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On Cell Problems for Nonlinear PDES and Its Application to Homogenization

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This proceeding is based on the lecture given by the author at GSIS international winter school 2017 on "Stochastic homogenization and its applications" at Tohoku University. The main purpose of this proceeding is to present ideas of the works of [5, 6] on (periodic) homogenization for the Hamilton–Jacobi equation, and give a recent result of [16] on the selection problem for the cell problem as simply as possible.

KEYWORDS: Hamilton–Jacobi equations, homogenization, cell problems, selection problems

1. Introduction

In this proceeding, we study the behavior, as ε (> 0) tends to 0, of the solutions u^ε of the Hamilton–Jacobi equation:

$$(\mathrm{HJ})_\varepsilon \qquad \begin{cases} u^\varepsilon_t + H\left(\dfrac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases}$$

where $T > 0$, the Hamiltonian $H(y,p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the initial function $u_0 : \mathbb{R}^n \to \mathbb{R}$, respectively, are given continuous functions. Here are our standing assumptions:

(A1) The function H is uniformly coercive in the y-variable, i.e.,

$$\lim_{r\to\infty} \inf\{H(y,p) \mid y \in \mathbb{R}^n, |p| \geq r\} = \infty.$$

(A2) The function $y \mapsto H(y,p)$ is $\mathbb{Z}^n$-periodic for each p, i.e.,

$$H(y+z,p) = H(y,p) \quad \text{for any } y,p \in \mathbb{R}^n, z \in \mathbb{Z}^n.$$

(A3) $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$.

This is a *singular limit problem* of differential equations, which is called *homogenization* with a background of the material science. The study of homogenization of nonlinear partial differential equations of the type $(\mathrm{HJ})_\varepsilon$ started from the famous unpublished work by Lions, Papanicolaou, Varadhan [12].

First, we heuristically derive the behavior of solutions of $(\mathrm{HJ})_\varepsilon$ as ε tends to 0. We consider a formal asymptotic expansion of solutions u^ε of $(\mathrm{HJ})_\varepsilon$ of the form

$$u^\varepsilon(x,t) := \overline{u}(x,t) + \varepsilon v\left(\frac{x}{\varepsilon}\right) + O(\varepsilon^2).$$

Of course, there are several possibility for ansatzs at this moment, but we will see below that this is a right one. Set $y := x/\varepsilon$. Plugging this into $(\mathrm{HJ})_\varepsilon$ and performing formal calculations, we achieve

$$\overline{u}_t + H(y, D_x\overline{u} + D_y v + \cdots) = 0.$$

Let $P = D_x\overline{u}(x,t)$, and $v(\cdot,P)$ be a $\mathbb{Z}^n$-periodic solution of the stationary problem:

$$(\mathrm{C})_P \qquad H(y, P + Dv(y,P)) = \overline{H}(P) \quad \text{in } \mathbb{R}^n,$$

where $\overline{H}(P)$ is a unknown constant. We call a problem to find a pair $(v(\cdot,P), \overline{H}(P)) \in C(\mathbb{R}^n) \times \mathbb{R}$ the *cell problem* for Hamilton–Jacobi equations.

This formal observation tells us that we can expect that u^ε converges to the function $\overline{u}$ as $\varepsilon \to 0$ which is the solution of

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$$\text{(H)} \qquad \begin{cases} \overline{u}_t + \overline{H}(D\overline{u}) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \overline{u}(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases}$$

We give a proof of the existence result for $(\mathrm{C})_P$ in the viscosity sense in Sect. 2. In Sect. 3, we explain an idea of the *perturbed test function method* which was introduced by Evans [5] and applied it to prove the homogenization of solutions of $(\mathrm{HJ})_\varepsilon$.

In Sect. 4, we revisit cell problem $(\mathrm{C})_P$. We emphasize here that in general, solutions to cell problem $(\mathrm{C})_P$ are not unique even up to additive constants. See examples in [16] and [11, Chapter 6] for instance. Therefore, if we consider an approximation procedure for $(\mathrm{C})_P$, then two natural questions appear:

(i) Does the whole family of approximate solutions converges?

(ii) If it converges, then which solution of the corresponding cell problem is the limit (which solution is selected)?

This type of questions is called a *selection problem* for cell problem $(\mathrm{C})_P$.

In Sect. 4, we consider a so-called *discounted approximation* (see (2.1)) for $(\mathrm{C})_P$, and investigate the selection problem for this approximation procedure. We give an answer for the above questions (i), (ii) based on a recent result in [16].

