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SHORT COMMUNICATION

Quadratic Embedding Constants of Wheel Graphs

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A connected graph is said to be of QE class if it admits a quadratic embedding in a Hilbert space, or equivalently if the distance matrix is conditionally negative definite, or equivalently if the quadratic embedding constant (QEC) of a graph is non-positive. The QEC of wheel graphs are calculated explicitly.

KEYWORDS: conditionally negative definite matrix, distance matrix, quadratic embedding, QE constant, wheel graph

1. Introduction

Let $G = (V, E)$ be a (finite or infinite) connected graph and $D = [d(x, y)]_{x,y\in V}$ the distance matrix. An interesting question is whether or not D is conditional negative definite. Following [6] the *QE constant* of G is defined by

$$\mathrm{QEC}(G) = \sup\{\langle f, Df\rangle \mid f \in C_0(V),\ \langle f, f\rangle = 1,\ \langle \mathbf{1}, f\rangle = 0\}, \tag{1.1}$$

where $C_0(V)$ is the space of $\mathbb{R}$-valued functions on V with finite supports, $\mathbf{1}$ is the constant function taking value one on V, and $\langle\cdot,\cdot\rangle$ is the canonical inner product. By the Schoenberg theorem [7, 8] a graph G admits a quadratic embedding in a Hilbert space if and only if the distance matrix is conditionally negative definite, that is, $\mathrm{QEC}(G) \le 0$. Thus, the QE constant is an interesting characteristic of a graph from the point of view of Euclidean distance geometry [4]. In this paper we determine $\mathrm{QEC}(W_n)$ for a wheel graph W_n on $n + 1$ vertices, $n \ge 3$. Other concrete examples of $\mathrm{QEC}(G)$ are found in [1–3, 5, 6]. The main result is stated in the following

Theorem 1.1. *For $n \ge 3$ we have*

$$\mathrm{QEC}(W_n) = 0 \quad \textit{if } n \textit{ is even}; \quad \mathrm{QEC}(W_n) = -4\sin^2\frac{\pi}{2n} \quad \textit{if } n \textit{ is odd}. \tag{1.2}$$

2. Join of Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two (finite or infinite, not necessarily connected) graphs that are disjoint, i.e., $V_1 \cap V_2 = \emptyset$. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is a graph on $V = V_1 \cup V_2$ with edge set

$$E = E_1 \cup E_2 \cup \{\{x, y\} \mid x \in V_1,\ y \in V_2\}.$$

The graph join $G_1 + G_2$ is always connected, and is not locally finite unless both G_1 and G_2 are finite. Let A_1 and A_2 be the adjacency matrices of G_1 and G_2, respectively. Then the adjacency matrix of $G = G_1 + G_2$ is given by

$$A = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}, \tag{2.1}$$

where J is the matrix whose entries are all one (this symbol is used without explicitly mentioning its size). The diameter of $G = G_1 + G_2$ verifies $1 \le \mathrm{diam}(G) \le 2$, and $\mathrm{diam}(G) = 1$ occurs if and only if both G_1 and G_2 are complete graphs.

Proposition 2.1. *Let $G = (V, E)$ be a (finite or infinite) connected graph with diameter $1 \le \mathrm{diam}(G) \le 2$. Let A be the adjacency matrix of G. Then,*

$$\mathrm{QEC}(G) = -2 - \inf\{\langle f, Af\rangle \mid f \in C_0(V),\ \langle f, f\rangle = 1,\ \langle \mathbf{1}, f\rangle = 0\}. \tag{2.2}$$

Proof. Let I denote the identity matrix (this symbol is used without explicitly mentioning its size). For a graph with diameter $1 \le \mathrm{diam}(G) \le 2$ we have the following obvious relation:

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$$D = 2J - 2I - A. \tag{2.3}$$

