alpha \hat{C} \neq 0.$$

Since π_W is a $\boldsymbol{T}_H$-homomorphism, it follows from (2.18) that

$$0 \neq \boldsymbol{E}_r \pi_W (\boldsymbol{A}^*_1)^{\ell-r} \boldsymbol{E}^*_\alpha \hat{C} = (\theta^*_{|\alpha|})^{\ell-r} \boldsymbol{E}_r \pi_W \boldsymbol{E}^*_\alpha \hat{C}.$$

Therefore, we must have $\ell = r$. However, this contradicts (4.13) since $\delta^* - \mu_r > r$ by (4.1). It follows that $\pi_W \boldsymbol{E}^*_\alpha \hat{C} = 0$, and the proof is complete.

4.3.2 Proof of Supplement 4.1.2

The most important step in the proof of Theorem 4.1.1 was to establish (4.13), and the first key observation (4.15) in this process was based on (4.14). By using Lemma 4.2.2, (4.14) can now be improved as follows:

$$\pi_W \boldsymbol{E}_\beta^* \hat{C} = 0 \;\text{ unless }\; \beta \in S_r \backslash K.$$

Hence it suffices to interpolate on $S_r \backslash K$, as desired.

4.3.3 Proof of Supplement 4.1.3

At the end of the proof of Theorem 4.1.1, we used (4.13) and assumption $\delta^* - \mu_r > r$ to show that $\pi_W \boldsymbol{E}_\alpha^* \hat{C} = 0$. Observe that we arrive at the same conclusion if we can instead prove that

$$\pi_W \boldsymbol{E}_\alpha^* \hat{C} \in \sum_{i=r+1}^{n-r} \boldsymbol{E}_i W. \tag{4.18}$$

Let δ_L^* denote the scalar in (4.3), and recall that we are assuming that $\delta_L^* - \mu_r > r$. Then (4.16) becomes

$$\pi_U \left(\hat{C} - \sum_{\beta \in L} \boldsymbol{E}_\beta \hat{C} \right) \in \sum_{i=\delta_L^*}^{n} \boldsymbol{E}_i V^{\otimes n},$$

from which it follows in the same manner that

$$\pi_W \boldsymbol{F}_\alpha^* \left(\hat{C} - \sum_{\beta \in L} \boldsymbol{E}_\beta \hat{C} \right) \in \sum_{i=\delta_L^* - \mu_r}^{n-r} \boldsymbol{E}_i W \subset \sum_{i=r+1}^{n-r} \boldsymbol{E}_i W, \tag{4.19}$$

where we abbreviate

$$\boldsymbol{F}^*_\alpha = f(\boldsymbol{D}^*_1, \boldsymbol{D}^*_2, \ldots, \boldsymbol{D}^*_s).$$

On the other hand, recall that the roles of $\boldsymbol{M}$ and $\boldsymbol{M}^*$ are interchanged when we work with the basis $\{\hat{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in X^{*n}\}$ of $V^{\otimes n}$, and observe that $\boldsymbol{E}_\beta \hat{C}$ is a scalar multiple of the characteristic vector of $(X^{*n})_\beta \cap C^\perp$ with respect to this basis; cf. (2.9). Hence, for any $\beta \in L$ and $0 \leqslant i \leqslant t$, it follows from Lemma 4.2.3 (applied to the dual) that

$$\begin{aligned} \boldsymbol{E}_i \pi_W \boldsymbol{F}^*_\alpha \boldsymbol{E}_\beta \hat{C} &= \boldsymbol{E}_i \pi_W \pi_{U_0} \boldsymbol{F}^*_\alpha \boldsymbol{E}_\beta \hat{C} \\ &= \pi_W \boldsymbol{E}_i \pi_{U_0} \boldsymbol{F}^*_\alpha \boldsymbol{E}_\beta \hat{C} \\ &\in \mathbb{C} \pi_W \boldsymbol{A}^*_i \hat{\boldsymbol{\iota}} \\ &= 0, \end{aligned}$$

where $\boldsymbol{\iota} = (\iota, \iota, \ldots, \iota)$ is the identity of X^{*n}, since $\boldsymbol{A}^*_i \hat{\boldsymbol{\iota}} = |X|^{n/2} \boldsymbol{E}_i \hat{\mathbf{0}}$ belongs to the primary $\boldsymbol{T}_H$-module $\boldsymbol{M}_H \hat{\mathbf{0}}$. (Recall that $\boldsymbol{x}_0 = \mathbf{0} = (0, 0, \ldots, 0)$ in this context.) Hence we have

$$\pi_W \boldsymbol{F}^*_\alpha \boldsymbol{E}_\beta \hat{C} \in \sum_{i=t+1}^{n-r} \boldsymbol{E}_i W \subset \sum_{i=r+1}^{n-r} \boldsymbol{E}_i W \quad (\beta \in L). \tag{4.20}$$

Combining (4.15), (4.19), and (4.20), we obtain (4.18), and this completes the proof.

4.4 Examples

In this section, we mainly discuss additive codes over various translation association schemes (so that $x_0 = 0$).

4.4.1 Codes with Hamming weight enumerators

Recall that the *Hamming weight* of $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in X^n$ is defined by

$$\mathrm{wt}(\boldsymbol{x}) = |\{\ell : x_\ell \neq 0\}|.$$

The *Hamming weight enumerator* of an additive code C in X^n is then defined by

$$\mathrm{hwe}_C(\xi_0, \xi_1) = \sum_{\boldsymbol{x} \in C} \xi_0^{n-\mathrm{wt}(\boldsymbol{x})} \xi_1^{\mathrm{wt}(\boldsymbol{x})}.$$

Thus, when working with the Hamming weight enumerator, we are considering codes over the 1-class association scheme $(X, \{R_0, (X \times X)\backslash R_0\})$ with eigenmatrices

$$P = Q = \begin{bmatrix} 1 & |X| - 1 \\ 1 & -1 \end{bmatrix},$$

whose extension of length n is the Hamming association scheme $H(n, |X|)$. In particular, we have $\boldsymbol{T} = \boldsymbol{T}_H$ in this case. Tanaka [60] showed the following:

Theorem 4.4.1 ([60, Theorem 5.2, Example 5.5])**.** *Let C be a code in X^n. Let*

$$\delta = \min\{i \neq 0 : \boldsymbol{E}_i^* \hat{C} \neq 0\}, \quad \delta^* = \min\{i \neq 0 : \boldsymbol{E}_i \hat{C} \neq 0\}.$$

Suppose that an integer t $(1 \leqslant t \leqslant n)$ is such that, for every $1 \leqslant r \leqslant t$, at least one of the following holds:

$$|\{i : r \leqslant i \leqslant n-r,\ \boldsymbol{E}_i^* \hat{C} \neq 0\}| \leqslant \delta^* - r, \tag{4.21}$$

$$|\{i : r \leqslant i \leqslant n-r,\ \boldsymbol{E}_i \hat{C} \neq 0\}| \leqslant \delta - r. \tag{4.22}$$

Then the multiset

$$\{\mathrm{supp}(\boldsymbol{x}) : \boldsymbol{x} \in (X^n)_i \cap C\}$$

is a t-design for every $0 \leqslant i \leqslant n$.

Observe that Theorem 4.4.1 strengthens the original Assmus–Mattson theorem (Theorem 4.0.1). In particular, it does not require the code C to be linear nor additive. The condition (4.21) agrees with (4.1) when $s = 1$. Indeed, the proof of Theorem 4.1.1 reduces to that of Theorem 4.4.1 for (4.21). The dual argument shows the result for the case (4.22). (It seems that the condition dual to (4.1) does not necessarily lead to the same conclusion as Theorem 4.1.1 when $s > 1$.) On the other hand, Supplements 4.1.2 and 4.1.3 refine [60, Remark 7.1], and prove useful as we will see below.