2. Existence Result for $(\mathrm{C})_P$

As seen in Introduction, cell problem $(\mathrm{C})_P$ plays an important role to determine the limit of solutions of $(\mathrm{HJ})_\varepsilon$. In this section, we consider cell problem $(\mathrm{C})_P$ for Hamilton–Jacobi equations for a fixed $P \in \mathbb{R}^n$, and prove the existence of a unknown pair $(v(\cdot,P), \overline{H}(P)) \in C(\mathbb{T}^n) \times \mathbb{R}$ so that $v(\cdot,P)$ solves $(\mathrm{C})_P$ in the viscosity sense. We recall the definition of viscosity solutions in Sect. 3.

In what follows in this section, we consider the situation that everything is assumed to be $\mathbb{Z}^n$-periodic with respect to the spatial variable x. As it is equivalent to consider the equations in the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, we always use this notation.

Theorem 2.1. *Assume that $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and that H satisfies* (A1), (A2). *For any $P \in \mathbb{R}^n$, there exists a pair $(v(\cdot,P), \overline{H}(P)) \in \mathrm{Lip}(\mathbb{T}^n) \times \mathbb{R}$ such that $v(\cdot,P)$ solves $(\mathrm{C})_P$ in the viscosity sense.*

This theorem was first proved in [12].

Proof. For $\delta > 0$, consider the following approximate problem

$$\delta v^\delta + H(x, P + Dv^\delta) = 0 \quad \text{in } \mathbb{T}^n. \tag{2.1}$$

Setting $M := \max_{x\in\mathbb{T}^n} |H(x,P)|$, it is easily seen that $\pm M/\delta$ are a supersolution and subsolution of (2.1), respectively. By the Perron method in the theory of viscosity solutions (we refer to [4], [1] for standard theory of viscosity solutions for instance), there exists a unique viscosity solution v^δ to (2.1) such that

$$|v^\delta| \le M/\delta,$$

which implies further that $H(x, P + Dv^\delta) \le M$. In view of coercivity assumption (A1), we get

$$|Dv^\delta| \le C \quad \text{for some } C > 0 \text{ independent of } \delta. \tag{2.2}$$

Therefore, we obtain that $\{v^\delta(\cdot) - v^\delta(x_0)\}_{\delta>0}$ is equi-Lipschitz continuous for a fixed $x_0 \in \mathbb{T}^n$. Moreover, noting that

$$|v^\delta(x) - v^\delta(x_0)| \le \|Dv^\delta\|_{L^\infty(\mathbb{T}^n)}|x - x_0| \le C,$$

we see that $\{v^\delta(\cdot) - v^\delta(x_0)\}_{\delta>0}$ is uniformly bounded in $\mathbb{T}^n$. Thus, in light of the Arzelà–Ascoli theorem, there exists a subsequence $\{\delta_j\}_j$ converging to 0 so that $v^{\delta_j}(\cdot) - v^{\delta_j}(x_0) \to v(\cdot,P)$ uniformly on $\mathbb{T}^n$ as $j \to \infty$. Since $|\delta_j v^{\delta_j}(x_0)| \le M$, by passing to another subsequence if necessary, we obtain that

$$\delta_j v^{\delta_j}(x_0) \to -\overline{H}(P) \quad \text{for some } c \in \mathbb{R}.$$

In view of the stability result of viscosity solutions, we get the conclusion. □

Remark 1. 1. Notice that $\overline{H}(P)$ is determined uniquely, which is called the *effective Hamiltonian*. 2. We notice that the approximation procedure above using (2.1) is called the *discounted approximation procedure*. It is a natural procedure in many senses. Firstly, the approximation makes Eq. (2.1) strictly monotone in v^δ, which fits perfectly in the well-posedness setting of viscosity solutions. Secondly, we can easily get a priori estimate on $|Dv^\delta|$ as seen in the proof of Theorem 2.1. Thus, in light of the Arzelà–Ascoli theorem, we get the existence result by a rather soft argument. We emphasize here that from this argument, we only know convergence of $\{v^{\delta_j} - v^{\delta_j}(x_0)\}_j$ via the subsequence $\{\delta_j\}_j$. It is not clear at all at this moment whether $\{v^\delta - v^\delta(x_0)\}_{\delta>0}$ converges uniformly as $\delta \to 0$ or not. We will come back to this question in Sect. 4.

3. Perturbed Test Function Method

In this section, we give a proof of the homogenization result for $(\mathrm{HJ})_\varepsilon$.

Theorem 3.1 (Homogenization Result). *Assume that $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and that H satisfies* (A1), (A2). *Let u^ε be the viscosity solution to* $(\mathrm{HJ})_\varepsilon$. *The function u^ε converges locally uniformly in $\mathbb{R}^n \times (0,T)$ to the function $u \in \mathrm{Lip}(\mathbb{R}^n \times [0,T))$ as $\varepsilon \to 0$, where u is the unique viscosity solution to* (H).

This theorem was first proved in [12]. Evans in [5] introduced the *perturbed test function method*, which is very versatile to study singular limit problems in differential equations, and applied it to obtain Theorem 3.1.