Then, for any $f \in C_0(V)$ satisfying $\langle f, f\rangle = 1$ and $\langle \mathbf{1}, f\rangle = 0$, we obtain

$$\langle f, Df\rangle = \langle f, (2J - 2I - A)f\rangle = 2\langle f, Jf\rangle - 2\langle f, f\rangle - \langle f, Af\rangle = -2 - \langle f, Af\rangle, \tag{2.4}$$

where the obvious relation $\langle f, Jf\rangle = \langle \mathbf{1}, f\rangle^2$ is used. Taking the supremum of both sides of (2.4), we come to (2.2). □

Proposition 2.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two arbitrary graphs that are disjoint, and A_1 and A_2 the adjacency matrices, respectively. Let $G = G_1 + G_2$ be their join, and let A and D be the adjacency and distance matrices of G, respectively. Then we have*

$$D = 2J - 2I - A = \begin{bmatrix} 2J - 2I - A_1 & J \\ J & 2J - 2I - A_2 \end{bmatrix} \tag{2.5}$$

and

$$\mathrm{QEC}(G) = -2 - \inf\left\{ \langle g, A_1 g\rangle + \langle h, A_2 h\rangle - 2\langle \mathbf{1}, h\rangle^2 \;\middle|\; \begin{array}{l} g \in C_0(V_1),\ \langle g, g\rangle + \langle h, h\rangle = 1, \\ h \in C_0(V_2),\ \langle \mathbf{1}, g\rangle + \langle \mathbf{1}, h\rangle = 0 \end{array} \right\}. \tag{2.6}$$

Proof. (2.5) is a direct consequence of (2.1) and (2.3). Also from (2.1) we see that (2.2) becomes

$$\mathrm{QEC}(G) = -2 - \inf\left\{ \left\langle \begin{bmatrix} g \\ h \end{bmatrix}, \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix} \right\rangle \;\middle|\; \begin{array}{l} g \in C_0(V_1),\ \langle g, g\rangle + \langle h, h\rangle = 1, \\ h \in C_0(V_2),\ \langle \mathbf{1}, g\rangle + \langle \mathbf{1}, h\rangle = 0 \end{array} \right\}.$$

Since $\langle g, Jh\rangle = \langle h, Jg\rangle = \langle \mathbf{1}, h\rangle\langle \mathbf{1}, g\rangle = -\langle \mathbf{1}, h\rangle^2$ under condition $\langle \mathbf{1}, g\rangle + \langle \mathbf{1}, h\rangle = 0$, we have

$$\left\langle \begin{bmatrix} g \\ h \end{bmatrix}, \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix} \right\rangle = \langle g, A_1 g\rangle + \langle h, A_2 h\rangle - 2\langle \mathbf{1}, h\rangle^2,$$

from which (2.6) follows immediately. □

3. Conditional Minimum

Let V be a finite set with $|V| = n \geq 3$, and T an arbitrary real symmetric matrix with index set V. We are interested in the conditional minimum:

$$M(T) = \min\{\langle f, Tf\rangle \mid f \in C(V),\ \langle f, f\rangle = 1,\ \langle \mathbf{1}, f\rangle = 0\}, \tag{3.1}$$

where $C(V)$ stands for the space of all $\mathbb{R}$-valued functions on V. Since V is a finite set with $|V| = n$, we have $C_0(V) = C(V) \cong \mathbb{R}^n$. Employing the method of Lagrange multipliers, we set

$$F(f, \lambda, \mu) = \langle f, Tf\rangle - \lambda(\langle f, f\rangle - 1) - \mu\langle \mathbf{1}, f\rangle. \tag{3.2}$$

Let $\mathcal{S}(T)$ be the set of all stationary points of F, namely, the set of $(f, \lambda, \mu) \in C(V) \times \mathbb{R} \times \mathbb{R}$ at which all the partial derivatives of F vanish. After simple observation we see that $\mathcal{S}(T)$ consists of (f, λ, μ) satisfying

$$(T - \lambda)f = \frac{\mu}{2}\mathbf{1}, \quad \langle f, f\rangle = 1, \quad \langle \mathbf{1}, f\rangle = 0. \tag{3.3}$$

The next result is useful for calculating $\mathrm{QEC}(G)$ for a finite graph G.