Example 4.4.2. The Assmus–Mattson-type theorem for additive codes over $\mathbb{F}_4$ given by Kim and Pless [38, Theorem 2.7] follows from Theorem 4.4.1, except their comment on the simplicity of the designs obtained from minimum weight codewords. The additive group of $\mathbb{F}_4$ is isomorphic to the Kleinian four group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and additive codes over $\mathbb{F}_4$ are the same thing as *linear Kleinian codes* studied by Höhn [32]. It should be noted that giving an (appropriate) inner product on $\mathbb{F}_4^n \cong (\mathbb{Z}_2 \times \mathbb{Z}_2)^n$, on which concepts like self-orthogonality and self-duality depend, amounts to choosing a group isomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \to (\mathbb{Z}_2 \times \mathbb{Z}_2)^*$ satisfying the symmetry (2.11). This last remark applies to all examples that follow.

Example 4.4.3. Recall that a binary Type II code is a self-dual binary code C which is doubly even (i.e. $\mathrm{wt}(u) \equiv 0 \pmod{4}\ \forall u \in C$). It is known that if there exists a binary Type II code of length n, then we can find $\mu \in \mathbb{N}$ and $\ell \in \{0, 1, 2\}$ such that

$$n = 24\mu + 8\ell.$$

A binary Type II code C is called *extremal* if its minimum weight is equal to $4\mu+4$. Suppose C is an extremal binary Type II code of length n. Then by Theorem 4.0.1 (or Theorem 4.4.1), we see that the words of any fixed weight in C yield a t-design with $t = 5-2\ell$. Using Bachoc's results on *harmonic weight enumerators* [3], Bannai, Koike, Shinohara, and Tagami [6, Theorem 6, Remark 5] showed that if one of these (non-trivial) designs is a $(t+1)$-design then so are the others. This observation is also immediate from Supplement 4.1.2. We note that similar observations hold for extremal Type III codes over $\mathbb{F}_3$ and extremal Type IV codes over $\mathbb{F}_4$. See also [43].

Example 4.4.4. Additive codes over $\mathbb{Z}_4$ are also referred to as $\mathbb{Z}_4$-*linear codes*. For a $\mathbb{Z}_4$-linear code C in $\mathbb{Z}_4^n$, let

$$C_2 = (2\mathbb{Z}_4^n) \cap C,$$

which may also be viewed as a binary linear code (called the *torsion code* of C) since $2\mathbb{Z}_4 \cong \mathbb{Z}_2$. We note that hwe_{C_2} is derived immediately from either the *complete* or the *symmetrized weight enumerators* of C; cf. Subsection 4.4.2. Shin, Kumar, and Helleseth [52, Theorem 10] proved an Assmus–Mattson-type theorem for $\mathbb{Z}_4$-linear codes, and we now claim that Theorem 4.4.1, together with Supplements 4.1.2 and 4.1.3, always gives at least as good estimate on t as their theorem. First, they assume that C_2 and $(C^{\perp})_2$ both satisfy the conclusion of Theorem 4.4.1. If a (Hamming) weight of C is not a weight of $C\backslash C_2$, then the corresponding words of C must all belong to C_2, and hence by Supplement 4.1.2 we can exclude that weight from the weights of C. The same comment applies to $C^{\perp}$. Second, they assume that the number of non-zero weights of the shortened code of $C^{\perp}\backslash(C^{\perp})_2$ at some t coordinates is bounded above by $\delta - t$. However, the conclusion of their theorem shows in the end that this number is equal to that of non-zero weights at most $n-t$ in $C^{\perp}\backslash(C^{\perp})_2$. Hence it follows that this second condition is not weaker than (4.22).

Remark 4.4.5. From the Assmus–Mattson-type theorem by Shin et al. mentioned above (or Theorem 4.4.1) it follows that the words of any fixed weight in the Goethals code or its dual (a Delsarte–Goethals code) over $\mathbb{Z}_4$ of length 2^m with m odd, support a 2-design. However, Shin et al. [52, Corollaries 7 and 8] showed that it is in fact a 3-design, based on what they call an *Assmus–Mattson-type approach*. See also [40].

4.4.2 Codes with complete/symmetrized weight enumerators

Let C be an additive code over the ring $\mathbb{Z}_k$. Besides hwe_C, it is also important to consider the *complete* and the *symmetrized weight enumerators* defined respectively by

$$\mathrm{cwe}_C(\xi_0, \xi_1, \ldots, \xi_{k-1}) = \sum_{\boldsymbol{x} \in C} \xi_0^{n_0(\boldsymbol{x})} \xi_1^{n_1(\boldsymbol{x})} \cdots \xi_{k-1}^{n_{k-1}(\boldsymbol{x})},$$
$$\mathrm{swe}_C(\xi_0, \xi_1, \ldots, \xi_e) = \sum_{\boldsymbol{x} \in C} \xi_0^{n_0(\boldsymbol{x})} \xi_1^{n_{\pm 1}(\boldsymbol{x})} \cdots \xi_s^{n_{\pm e}(\boldsymbol{x})},$$

where $e = \lfloor k/2 \rfloor$,

$$n_i(\boldsymbol{x}) = |\{\ell : x_\ell = i\}| \quad (0 \leqslant i \leqslant k-1),$$
$$n_{\pm i}(\boldsymbol{x}) = n_i(\boldsymbol{x}) + n_{k-i}(\boldsymbol{x}) \quad (1 \leqslant i \leqslant \lfloor (k-1)/2 \rfloor),$$

and we understand that $n_{\pm e}(\boldsymbol{x}) = n_e(\boldsymbol{x})$ if k is even. Thus, for cwe_C, the initial association scheme $(X, \mathscr{R})$ is the *group association scheme* of $\mathbb{Z}_k$, which is the translation association scheme on $\mathbb{Z}_k$ defined by the partition (cf. (2.5))

$$\mathbb{Z}_k = \{0\} \sqcup \{1\} \sqcup \cdots \sqcup \{k-1\},$$

and has eigenmatrices

$$P = \left[\zeta_k^{ij}\right]_{i,j=0}^{k-1}, \quad Q = \left[\zeta_k^{-ij}\right]_{i,j=0}^{k-1},$$

where $\zeta_k \in \mathbb{C}$ is a primitive k^{th} root of unity. For swe_C, the initial association scheme $(X, \mathscr{R})$ is the association scheme of the *ordinary k-cycle*, which is defined similarly by the partition

$$\mathbb{Z}_k = \{0\} \sqcup \{\pm 1\} \sqcup \cdots \sqcup \{\pm e\},$$

and has eigenmatrices

$$P = Q = \left[(1+\delta_{0,2j})^{-1}\left(\zeta_k^{ij} + \zeta_k^{-ij}\right)\right]_{i,j=0}^{e},$$

where $\delta_{0,2j}$ is evaluated in $\mathbb{Z}_{2k}$. Extensions of the ordinary k-cycle are referred to as *Lee association schemes* (see [54, 62]). We note that

$$\mathrm{swe}_C(\xi_0, \xi_1, \ldots, \xi_e) = \mathrm{cwe}_C(\xi_0, \xi_1, \xi_2, \xi_3, \ldots, \xi_2, \xi_1),$$
$$\mathrm{hwe}_C(\xi_0, \xi_1) = \mathrm{swe}_C(\xi_0, \xi_1, \ldots, \xi_1),$$

and that

$$\mathrm{hwe}_{C_2}(\xi_0, \xi_1) = \mathrm{swe}_C(\xi_0, 0, \xi_1) \quad \text{when } k = 4.$$

Example 4.4.6. Our main results are in fact modeled after the Assmus–Mattson-type theorem due to Tanabe [59, Theorem 2] for $\mathbb{Z}_4$-linear codes with respect to the symmetrized weight enumerator, so that the latter is a special case of the former. In particular, we can easily find 5-designs from the lifted Golay code over $\mathbb{Z}_4$ of length 24 as discussed in [59] (see also [9]). On the other hand, it is not clear at present whether Tanabe's original version of his theorem [57, Theorem 3] is a consequence of our results. It would be an interesting problem to understand [57, Theorem 3] in terms of the irreducible $\boldsymbol{T}$-modules (see [45]).

See [31] for a survey on t-designs constructed from $\mathbb{Z}_4$-linear codes.