To have a look at ideas of Evans as simply as possible, let us first formally suppose that u^ε is smooth. Notice that this is a completely formal assumption as we cannot expect a global smooth solution u of Hamilton–Jacobi equations in general. Moreover, take a sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$ as $j \to \infty$ so that $u^{\varepsilon_j} \to \overline{u}$, and we will observe how we prove that $\overline{u}$ satisfies the limit problem (H). Note that $\overline{u}$ may depend on the choice of a subsequence at this moment, but the uniqueness of viscosity solutions of (H) easily implies the convergence (for a whole sequence).

First attempt. Take a test function $\varphi \in C^1(\mathbb{R}^n \times [0,T))$ and let $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ be a point such that $\max_{(x,t)\in\mathbb{R}^n\times[0,T)}(\overline{u}-\varphi)(x,t) = (\overline{u}-\varphi)(x_0,t_0)$. Take points $(x_{\varepsilon_j}, t_{\varepsilon_j}) \in \mathbb{R}^n \times (0,T)$ satisfying $\max_{(x,t)\in\mathbb{R}^n\times[0,T)}(u^{\varepsilon_j} - \varphi)(x,t) = (u^{\varepsilon_j}-\varphi)(x_{\varepsilon_j}, t_{\varepsilon_j})$, and $(x_{\varepsilon_j}, t_{\varepsilon_j}) \to (x_0,t_0)$ as $j \to \infty$. Then, we have

$$0 = u_t^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j}) + H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, Du^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j})\right) = \varphi_t(x_{\varepsilon_j}, t_{\varepsilon_j}) + H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, D\varphi(x_{\varepsilon_j}, t_{\varepsilon_j})\right).$$

We realize here that because of the term $x_{\varepsilon_j}/\varepsilon_j$ in the equation, we cannnot see what happens if we send $j \to \infty$ here. Thus, we see that this attempt is a failure.

Second attempt. Take a test function $\varphi \in C^1(\mathbb{R}^n \times [0,T))$ and let $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ be a point such that $\max_{(x,t)\in\mathbb{R}^n\times[0,T)}(\overline{u}-\varphi)(x,t) = (\overline{u}-\varphi)(x_0,t_0)$. Let $P \in \mathbb{R}^n$ (which will be chosen later), and $v(\cdot,P)$ be a solution to $(\mathrm{C})_P$. Suppose here again that $v(\cdot,P)$ is smooth. Consider

$$\max_{(x,t)\in\mathbb{R}^n\times[0,T)}\left(u^{\varepsilon_j}(x,t) - \varphi(x,t) - \varepsilon_j v\left(\frac{x}{\varepsilon_j}, P\right)\right). \tag{3.1}$$

Let $(x_{\varepsilon_j}, t_{\varepsilon_j}) \in \mathbb{R}^n \times (0,T)$ be a point where the maximum is attained at. Then,

$$\begin{aligned}
0 &= u_t^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j}) + H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, Du^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j})\right)\\
&= \varphi_t(x_{\varepsilon_j}, t_{\varepsilon_j}) + H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, D\varphi(x_{\varepsilon_j}, t_{\varepsilon_j}) + D_y v\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, P\right)\right)\\
&\geq \varphi_t(x_{\varepsilon_j}, t_{\varepsilon_j}) + H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, D\varphi(x_0, t_0) + D_y v\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, P\right)\right) - \omega(\varepsilon_j),
\end{aligned}$$

where $y = x/\varepsilon$, and ω is a continuous function with $\omega(0)=0$.

Here, letting $P = D\varphi(x_0,t_0)$, we see that

$$\overline{H}(P) = H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, P + D_y v\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, P\right)\right) = H\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, D\varphi(x_0,t_0) + D_y v\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}, P\right)\right),$$

since $v(\cdot,P)$ is a solution to $(\mathrm{C})_P$. Therefore,

$$\varphi_t(x_{\varepsilon_j}, t_{\varepsilon_j}) + \overline{H}(D\varphi(x_0,t_0)) \leq \omega(\varepsilon_j).$$

Sending $j \to \infty$ yields that

$$\overline{u}_t(x_0,t_0) + \overline{H}(D\overline{u}(x_0,t_0)) \leq 0.$$

Similarly, we can formally prove that $\overline{u}$ is a supersolution of (H).

In this argument, we perturb a test function φ by using a solution $v(\cdot,P)$ to $(\mathrm{C})_P$, which is called a *corrector*. Therefore, we call this argument a perturbed test function method.

Now, to make this argument rigorous, let us recall the definition of the viscosity solution.