Lemma 3.1. *$M(T) = \min \Lambda(T)$, where $\Lambda(T) = \{\lambda \in \mathbb{R} \mid (f, \lambda, \mu) \in \mathcal{S}(T)$ for some $f \in C(V)$ and $\mu \in \mathbb{R}\}$.*

Proof. We first note that the conditions $\langle f, f\rangle = 1$ and $\langle \mathbf{1}, f\rangle = 0$ define a sphere of $n - 2$ dimension in $C(V) \cong \mathbb{R}^n$, which is a smooth compact manifold for $n \geq 3$. Since the quadratic function $\langle f, Tf\rangle$ is smooth, the conditional minimum $M(T)$ is attained at a certain $f \in C(V)$ appearing in $\mathcal{S}(T)$. Namely,

$$M(T) = \min\{\langle f, Tf\rangle \mid f \in C(V) \text{ with } (f, \lambda, \mu) \in \mathcal{S}(T) \text{ for some } \lambda \in \mathbb{R} \text{ and } \mu \in \mathbb{R}\}. \tag{3.4}$$

On the other hand, for $(f, \lambda, \mu) \in \mathcal{S}(T)$ we have

$$\langle f, Tf\rangle = \left\langle f, \lambda f + \frac{\mu}{2}\mathbf{1} \right\rangle = \lambda\langle f, f\rangle + \frac{\mu}{2}\langle f, \mathbf{1}\rangle = \lambda. \tag{3.5}$$

Combining (3.4) and (3.5) we get the assertion. □

Remark 3.2. In a similar manner as in the proof of Lemma 3.1 the following relation holds:

$$\max\{\langle f, Tf\rangle \mid f \in C(V),\ \langle f, f\rangle = 1,\ \langle \mathbf{1}, f\rangle = 0\} = \max \Lambda(T).$$

4. Wheel Graphs

Let $n \geq 3$. A *wheel graph* W_n is a graph on $n+1$ vertices, defined as the join of a cycle C_n and a singleton graph K_1. In what follows, the vertex set of W_n is taken to be $\{0, 1, 2, \dots, n-1, n\}$, where $\{0, 1, 2, \dots, n-1\}$ constitutes a cycle C_n with edge set $\{\{0,1\}, \{1,2\}, \dots, \{n-2, n-1\}, \{n-1, 0\}\}$ and the vertex n becomes a hub of the wheel. The adjacency matrices of C_n and K_1 are given respectively by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = [0],$$

with which the adjacency matrix of W_n is of the form (2.1). Then, using Proposition 2.2 we obtain

$$\mathrm{QEC}(W_n) = -2 - \min\{\langle g, A_1 g\rangle - 2h^2 \mid g \in \mathbb{R}^n,\ h \in \mathbb{R},\ \langle g, g\rangle + h^2 = 1,\ \langle \mathbf{1}, g\rangle + h = 0\}. \tag{4.1}$$

We will calculate the conditional minimum in (4.1). Setting T to be the block diagonal matrix with blocks A_1 and $-2I$, we employ the method introduced in Sect. 3. Let $\mathcal{S}$ be the set of all stationary points of

$$F(g, h, \lambda, \mu) = \langle g, A_1 g\rangle - 2h^2 - \lambda(\langle g, g\rangle + h^2 - 1) - \mu(\langle \mathbf{1}, g\rangle + h), \tag{4.2}$$

namely, the set of $(g, h, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfying

$$(A_1 - \lambda)g = \frac{\mu}{2}\mathbf{1}, \tag{4.3}$$

$$(-2 - \lambda)h = \frac{\mu}{2}, \tag{4.4}$$

$$\langle g, g\rangle + h^2 = 1, \tag{4.5}$$

$$\langle \mathbf{1}, g\rangle + h = 0, \tag{4.6}$$

and put $\Lambda = \{\lambda \in \mathbb{R} \mid (g, h, \lambda, \mu) \in \mathcal{S}$ for some $g \in \mathbb{R}^n$, $h \in \mathbb{R}$ and $\mu \in \mathbb{R}\}$. Then by Lemma 3.1 we have