Below we discuss the extended quadratic residue codes XQ_{11} of length 12 over small finite fields. That these codes support 3-designs follows from the fact that their automorphism groups contain $\mathrm{PSL}(\mathbb{F}_{11}^2)$ and hence are 3-homogeneous on the

12 coordinates, but we include these examples in order to demonstrate the use of our results further. Recall again that we only look at the weight enumerators (and linearity) of these self-dual codes. We aim at doing the relevant computations *by hand*. The first example is a warm-up:

Example 4.4.7. Consider $C = XQ_{11}$ over $\mathbb{F}_3 = \mathbb{Z}_3$, which is the extended ternary Golay code. We have hwe_C and cwe_C as follows:

wt	n_1	n_2	#words
0	0	0	1
6	6	0	22
	0	6	22
	3	3	220
9	6	3	220
	3	6	220
12	12	0	1
	0	12	1
	6	6	22

As is well known, the words of fixed (Hamming) weight 6 or 9 support 5-designs by Theorem 4.0.1. (The one with block size 9 is the non-simple trivial design with constant multiplicity 2.) Set $t = 3$. We have $\delta^* = 6$ and

$$S_1 = S_2 = S_3 = \{(6,0),(0,6),(3,3),(6,3),(3,6)\}.$$

Observe that the words with $(n_1, n_2) = (6,3)$ and those with $(n_1, n_2) = (3,6)$ come in pairs by the correspondence $\boldsymbol{x} \mapsto -\boldsymbol{x}$, so that the words with each of these two complete weight types support the (simple!) trivial design. Hence we may disregard them by Supplement 4.1.2, i.e., we set $K = \{(6,3),(3,6)\}$. Then $S_3 \backslash K$ consists of three collinear points in $\mathbb{R}^2$, and thus we have $\mu(S_3\backslash K) = 2$. Since $2 < 6-3$, it follows from Theorem 4.1.1 that the non-trivial 5-design with block size 6 is partitioned into two 3-designs (after discarding repeated blocks).

Example 4.4.8. Consider $C = XQ_{11}$ over $\mathbb{F}_5 = \mathbb{Z}_5$. We have hwe_C and swe_C as follows:

wt	$n_{\pm 1}$	$n_{\pm 2}$	#words
0	0	0	1
6	3	3	440
7	6	1	264
	1	6	264
8	4	4	2640
9	7	2	1320
	2	7	1320
10	5	5	5544
11	8	3	1320
	3	8	1320
12	11	1	24
	1	11	24
	6	6	1144

We have $\delta^* = 6$. Observe that Theorem 4.0.1 nor Theorem 4.4.1 cannot find designs from the supports of the codewords in this case. On the other hand, set $t = 3$, and take $\sigma \in \mathrm{GL}(\mathbb{R}^2)$ such that $\sigma(i,j) = (1/5)(2i+3j, i-j)$. Then we have

$$\sigma(S_1) = \left\{ \begin{array}{r} (6,-1), \\ (5,-1),(5,0),(5,1), \\ (4,-1),(4,0),(4,1), \\ (3,0),(3,1) \end{array} \right\},$$

$$\sigma(S_2) = \left\{ \begin{array}{r} (5,-1),(5,0), \\ (4,-1),(4,0),(4,1), \\ (3,0),(3,1) \end{array} \right\},$$

$$\sigma(S_3) = \left\{ \begin{array}{r} (5,-1), \\ (4,-1),(4,0),(4,1), \\ (3,0),(3,1) \end{array} \right\}.$$

From Supplement 4.1.5 it follows that $\mu(S_1) \leqslant 4$ and $\mu(S_2) \leqslant 3$. If we apply Supplement 4.1.5 directly to $\sigma(S_3)$ then we would only obtain $\mu(S_3) \leqslant 3$, but indeed it follows that $\mu(S_3) = 2$. To see this, let

$$\begin{aligned}
f_{(5,-1)} &= (\xi_1 - 3)(\xi_1 - 4)/2, \\
f_{(4,-1)} &= -(\xi_1 + \xi_2 - 4)(\xi_1 - \xi_2 - 3)/2, \\
f_{(4,1)} &= (\xi_1 + \xi_2 - 3)(\xi_1 + \xi_2 - 4)/2, \\
f_{(3,0)} &= (\xi_1 - 4)(\xi_1 + \xi_2 - 4), \\
f_{(3,1)} &= -(\xi_1 - 4)(\xi_1 + 2\xi_2 - 3)/2, \\
f_{(4,0)} &= 1 - f_{(5,-1)} - f_{(4,-1)} - f_{(4,1)} - f_{(3,0)} - f_{(3,1)}.
\end{aligned}$$

Then we have

$$f_\alpha(\beta) = \delta_{\alpha\beta} \quad (\alpha, \beta \in \sigma(S_3)),$$

from which it follows that the linear span of the f_α $(\alpha \in \sigma(S_3))$ is an interpolation space with respect to $\sigma(S_3)$. This shows $\mu(S_3) = 2$, as desired. Thus, the condition (4.1) is satisfied for $r \in \{1, 2, 3\}$. Theorem 4.1.1 now shows that the codewords of any fixed symmetrized weight type support 3-designs. This example tells us that looking at swe_C may sometimes give a better estimate on t than hwe_C, even when Supplement 4.1.2 is not applicable.

Finally, we consider $C = XQ_{11}$ over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. Note that cwe_C makes sense by defining $n_\omega(\boldsymbol{x})$ and $n_{\omega^2}(\boldsymbol{x})$ in the same manner as above. The eigenmatrices of the group association scheme of $\mathbb{F}_4$ are given by

$$P = Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Example 4.4.9. Consider $C = XQ_{11}$ over $\mathbb{F}_4$. We have hwe_C and cwe_C as follows:

wt	n_1	n_ω	n_{ω^2}	#words
0	0	0	0	1
6	2	2	2	330
7	5	1	1	132
	1	5	1	132
	1	1	5	132
8	4	4	0	165
	4	0	4	165
	0	4	4	165
9	3	3	3	1320
10	6	2	2	330
	2	6	2	330
	2	2	6	330
11	5	5	1	132
	5	1	5	132
	1	5	5	132
12	12	0	0	1
	0	12	0	1
	0	0	12	1
	4	4	4	165

We have $\delta^* = 6$. Again, Theorem 4.0.1 cannot find designs from the supports of the codewords. Take $\sigma \in \mathrm{GL}(\mathbb{R}^3)$ such that

$$\sigma(i, j, k) = (1/4)(2i + j + k, i + 2j + k, i + j + 2k).$$

Then we have

$$\sigma(S_3) = \left\{ \begin{matrix} (2,2,2),(2,2,3),(2,3,2),(2,3,3), \\ (3,2,2),(3,2,3),(3,3,2),(3,3,3) \end{matrix} \right\},$$

$$\sigma(S_2) = \sigma(S_3) \sqcup \{(4,3,3),(3,4,3),(3,3,4)\},$$

$$\sigma(S_1) = \sigma(S_2) \sqcup \{(4,4,3),(4,3,4),(3,4,4)\}.$$

We claim that $\mu(S_1) \leqslant 4$ and that $\mu(S_2) = \mu(S_3) = 3$. First, it is easy to see that $\mu(S_3) = 3$ as $\sigma(S_3)$ forms a cube. Next, let

$$f_{(4,3,3)} = (\xi_1 - 2)(\xi_1 - 3)/2.$$

Then we have

$$f_{(4,3,3)}(\alpha) = \delta_{(4,3,3),\alpha} \quad (\alpha \in \sigma(S_2)).$$

We similarly define $f_{(3,4,3)}$ and $f_{(3,3,4)}$. Recall that $\mathscr{M}(\sigma(S_3))$ denotes a minimal degree interpolation space with respect to $\sigma(S_3)$. Then it is immediate to see that

$$\mathscr{M}(\sigma(S_3)) + \mathbb{R}f_{(4,3,3)} + \mathbb{R}f_{(3,4,3)} + \mathbb{R}f_{(3,3,4)}$$

is an interpolation space with respect to $\sigma(S_2)$. Since $\mu(S_3) \leqslant \mu(S_2)$, we have $\mu(S_2) = 3$. Finally, let for example

$$f_{(4,4,3)} = (\xi_1 + \xi_2 - 4)(\xi_1 + \xi_2 - 5)(\xi_1 + \xi_2 - 6)(\xi_1 + \xi_2 - 7)/24,$$

so that we have

$$f_{(4,4,3)}(\alpha) = \delta_{(4,4,3),\alpha} \quad (\alpha \in \sigma(S_1)),$$

and a similar argument establishes $\mu(S_1) \leqslant 4$, as desired. Thus, the condition (4.1) is satisfied for $r \in \{1, 2\}$ but fails for $r = 3$. Theorem 4.1.1 now shows that the codewords of any fixed complete weight type support 2-designs. Though this is not the best estimate (i.e., $t = 3$), Theorem 4.1.1 still outperforms Theorem 4.0.1 for this example.

Chapter 5

Asymptotic Spectral Distributions for Cartesian Powers of Strongly Regular Graphs

Let $G = (X, R)$ be a finite simple graph with vertex set X and edge set R. Let A be the adjacency matrix of G. By an *eigenvalue* of G we mean an eigenvalue of A. Likewise, we speak of the *spectrum* of G.

Spectra of graphs have been receiving attention from the point of view of quantum probability theory. Recall that an *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a *state*, i.e., a linear map such that $\varphi(1) = 1$ and that $\varphi(a^*a) \geqslant 0$ for every $a \in \mathcal{A}$. The elements of $\mathcal{A}$ are referred to as (*algebraic*) *random variables*. We call $a \in \mathcal{A}$ *real* if $a^* = a$. For a real random variable $a \in \mathcal{A}$, we are interested in finding, and discussing the uniqueness of, a probability measure ν on $\mathbb{R}$ such that

$$\varphi(a^j) = M_j(\nu) := \int_{\mathbb{R}} x^j \,\nu(dx) \quad (j = 0, 1, 2, \dots). \tag{5.1}$$

The adjacency algebra $\mathbb{C}[A]$ of the graph G is the commutative subalgebra of the full matrix algebra generated by A. It is natural to consider the *tracial state* φ_{tr} defined by[2]

$$\varphi_{\mathrm{tr}}(Z) = \frac{1}{|Z|}\,\mathrm{tr}(Z) \quad (Z \in \mathbb{C}[A]),$$

and we view A as a random variable in the algebraic probability space $(\mathbb{C}[A], \varphi_{\mathrm{tr}})$. Suppose for simplicity that G is k-regular, so that A has mean 0 and variance k. The probability measure $\nu = \nu_G$ in (5.1) for $a = A/\sqrt{k}$ is then the (normalized) *spectral distribution* of G. It is interesting to find the limit of this normalized spectral distribution when G grows, as an analogue of the classical central limit theorem. Notable works in this area are done by Hora [33], and by Hora and Obata [34].

The objective of this chapter is to give a concrete bivariate example of this sort, as an attempt towards a multivariate extension of the theory. Consider again a general algebraic probability space $(\mathcal{A}, \varphi)$. We now pick two commuting real random variables $a, b \in \mathcal{A}$, and discuss a probability measure ν on $\mathbb{R}^2$ such that

$$\varphi(a^j b^h) = M_{j,h}(\nu) := \int_{\mathbb{R}^2} x^j y^h \,\nu(dxdy) \quad (j, h = 0, 1, 2, \ldots). \tag{5.2}$$

We take another regular graph H with valency ℓ on the vertex set X such that the adjacency matrix B of H commutes with A. This occurs for instance when H is the complement of G. We view A and B as real random variables in the algebraic probability space $(\mathbb{C}[A, B], \varphi_{\mathrm{tr}})$. The probability measure $\nu = \nu_{G,H}$ in (5.2) for

[2]Another important example is the *vacuum state* $\varphi_x(Z) = Z_{x,x}$ $(Z \in \mathbb{C}[A])$ at a fixed origin $x \in X$. We note that the matrix $*$-algebras we will discuss in this chapter all have the property that every element has constant diagonal entries, so that the two states φ_{tr} and φ_x turn out to be equal on them.

$a = A/\sqrt{k}$ and $b = B/\sqrt{\ell}$ is then the (normalized) *joint spectral distribution* of G and H. We are interested in the limit of this joint spectral distribution when G and H both grow, as an analogue of the bivariate central limit theorem.

Our main result (Theorem 5.1.1) is indeed a bivariate version of the result of Hora [33] for the Hamming graphs. We will consider the pair $(G^{\square n}, \overline{G}^{\square n})$ of the n^{th} Cartesian powers of a strongly regular graph G and its complement $\overline{G}$, and obtain as limits a bivariate Poisson distribution and the standard bivariate Gaussian distribution, together with an intermediate distribution.

This chapter is organized as follows: We recall basic facts about graphs and state the main theorem in Section 5.1. We review basic properties of strongly regular graphs in Section 5.2. We prove the main theorem in Section 5.3. In Section 5.4, we demonstrate the main theorem with some specific families of strongly regular graphs. The entire chapter is based on [46].

5.1 Basic definitions and the main result

Let $G = (X, R)$ be a graph with vertex set X and edge set R. All the graphs we consider in this chapter are finite and simple. Thus, X is a finite set and R is a subset of $\binom{X}{2}$, the set of 2-element subsets of X. Two vertices $x, y \in X$ are called *adjacent* (and written $x \sim y$) if $\{x, y\} \in R$. The graph G is called k*-regular* if every vertex is adjacent to precisely k vertices. It is called *connected* if for any two vertices x and y, there is a sequence $x = x_0, x_1, \ldots, x_m = y$ of vertices such that $x_{j-1} \sim x_j$ for $j = 1, 2, \ldots, m$. Recall that a *complete graph* K_v is a graph on $|X| = v$ vertices such that $R = \binom{X}{2}$. We note that if G is k-regular then every eigenvalue θ of G satisfies $|\theta| \leqslant k$.

From now on, suppose that G is k-regular and has $|X| = v$ vertices. We call G *strongly regular with parameters* (v, k, λ, μ) if G is not complete or edgeless (i.e., $0 < k < v - 1$), and if every pair of adjacent (resp. non-adjacent) vertices has precisely λ (resp. μ) common adjacent vertices; cf. [14, §9.1]. In matrix terms, this means that

$$A^2 = kI + \lambda A + \mu(J - A - I), \tag{5.3}$$

where I and J denote the identity and the all-ones matrix, respectively. It is clear that G is a disconnected strongly regular graph precisely when it is the disjoint union pK_q of p complete graphs K_q for some integers $p, q \geqslant 2$.

The *complement* $\overline{G}$ of G is the graph with the same vertex set X as G, where two distinct vertices are adjacent if and only if they are non-adjacent in G. Thus, $\overline{G}$ has adjacency matrix $\overline{A} := J - A - I$. Since $AJ = JA = kJ$, we have $A\overline{A} = \overline{A}A$. It is easy to see that if G is strongly regular as above then $\overline{G}$ is again strongly regular with parameters $(v, \overline{k}, \overline{\lambda}, \overline{\mu})$, where

$$\overline{k} = v - k - 1, \quad \overline{\lambda} = v - 2k + \mu - 2, \quad \overline{\mu} = v - 2k + \lambda. \tag{5.4}$$

Thus, strongly regular graphs always exist in pairs. The complement of pK_q is the *complete multipartite graph* $K_{p\times q}$.

Observe that G is complete if and only if the linear span $\langle I, A\rangle$ equals $\langle I, J\rangle$, which is the Bose–Mesner algebra of a one-class association scheme. Likewise, from (5.3) it follows that G is strongly regular if and only if $\langle I, A, \overline{A}\rangle = \langle I, A, J\rangle$ is the Bose–Mesner algebra of a two-class association scheme. In particular, if this is the case then there are precisely three (maximal) common eigenspaces for $(A, \overline{A})$, one of which corresponds to the eigenvalues $(k, \overline{k})$ and is spanned by the all-ones vector $\mathbf{1}$ in $\mathbb{C}^v$. (Note that these eigenspaces diagonalize J as well.) Let $(r, \overline{s})$ and $(s, \overline{r})$ be the eigenvalues corresponding to the other two, where $\overline{s} = -r-1$ and $\overline{r} = -s-1$. We will assume $r > s$, or equivalently, $\overline{r} > \overline{s}$. We have $s, \overline{s} < 0$ as $\varphi_{\mathrm{tr}}(A) = \varphi_{\mathrm{tr}}(\overline{A}) = 0$, so that[3]

$$-1 < r \leqslant k, \quad -k \leqslant s < 0, \quad -1 < \overline{r} \leqslant \overline{k}, \quad -\overline{k} \leqslant \overline{s} < 0. \tag{5.5}$$

We call r and s (resp. $\overline{r}$ and $\overline{s}$) the *restricted*[4] eigenvalues of G (resp. $\overline{G}$).