$$\min\{\langle g, A_1 g\rangle - 2h^2 \mid g \in \mathbb{R}^n,\ h \in \mathbb{R},\ \langle g, g\rangle + h^2 = 1,\ \langle \mathbf{1}, g\rangle + h = 0\} = \min \Lambda. \tag{4.7}$$

Now we will determine the set Λ. Taking the inner product of (4.3) with $\mathbf{1}$, we obtain $\langle \mathbf{1}, (A_1 - \lambda)g\rangle = (\mu/2)\langle \mathbf{1}, \mathbf{1}\rangle$. Then using $\langle \mathbf{1}, A_1 g\rangle = \langle A_1 \mathbf{1}, g\rangle = 2\langle \mathbf{1}, g\rangle$, $\langle \mathbf{1}, \mathbf{1}\rangle = n$ and $\langle \mathbf{1}, g\rangle = -h$ by (4.6) we obtain

$$(\lambda - 2)h = \frac{\mu n}{2}. \tag{4.8}$$

On the other hand, for (4.3) we consider the difference equation:

$$g_{k-1} - \lambda g_k + g_{k+1} = \frac{\mu}{2}, \quad k \in \mathbb{Z}. \tag{4.9}$$

Any solution to (4.9) satisfying the periodic condition $g_k = g_{n+k}$ gives rise to a solution $g = [g_0\ g_1\ \dots\ g_{n-1}]^{\mathrm{T}} \in \mathbb{R}^n$ to (4.3), and vice versa. The characteristic equation of (4.9) is given by $\xi^2 - \lambda\xi + 1 = 0$, and we consider three cases according to the characteristic roots.

(Case 1) $\lambda = 2$ and the characteristic root is $\xi = 1$ (multiplicity two). We see first from (4.8) that $\mu = 0$. Using the characteristic root $\xi = 1$ and periodicity, we see that a general solution to (4.9) is given by $g_k = C$ (constant), and hence $g = C\mathbf{1}$. On the other hand, we have $h = 0$ by $\mu = 0$ and (4.4). Then (4.5) and (4.6) become $C^2\langle \mathbf{1}, \mathbf{1}\rangle = 1$ and $C\langle \mathbf{1}, \mathbf{1}\rangle = 0$, respectively, from which we come to contradiction. Consequently, $\lambda = 2 \notin \Lambda$.

(Case 2) $\lambda = -2$ and the characteristic root is $\xi = -1$ (multiplicity two). Note first that $\mu = 0$ by (4.4), and hence $h = 0$ by (4.8). In this case a general solution to (4.9) is given by $g_k = (C_1 + C_2 k)(-1)^k$, where C_1 and C_2 are constants. Taking the periodicity into account, we obtain

$$g_k = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ C_1(-1)^k, & \text{if } n \text{ is even}. \end{cases}$$

Then (4.5) is not fulfilled if n is odd, and it is fulfilled with $C_1 = 1/\sqrt{n}$ if n is even. Consequently, $\lambda = -2 \notin \Lambda$ if n is odd, and $\lambda = -2 \in \Lambda$ if n is even.

(Case 3) $\lambda \neq \pm 2$. Let α, β be the characteristic roots, where $\alpha \neq \beta$, $\alpha + \beta = \lambda$ and $\alpha\beta = 1$. Then a general solution to (4.9) is given by

$$g_k = C_1\alpha^k + C_2\beta^k + \frac{\mu}{2(2-\lambda)}, \tag{4.10}$$

where C_1 and C_2 are constants. The periodic conditions $g_0 = g_n$ and $g_1 = g_{n+1}$ give rise to

$$\begin{cases} (1-\alpha^n)C_1 + (1-\beta^n)C_2 = 0, \\ \alpha(1-\alpha^n)C_1 + \beta(1-\beta^n)C_2 = 0. \end{cases} \tag{4.11}$$

The determinant of the coefficient matrix is $(1-\alpha^n)(1-\beta^n)(\beta-\alpha)$.