The *Cartesian product* $G_1 \square G_2$ of two graphs $G_j = (X_j, R_j)$ $(j = 1, 2)$ is the graph with vertex set $X_1 \times X_2$, where $(x_1, x_2) \sim (y_1, y_2)$ if and only if either $x_1 \sim y_1$ and $x_2 = y_2$, or $x_1 = y_1$ and $x_2 \sim y_2$; cf. [14, §1.4.6]. For a positive integer n, the Cartesian power $G \square G \square \cdots \square G$ (n times) will be denoted by $G^{\square n}$. For example, we already mentioned that $H(n, q) = K_q^{\square n}$. We note that $G^{\square n}$ is nk-regular. The adjacency matrix $\boldsymbol{A}$ of $G^{\square n}$ is given by

$$\boldsymbol{A} = \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \underset{\frown \atop j}{A} \otimes I \otimes \cdots \otimes I. \tag{5.6}$$

[3] In fact, it follows that $r, \overline{r} \geqslant 0$ and $s, \overline{s} \leqslant -1$; cf. Lemma 5.2.1.

[4] More generally, an eigenvalue of a (not necessarily regular) graph is called *restricted* if it has an eigenvector which is not a scalar multiple of $\mathbf{1}$.

Let $\overline{\boldsymbol{A}}$ denote the adjacency matrix of $\overline{G}^{\square n}$. Note that $G^{\square n}$ and $\overline{G}^{\square n}$ are relations in the extension of length n of the two-class association scheme induced by G and $\overline{G}$, and that $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$ together generate its Bose–Mesner algebra; see the first paragraph of Section 4.3.1, and also Lemma 3.3.10. Also, observe that $\boldsymbol{A}\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}\boldsymbol{A}$ and the covariance $\varphi_{\mathrm{tr}}(\boldsymbol{A}\overline{\boldsymbol{A}}) = 0$.

Notation & Assumption. *We consider an infinite family of pairs of Cartesian powers of graphs $(G^{\square n}, \overline{G}^{\square n})$, where n ranges over an infinite set of positive integers, and G is strongly regular and may depend on n. To simplify notation, we think of $G, v, k, \overline{k}, r, s$, etc., as functions in n. We will assume that*

$$\frac{k}{n} \to k', \quad \frac{\overline{k}}{n} \to \overline{k}', \quad \frac{r}{n} \to r', \quad \frac{s}{n} \to s'$$

as $n \to \infty$, where $k', \overline{k}', r'$, and s' are finite. We note that

$$\frac{v}{n} \to v' := k' + \overline{k}'.$$

The following is our main result which describes the possible limits of the joint spectral distribution $\nu_{G^{\square n}, \overline{G}^{\square n}}$.

Theorem 5.1.1. *With the above notation and assumption, we have $r' = 0$ or $s' = 0$, and one of the following holds:*

i. $k' > 0$, $\overline{k}' = -s' > 0$, $r' = 0$, and $\nu_{G^{\square n}, \overline{G}^{\square n}}$ converges weakly to an affine transformation ν of a bivariate Poisson distribution given by

$$\nu\left(\left\{\left(\frac{k'j - \overline{k}'h}{\sqrt{k'}}, \frac{\overline{k}'j + \overline{k}'h - 1}{\sqrt{\overline{k}'}}\right)\right\}\right) = e^{-1/\overline{k}'}\left(\frac{1}{v'}\right)^{j}\left(\frac{k'}{v'\overline{k}'}\right)^{h}\frac{1}{j!h!}$$

for non-negative integers j and h. In this case, G is a complete multipartite graph for all but finitely many values of n.

ii. *$k' = r' > 0$, $\overline{k}' > 0$, $s' = 0$, and $\nu_{G^{\square n},\overline{G}^{\square n}}$ converges weakly to an affine transformation ν of a bivariate Poisson distribution given by*

$$\nu\left(\left\{\left(\frac{k'j + k'h - 1}{\sqrt{k'}}, \frac{\overline{k}'j - k'h}{\sqrt{\overline{k}'}}\right)\right\}\right) = e^{-1/k'}\left(\frac{1}{v'}\right)^j\left(\frac{\overline{k}'}{v'k'}\right)^h\frac{1}{j!h!}$$

for non-negative integers j and h. In this case, G is a disjoint union of complete graphs for all but finitely many values of n.

iii. *$k' > 0$ or $\overline{k}' > 0$, and $r' = s' = 0$, and $\nu_{G^{\square n},\overline{G}^{\square n}}$ converges weakly to an affine transformation ν of the product measure of a Poisson distribution and a Gaussian distribution given by*

$$\int_{\mathbb{R}^2} f(x)\,\nu(dx) = \sqrt{\frac{v'}{2\pi}}\,e^{-1/v'}\sum_{h=0}^{\infty}\left(\frac{1}{v'}\right)^h\frac{1}{h!}\int_{-\infty}^{\infty} f(x_{h,t})\,e^{-v't^2/2}\,dt$$

for every Borel function $f : \mathbb{R}^2 \to \mathbb{R}$, where

$$x_{h,t} = \left(\sqrt{k'}\,h + \sqrt{\overline{k}'}\,t - \frac{\sqrt{k'}}{v'}, \sqrt{\overline{k}'}\,h - \sqrt{k'}\,t - \frac{\sqrt{\overline{k}'}}{v'}\right).$$

iv. *$k' = \overline{k}' = r' = s' = 0$, and $\nu_{G^{\square n},\overline{G}^{\square n}}$ converges weakly to the standard bivariate Gaussian distribution.*

5.2 Preliminaries on strongly regular graphs

In this section, we collect necessary facts about strongly regular graphs. See [14, Chapter 9] for more details. Throughout this section, let G be a (fixed) strongly regular graph with parameters (v, k, λ, μ), and let $\overline{G}$ be the complement of G, having parameters $(v, \overline{k}, \overline{\lambda}, \overline{\mu})$ (cf. (5.4)). Let A (resp. $\overline{A}$) be the adjacency matrix of G (resp. $\overline{G}$). For convenience, we let

$$\kappa = (k, \overline{k}), \quad \rho = (r, \overline{s}), \quad \sigma = (s, \overline{r}).$$

Let $\mathbb{U}_\kappa, \mathbb{U}_\rho$, and $\mathbb{U}_\sigma$ be the common eigenspaces of $(A, \overline{A})$ associated with κ, ρ, and σ, respectively. Recall that $\mathbb{U}_\kappa = \langle \mathbf{1} \rangle$. Let $f = \dim(\mathbb{U}_\rho)$, $g = \dim(\mathbb{U}_\sigma)$.

There are a number of standard identities involving these scalars. What we will need are the following:

$$v = 1 + k + \overline{k} = 1 + f + g, \tag{5.7}$$

$$0 = 1 + r + \overline{s} = 1 + s + \overline{s}, \tag{5.8}$$

$$0 = k + rf + sg = \overline{k} + \overline{s}f + \overline{r}g, \tag{5.9}$$

$$k^2 + r^2 f + s^2 g = kv, \tag{5.10}$$

$$k\overline{k} + r\overline{s}f + s\overline{r}g = 0, \tag{5.11}$$

$$\overline{k}^2 + \overline{s}^2 f + \overline{r}^2 g = \overline{k}v, \tag{5.12}$$

$$f = \frac{(v-1)s + k}{s - r}, \quad g = \frac{(v-1)r + k}{r - s}, \tag{5.13}$$

$$fg = \frac{k\overline{k}v}{(r-s)^2}. \tag{5.14}$$

The proofs of (5.7)–(5.13) are straightforward: (5.7) is clear; (5.8) is already mentioned and follows from $I + A + \overline{A} = J$; (5.9)–(5.12) are the values of $\mathrm{tr}(A)$, $\mathrm{tr}(\overline{A})$, $\mathrm{tr}(A^2)$, $\mathrm{tr}(A\overline{A})$, and $\mathrm{tr}(\overline{A}^2)$; (5.13) is immediate from (5.7) and (5.9). To show (5.14), count the triples of distinct vertices x, y, z such that $x \sim y \sim z \not\sim x$ (in G) in two ways to get

$$k(k - 1 - \lambda) = \overline{k}\mu, \tag{5.15}$$

and then use (5.13) together with the fact that r and s are the solutions of the quadratic equation

$$\xi^2 - (\lambda - \mu)\xi + \mu - k = 0, \tag{5.16}$$

where ξ is an indeterminate (cf. (5.3)).

Lemma 5.2.1. *If r and s are non-integral then $f = g$ and we have*

$$v = 4\ell + 1, \quad k = 2\ell, \quad \lambda = \ell - 1, \quad \mu = \ell \tag{5.17}$$

for some positive integer ℓ. Moreover, in this case we have

$$r = \frac{-1 + \sqrt{1 + 4\ell}}{2}, \quad s = \frac{-1 - \sqrt{1 + 4\ell}}{2}. \tag{5.18}$$

Proof. See [14, p. 118]. From (5.16) it follows that r and s are algebraic conjugates, so that we have $f = g$. Using (5.13) and (5.16), we then have $(v - 1)(\mu - \lambda) = 2k$. Since $k < v - 1$, this is possible only when $\mu - \lambda = 1$ and $v - 1 = 2k$. In particular, we have $\overline{k} = k$ and hence it follows from (5.15) that $k = 2\mu$, as desired. □

Strongly regular graphs with parameters of the form (5.17) are called *conference* *graphs*.

We say that G is *imprimitive* if either G or $\overline{G}$ is disconnected, and *primitive* otherwise. Thus, G is imprimitive if and only if $G = pK_q$ or $G = K_{p\times q}$ for some integers $p, q \geqslant 2$.

Example 5.2.2 (imprimitive graphs)**.** Let p and q be integers such that $p, q \geqslant 2$. The disjoint union pK_q is strongly regular with parameters $(pq, q-1, q-2, 0)$ and restricted eigenvalues $r = q-1$, $s = -1$. The complete multipartite graph $K_{p\times q}$ is strongly regular with parameters $(pq, (p-1)q, (p-2)q, (p-1)q)$ and restricted eigenvalues $r = 0$, $s = -q$.

We now introduce two more families of strongly regular graphs. See [14, Sections 9.1.10–9.1.13]. Recall that an *incidence structure* is a triple $(\mathscr{P}, \mathscr{B}, \mathscr{I})$, where $\mathscr{P}$ and $\mathscr{B}$ are finite sets whose elements are called *points* and *blocks*, respectively, and where $\mathscr{I} \subset \mathscr{P} \times \mathscr{B}$. If $(\mathfrak{p}, \mathfrak{b}) \in \mathscr{I}$ then we say that $\mathfrak{p}$ and $\mathfrak{b}$ are *incident*, or $\mathfrak{p}$ is *contained* in $\mathfrak{b}$, and so on. The *block graph* of $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ is the graph $G = (X, R)$ with $X = \mathscr{B}$ where two distinct blocks are adjacent if and only if they contain a point in common.

Example 5.2.3 (Steiner graphs)**.** Let m and d be integers such that $2 \leqslant m < d$. A *Steiner system* $S(2, m, d)$ is an incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ with $|\mathscr{P}| = d$ such that every block contains precisely m points, and that any two distinct points are contained in a unique block. The block graph of an $S(2, m, d)$ is called a *Steiner graph* and is strongly regular with parameters (v, k, λ, μ) provided $d \geqslant 4$, where

$$v = \frac{d(d-1)}{m(m-1)}, \quad k = \frac{(d-m)m}{m-1}, \quad \lambda = (m-1)^2 + \frac{d-2m+1}{m-1}, \quad \mu = m^2,$$

and with restricted eigenvalues

$$r = \frac{d - m^2}{m - 1}, \quad s = -m.$$

Example 5.2.4 (Latin square graphs)**.** Let m and e be integers with $m, e \geqslant 2$. A *transversal design* $TD(m, e)$ is an incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ where the point set is given a partition $\mathscr{P} = \mathscr{P}_1 \sqcup \cdots \sqcup \mathscr{P}_m$ into m *groups* of the same size e (so $|\mathscr{P}| = me$), such that every block is incident with every group in exactly one point, and that any two points from distinct groups are contained in a unique block. The block graph of a $TD(m, e)$ is called a *Latin square graph* and is strongly regular with parameters (v, k, λ, μ) provided $m \leqslant e$, where

$$v = e^2, \quad k = m(e-1), \quad \lambda = (m-1)(m-2) + e - 2, \quad \mu = m(m-1),$$

and with restricted eigenvalues

$$r = e - m, \quad s = -m.$$

The following fundamental result is due to Neumaier [48].

Proposition 5.2.5 ([48])**.** *For any fixed integer $m > 0$, there are only finitely many primitive strongly regular graphs with $s = -m$, other than Steiner graphs and Latin square graphs.*

5.3 Proof of Theorem 5.1.1

To prove Theorem 5.1.1, we invoke Lévy's continuity theorem (see [39, p. 225]) concerning the pointwise convergence of the characteristic functions. Thus, we fix $(\xi_1, \xi_2) \in \mathbb{R}^2$ throughout the proof.

Recall that G depends on n in general. Let $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$ be the adjacency matrices of $G^{\square n}$ and $\overline{G}^{\square n}$, respectively. For $j, h = 0, 1, \ldots, n$ with $j + h \leqslant n$, consider the subspace

$$\bigoplus_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \mathbb{U}_{\varepsilon_1} \otimes \mathbb{U}_{\varepsilon_2} \otimes \cdots \otimes \mathbb{U}_{\varepsilon_n}$$

of $\mathbb{C}^{v^n} \cong (\mathbb{C}^v)^{\otimes n}$, where the sum is over $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\kappa, \rho, \sigma\}$ such that

$$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} = \{\kappa^{n-j-h}, \rho^j, \sigma^h\}$$

as multisets. It has dimension

$$\binom{n}{n-j-h, j, h} f^j g^h.$$

By virtue of (5.6), this subspace is a common eigenspace[5] of $(\boldsymbol{A}, \overline{\boldsymbol{A}})$ with eigenvalues $(\theta_{j,h}, \overline{\theta}_{j,h})$, where

$$\theta_{j,h} = (n - j - h)k + jr + hs, \quad \overline{\theta}_{j,h} = (n - j - h)\overline{k} + j\overline{s} + h\overline{r}.$$

[5]Since $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$ are generators of the Bose–Mesner algebra of the extension scheme, the pairs $(\theta_{j,h}, \overline{\theta}_{j,h})$ $(j, h = 0, 1, \ldots, n,\ j + h \leqslant n)$ are mutually distinct, and these subspaces are indeed the *maximal* common eigenspaces of $(\boldsymbol{A}, \overline{\boldsymbol{A}})$. However, this fact is not necessary in the computation of (5.19) below.

Note that $G^{\square n}$ and $\overline{G}^{\square n}$ are nk-regular and $n\overline{k}$-regular, respectively. Hence it follows from the above comments that the value of the characteristic function of $\nu_{G^{\square n},\overline{G}^{\square n}}$ at (ξ_1, ξ_2) is given by

$$\begin{aligned}\varphi_{\mathrm{tr}}\bigg(\exp\bigg(\frac{i\xi_1}{\sqrt{nk}}\boldsymbol{A} + \frac{i\xi_2}{\sqrt{n\overline{k}}}\overline{\boldsymbol{A}}\bigg)\bigg) \\ &= \frac{1}{v^n}\sum_{j,h}\exp\bigg(\frac{i\xi_1\theta_{j,h}}{\sqrt{nk}} + \frac{i\xi_2\overline{\theta}_{j,h}}{\sqrt{n\overline{k}}}\bigg)\binom{n}{n-j-h, j, h} f^j g^h \\ &= \bigg(\frac{1}{v}e^{\Delta_\kappa} + \frac{f}{v}e^{\Delta_\rho} + \frac{g}{v}e^{\Delta_\sigma}\bigg)^n \\ &= \exp\bigg(n\log\bigg(\frac{1}{v}e^{\Delta_\kappa} + \frac{f}{v}e^{\Delta_\rho} + \frac{g}{v}e^{\Delta_\sigma}\bigg)\bigg), \end{aligned} \tag{5.19}$$

where

$$\Delta_\kappa = \frac{i\xi_1 k}{\sqrt{nk}} + \frac{i\xi_2\overline{k}}{\sqrt{n\overline{k}}}, \quad \Delta_\rho = \frac{i\xi_1 r}{\sqrt{nk}} + \frac{i\xi_2\overline{s}}{\sqrt{n\overline{k}}}, \quad \Delta_\sigma = \frac{i\xi_1 s}{\sqrt{nk}} + \frac{i\xi_2\overline{r}}{\sqrt{n\overline{k}}}.$$