(Case 3-1) $\alpha^n \neq 1$, also $\beta^n \neq 1$ by $\alpha\beta = 1$. We then see from (4.11) that $C_1 = C_2 = 0$. If $\mu = 0$, we have $g = 0$ by (4.10) and also $h = 0$ by (4.4). Then (4.5) is not fulfilled. Hence $\mu \neq 0$, and we have

$$g = \frac{\mu}{2(2-\lambda)}\,\mathbf{1}, \quad h = -\frac{\mu}{2(\lambda+2)}.$$

These satisfy (4.5) and (4.6) if and only if

$$\lambda = \frac{2-2n}{n+1} = -2 + \frac{4}{n+1}, \quad \mu^2 = \frac{64n}{(n+1)^3}.$$

Consequently, the above λ belongs to Λ. (This corresponds to the case of $g_k = \text{const.}$)

(Case 3-2) $\alpha^n = \beta^n = 1$. In this case we may set $\alpha = e^{2\pi ip/n}$ and $\beta = e^{-2\pi ip/n}$ with $p = 0, 1, 2, \dots, n-1$. Then,

$$\lambda = \alpha + \beta = 2\cos\frac{2p\pi}{n}. \tag{4.12}$$

Since $\lambda \neq \pm 2$ by assumption, we choose p from $\{1, 2, \dots, n-1\}\backslash\{n/2\}$ and we have $\alpha, \beta \notin \{\pm 1\}$. Now we show that

$$\mu = 0, \quad h = 0, \quad g_k = \frac{\alpha^k + \beta^k}{\sqrt{2n}}$$

together with (4.12) satisfy (4.3)–(4.6). In fact, (4.3) follows since g_k in (4.10) is periodic for any choice of C_1 and C_2. (4.4) is obvious. (4.5) follows from the obvious relations $1 + \alpha + \cdots + \alpha^{n-1} = 1 + \beta + \cdots + \beta^{n-1} = 0$. Similarly, (4.6) is verified. Consequently, every λ in (4.12) belongs to Λ.

Noting that $\lambda = -2$ is obtained by setting $p = n/2$ in (4.12), we may summarize the above three cases as follows:

$$\Lambda = \left\{-2 + \frac{4}{n+1}\right\} \cup \left\{2\cos\frac{2p\pi}{n} \,\middle|\, 1 \le p \le n-1\right\}.$$

Now we compute $\mathrm{QEC}(W_n) = -2 - \min\Lambda$, see (4.1) and (4.7). If n is even, we have $\min\Lambda = -2$ so that $\mathrm{QEC}(W_n) = 0$. Suppose that n is odd, say, $n = 2m-1$ with $m \ge 2$. Note that

$$\min\left\{2\cos\frac{2p\pi}{n} \,\middle|\, 1 \le p \le n-1\right\} = 2\cos\frac{2m\pi}{2m-1} = -2\cos\frac{\pi}{2m-1} = -2 + 4\sin^2\frac{\pi}{2(2m-1)}. \tag{4.13}$$

Using the obvious inequality $\sin\theta \le \theta$ for $\theta \ge 0$, we see by easy calculus that

$$4\sin^2\frac{\pi}{2(2m-1)} \le \frac{4}{(2m-1)+1}, \quad m \ge 2.$$

Therefore, $\min\Lambda$ is given by (4.13) and

$$\mathrm{QEC}(W_{2m-1}) = -2 - \min\Lambda = -4\sin^2\frac{\pi}{2(2m-1)} = -4\sin^2\frac{\pi}{2n},$$

which completes the proof of Theorem 1.1.

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