Note that

$$\frac{\overline{r}}{n} \to -s', \quad \frac{\overline{s}}{n} \to -r', \tag{5.20}$$

and that (cf. (5.5))

$$-\min\{k', \overline{k}'\} \leqslant s' \leqslant 0 \leqslant r' \leqslant \min\{k', \overline{k}'\}. \tag{5.21}$$

5.3.1 The case $r' > 0$ or $s' < 0$

First we consider the case where $r' > 0$ or $s' < 0$. Then we have $k' > 0$, $\overline{k}' > 0$ by (5.21), and moreover $1/v = O(1/n)$. Note that each of Δ_κ, Δ_ρ, and Δ_σ converges by (5.20). On the one hand, by (5.14) we have

$$\frac{fg}{n} \to \frac{k'\overline{k}'v'}{(r'-s')^2} < \infty, \tag{5.22}$$

so that

$$\frac{fg}{n^2} \to 0.$$

On the other hand, by (5.13) we have

$$\frac{f}{n} \to \frac{v's'}{s'-r'}, \quad \frac{g}{n} \to \frac{v'r'}{r'-s'}.$$

Hence we have $r' = 0$ or $s' = 0$.

For the moment, assume that $r' = 0$ and $s' < 0$, so that

$$\frac{f}{n} \to v', \quad \frac{g}{n} \to 0.$$

Then it follows from (5.22) that

$$g \to g_\infty := \frac{k'\overline{k}'}{s'^2}.$$

In particular, g is bounded. Moreover, by (5.9) we have

$$r = -\frac{k+sg}{f} \to r_\infty := -\frac{k'+s'g_\infty}{v'},$$

so that r is also bounded, and therefore $\Delta_\rho = O(1/n)$. Since

$$\frac{f}{v}e^{\Delta_\rho} = e^{\Delta_\rho} - \frac{1+g}{v}e^{\Delta_\rho} = 1 + \Delta_\rho - \frac{1+g}{v} + O\left(\frac{1}{n^2}\right),$$

it follows that (5.19) is equal to

$$\begin{aligned}
&\exp\left(n\left(\frac{1}{v}e^{\Delta_\kappa} + \frac{g}{v}e^{\Delta_\sigma} + \Delta_\rho - \frac{1+g}{v} + O\left(\frac{1}{n^2}\right)\right)\right) \\
&\quad = \exp\left(\frac{n}{v}e^{\Delta_\kappa} + \frac{ng}{v}e^{\Delta_\sigma} + n\Delta_\rho - \frac{n(1+g)}{v} + O\left(\frac{1}{n}\right)\right) \\
&\quad \to \exp\left(\exp\left(i\xi_1\sqrt{k'} + i\xi_2\sqrt{\overline{k}'}\right)\frac{1}{v'} + \exp\left(\frac{i\xi_1 s'}{\sqrt{k'}} - \frac{i\xi_2 s'}{\sqrt{\overline{k}'}}\right)\frac{g_\infty}{v'}\right. \\
&\qquad\qquad \left. + \frac{i\xi_1 r_\infty}{\sqrt{k'}} - \frac{i\xi_2(r_\infty+1)}{\sqrt{\overline{k}'}} - \frac{1+g_\infty}{v'}\right). \qquad (5.23)
\end{aligned}$$

We note that (5.23) is (a value of) the characteristic function of an affine transformation of a bivariate Poisson distribution.

We now show that G is a complete multipartite graph for $n \gg 0$, so that we have $s' = -\overline{k}'$, $r_\infty = 0$, $g_\infty = k'/\overline{k}'$, and (5.23) becomes

$$\exp\left(\exp\left(i\xi_1\sqrt{k'} + i\xi_2\sqrt{\overline{k}'}\right)\frac{1}{v'} + \exp\left(-\frac{i\xi_1\overline{k}'}{\sqrt{k'}} + i\xi_2\sqrt{\overline{k}'}\right)\frac{k'}{v'\overline{k}'} - \frac{i\xi_2}{\sqrt{\overline{k}'}} - \frac{1}{\overline{k}'}\right),$$

which corresponds to the distribution ν given in Theorem 5.1.1 (i). Observe that $\overline{s} = -r - 1$ is bounded. By virtue of Proposition 5.2.5 and Lemma 5.2.1, $\overline{G}$ is one of the following for $n \gg 0$: ($\overline{G}_1$) a conference graph; ($\overline{G}_2$) a disjoint union pK_q of complete graphs; ($\overline{G}_3$) a complete multipartite graph $K_{p\times q}$; ($\overline{G}_4$) a Steiner graph of an $S(2, m, d)$; ($\overline{G}_5$) a Latin square graph of a $TD(m, e)$. Case ($\overline{G}_1$) is impossible as v would also be bounded. For Case ($\overline{G}_3$), we have $s' = 0$, a contradiction. If $\overline{G}$ is a

Steiner graph of an $S(2, m, d)$ as in Case ($\overline{G}_4$), then m is bounded since $\overline{s} = -m$. However, since $\overline{k}$ and v are linear and quadratic in d, respectively, $\overline{k}'$ and v' cannot be both finite and non-zero, a contradiction. The same argument shows that Case ($\overline{G}_5$) is also impossible. Hence we are left with Case ($\overline{G}_2$), so that we have (i) in Theorem 5.1.1.

If $r' > 0$ and $s' = 0$, then switching the roles of G and $\overline{G}$ gives (ii) in Theorem 5.1.1. This completes the case where $r' > 0$ or $s' < 0$.

5.3.2 The case $r' = s' = 0$

Next we deal with the case where $r' = s' = 0$. Note that Δ_κ converges, $\Delta_\rho \to 0$, and $\Delta_\sigma \to 0$. From (5.10), (5.11), and (5.12), it follows that

$$r^2 f \leqslant kv, \quad -r\overline{s}f \leqslant k\overline{k}, \quad \overline{s}^2 f \leqslant \overline{k}v,$$

from which it follows that

$$\begin{aligned} \left|\Delta_\rho^2\right| f &\leqslant \left(\frac{|\xi_1^2| r^2}{nk} - 2\frac{|\xi_1\xi_2| r\overline{s}}{n\sqrt{k\overline{k}}} + \frac{|\xi_2^2|\overline{s}^2}{n\overline{k}} \right) f \\ &\leqslant |\xi_1^2|\frac{v}{n} + 2|\xi_1\xi_2|\frac{\sqrt{k\overline{k}}}{n} + |\xi_2^2|\frac{v}{n}. \end{aligned} \tag{5.24}$$

Hence $\Delta_\rho^2 f$ is bounded. Likewise, we can show that $\Delta_\sigma^2 g$ is bounded. We also need the following identities:

$$\Delta_\kappa + \Delta_\rho f + \Delta_\sigma g = 0, \tag{5.25}$$

$$\Delta_\kappa^2 + \Delta_\rho^2 f + \Delta_\sigma^2 g = -\frac{\xi_1^2 v}{n} - \frac{\xi_2^2 v}{n}. \tag{5.26}$$

For the moment, assume that $k' > 0$ or $\overline{k}' > 0$. Note that $1/v = O(1/n)$ in this case. Moreover, we have

$$\begin{aligned}\frac{f}{v}e^{\Delta_\rho} &= \left(1 + \Delta_\rho + \frac{\Delta_\rho^2}{2} + O\big(\Delta_\rho^3\big)\right)\frac{f}{v} \\ &= \left(1 + \Delta_\rho + \frac{\Delta_\rho^2}{2}\right)\frac{f}{v} + o\left(\frac{1}{n}\right),\end{aligned}$$

and similarly for ge^{Δ_σ}/v. Hence it follows from (5.25) and (5.26) that (5.19) is equal to

$$\begin{aligned}&\exp\left(n\log\left(\frac{1}{v}e^{\Delta_\kappa} + \left(1 + \Delta_\rho + \frac{\Delta_\rho^2}{2}\right)\frac{f}{v}\right.\right. \\ &\qquad\qquad\qquad \left.\left. + \left(1 + \Delta_\sigma + \frac{\Delta_\sigma^2}{2}\right)\frac{g}{v} + o\left(\frac{1}{n}\right)\right)\right) \\ &= \exp\left(n\log\left(1 + \frac{1}{v}e^{\Delta_\kappa} - \frac{\Delta_\kappa}{v}\right.\right. \\ &\qquad\qquad\qquad \left.\left. + \left(\Delta_\rho^2 f + \Delta_\sigma^2 g\right)\frac{1}{2v} - \frac{1}{v} + o\left(\frac{1}{n}\right)\right)\right) \\ &= \exp\left(n\left(\frac{1}{v}e^{\Delta_\kappa} - \frac{\Delta_\kappa}{v} + \left(\Delta_\rho^2 f + \Delta_\sigma^2 g\right)\frac{1}{2v} - \frac{1}{v} + o\left(\frac{1}{n}\right)\right)\right) \\ &\to \exp\left(\exp\left(i\xi_1\sqrt{k'} + i\xi_2\sqrt{\overline{k}'}\right)\frac{1}{v'} - \frac{i\xi_1\sqrt{k'} + i\xi_2\sqrt{\overline{k}'}}{v'}\right. \\ &\qquad\qquad\qquad \left. - \frac{\big(\xi_1\sqrt{\overline{k}'} - \xi_2\sqrt{k'}\big)^2}{2v'} - \frac{1}{v'}\right).\end{aligned}$$

This corresponds to the distribution ν in Theorem 5.1.1 (iii).

Finally, assume that $k' = \overline{k}' = 0$. In this case, we have $v' = 0$, i.e.,

$$\frac{v}{n} \to 0.$$

Note that $\Delta_\kappa = O((v/n)^{1/2})$, $\Delta_\rho = O((v/n)^{1/2})$, and $\Delta_\sigma = O((v/n)^{1/2})$. Moreover, by virtue of (5.24), it follows that $\Delta_\rho^2 f = O(v/n)$ and also $\Delta_\sigma^2 g = O(v/n)$. Hence it follows from (5.25) and (5.26) that (5.19) is equal to

$$\begin{aligned}
&\exp\left(n\log\left(\left(1+\Delta_\kappa+\frac{\Delta_\kappa^2}{2}\right)\frac{1}{v}+\left(1+\Delta_\rho+\frac{\Delta_\rho^2}{2}\right)\frac{f}{v}\right.\right.\\
&\qquad\qquad\left.\left.+\left(1+\Delta_\sigma+\frac{\Delta_\sigma^2}{2}\right)\frac{g}{v}+O\left(\left(\frac{v}{n}\right)^{\frac{3}{2}}\right)\frac{1}{v}\right)\right)\\
&=\exp\left(n\log\left(1-\frac{\xi_1^2}{2n}-\frac{\xi_2^2}{2n}+O\left(\left(\frac{v}{n}\right)^{\frac{3}{2}}\right)\frac{1}{v}\right)\right)\\
&=\exp\left(n\left(-\frac{\xi_1^2}{2n}-\frac{\xi_2^2}{2n}+O\left(\left(\frac{v}{n}\right)^{\frac{3}{2}}\right)\frac{1}{v}\right)\right)\\
&=\exp\left(-\frac{\xi_1^2}{2}-\frac{\xi_2^2}{2}+O\left(\left(\frac{v}{n}\right)^{\frac{1}{2}}\right)\right)\\
&\to\exp\left(-\frac{\xi_1^2}{2}-\frac{\xi_2^2}{2}\right).
\end{aligned}$$

This corresponds to the standard bivariate Gaussian distribution, and hence we have (iv) in Theorem 5.1.1.

This completes the proof of Theorem 5.1.1.

5.4 Examples

The graph G (which we recall is a function in n) is already identified for (i) and (ii) in Theorem 5.1.1, whereas (iv) is a degenerate case and is easily realized as it only requires $v/n \to 0$. Below are some examples for (iii) in Theorem 5.1.1, i.e., such that $k' > 0$ or $\overline{k}' > 0$, and $r' = s' = 0$.

Example 5.4.1. Consider the imprimitive strongly regular graphs pK_q and $K_{p\times q}$. Assume that pq is (essentially) linear in n and that $q/n \to 0$. Then we have $k' = 0$

and $\overline{k}' > 0$ in Theorem 5.1.1 (iii) for pK_q, and $k > 0$ and $\overline{k}' = 0$ for $K_{p\times q}$.

Example 5.4.2 (Paley graphs)**.** Let q be a prime power with $q \equiv 1 \pmod 4$. The *Paley graph* $\mathrm{Paley}(q)$ has vertex set $\mathbb{F}_q$ (the finite field with q elements), where two distinct vertices are adjacent if and only if their difference is a square. It is easy to see that $\mathrm{Paley}(q)$ is a conference graph. See [14, Sections 9.1.1 and 9.1.2]. Hence if we take n to be linear in q, then it follows from (5.17) and (5.18) that we are in Theorem 5.1.1 (iii) with $k' = \overline{k}' > 0$.

Example 5.4.3 (Symplectic graphs)**.** Let q be a prime power, and let ℓ be an integer at least two. Endow $\mathbb{F}_q^{2\ell}$ with a non-degenerate symplectic form. The *Symplectic graph* $Sp_{2\ell}(q)$ has as vertex set the set of one-dimensional subspaces (i.e., projective points) of $\mathbb{F}_q^{2\ell}$, where two distinct vertices are adjacent if and only if they are orthogonal. The graph $Sp_{2\ell}(q)$ is strongly regular with parameters (v, k, λ, μ), where

$$v = \frac{q^{2\ell}-1}{q-1}, \quad k = \frac{q(q^{2\ell-2}-1)}{q-1}, \quad \lambda = \frac{q^2(q^{2\ell-4}-1)}{q-1} + q - 1, \quad \mu = \frac{q^{2\ell-2}-1}{q-1},$$

and with restricted eigenvalues

$$r = q^{\ell-1} - 1, \quad s = -q^{\ell-1} - 1.$$

Fix q and let $\ell \to \infty$. If n is linear in $q^{2\ell}$ then again we are in Theorem 5.1.1 (iii) with $k' = \overline{k}'/(q-1) > 0$. There are many other infinite families of strongly regular graphs related to finite geometry; cf. [12].

Example 5.4.4. Let q be a prime power. Let $H_1, H_2, \ldots, H_m$ be distinct one-dimensional subspaces of $\mathbb{F}_q^2$, where $1 \leqslant m \leqslant q$. For $j = 1, 2, \ldots, m$, let $\mathscr{P}_j$

be the set of q parallel affine subspaces of $\mathbb{F}_q^2$ with direction H_j, i.e.,

$$\mathscr{P}_j = \{H_j + x : x \in \mathbb{F}_q^2\} \quad (j = 1, 2, \ldots, m).$$

Let $\mathscr{P} = \mathscr{P}_1 \sqcup \cdots \sqcup \mathscr{P}_m$ and $\mathscr{B} = \mathbb{F}_q^2$. Consider an incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{I})$, where a point $H_j + x$ and a block y are incident if and only if $y \in H_j + x$. Then it is easy to see that $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ is a $TD(q, e)$. Hence if we take both q^2 and mq to be linear in n, then the corresponding Latin square graph attains Theorem 5.1.1 (iii) with $k' > 0$. We may view $\mathrm{Paley}(q^2)$ in this way with $m = (q+1)/2$, as $\mathbb{F}_{q^2} \cong \mathbb{F}_q^2$. We note that, unlike the previous examples, *any* $k' > 0$ and $\overline{k}' \geqslant 0$ can be achieved here as limits. There is also a more general construction of strongly regular graphs from cyclotomy, all giving rise to examples of Theorem 5.1.1 (iii); cf. [14, §9.8.5].

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