Definition 1 (Viscosity subsolutions, supersolutions, solutions). *An upper (resp., lower) semicontinuous function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of the intial-value problem:*

$$(C) \qquad \begin{cases} u_t + H(x,Du) = 0 & \text{in } \mathbb{R}^n \times (0,T),\\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases}$$

provided that $u(\cdot,0) \leq u_0$ (resp., $u(\cdot,0) \geq u_0$) on $\mathbb{R}^n$, and for each $\varphi \in C^1(\mathbb{R}^n \times (0,T))$, if $u - \varphi$ has a local maximum (resp., minimum) at $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ then

$$\varphi_t(x_0,t_0) + H(x_0, D\varphi(x_0,t_0)) \le \text{(resp., } \ge\text{) } 0.$$

A function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *is called a viscosity solution of the intial-value problem* (C) *if* u *is both a viscosity subsolution, and a viscosity supersolution (hence continuous) of* (C).

To prove Theorem 3.1, we prepare two lemmas.

Lemma 3.2. *Set*

$$W(x,t) := \lim_{\varepsilon\to 0} \sup\{u^\delta(y,s) \mid |x-y| \le \varepsilon, |t-s| \le \varepsilon, \delta \le \varepsilon\} = \limsup_{\varepsilon\to 0}{}^* u^\varepsilon(x,t),$$
$$w(x,t) := \lim_{\varepsilon\to 0} \inf\{u^\delta(y,s) \mid |x-y| \le \varepsilon, |t-s| \le \varepsilon, \delta \le \varepsilon\} = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon(x,t).$$

Then, W *and* w *are, respectively, a viscosity subsolution and a supersolution of* (H).

Proof. We can easily check $W(\cdot,0) = w(\cdot,0) = u_0(x)$ on $\mathbb{R}^n$. We employ the perturbed test function method to prove that W and w are a viscosity subsolution and supersolution, respectively, of the equation in (H). We only prove that W is a viscosity subsolution since by symmetry we can prove that w is a viscosity supersolution. We take a test function $\varphi \in C^1(\mathbb{R}^n \times (0,T))$ such that $W - \varphi$ has a strict maximum at $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$. Let $P := D\varphi(x_0,t_0)$. Choose a sequence $\varepsilon_m \to 0$, $(x_m,t_m) \in \mathbb{R}^n \times (0,T)$ such that

$$W(x_0,t_0) = \lim_{m\to\infty} u^{\varepsilon_m}(x_m,t_m).$$

Set

$$\psi^{\varepsilon,\alpha}(x,y,t) := \varphi(x,t) + \varepsilon v\left(\frac{y}{\varepsilon}\right) + \frac{|x-y|^2}{2\alpha^2},$$

for $\alpha > 0$, where v is a viscosity solution of $(\mathrm{C})_P$. For every $m \in \mathbb{N}, \alpha > 0$, there exist $(x_{m,\alpha}, y_{m,\alpha}, t_{m,\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,T)$ such that

$$\max_{\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} [u^{\varepsilon_m}(x,t) - \psi^{\varepsilon_m,\alpha}(x,y,t)] = u^{\varepsilon_m}(x_{m,\alpha},t_{m,\alpha}) - \psi^{\varepsilon_m,\alpha}(x_{m,\alpha},y_{m,\alpha},t_{m,\alpha}) \tag{3.2}$$

and up to passing some subsequences

$$(x_{m,\alpha}, y_{m,\alpha}, t_{m,\alpha}) \to (x_m, x_m, t_m) \text{ as } \alpha \to 0,$$
$$(x_m,t_m) \to (x_0,t_0) \text{ as } m \to \infty,$$
$$\lim_{m\to\infty}\lim_{\alpha\to 0} u^{\varepsilon_m}(x_{m,\alpha},t_{m,\alpha}) = W(x_0,t_0).$$

By the definition of viscosity subsolutions, we have

$$\varphi_t(x_{m,\alpha},t_{m,\alpha}) + H\left(\frac{x_{m,\alpha}}{\varepsilon_m}, D\varphi(x_{m,\alpha},t_{m,\alpha}) + \frac{x_{m,\alpha}-y_{m,\alpha}}{\alpha^2}\right) \le 0. \tag{3.3}$$

Since $v(\cdot,P)$ is a viscosity supersolution of $(\mathrm{C})_P$, we have

$$H\left(\frac{y_{m,\alpha}}{\varepsilon_m}, P + \frac{x_{m,\alpha}-y_{m,\alpha}}{\alpha^2}\right) \ge \overline{H}(P). \tag{3.4}$$

Note that due the Lipschitz continuity of v, we have

$$\left|\frac{x_{m,\alpha}-y_{m,\alpha}}{\alpha^2}\right| \le C \quad \text{for some } C > 0,$$

where C is independent of m, α. Thus, by taking a subsequence if necessary, we can assume that $(x_{m,\alpha} - y_{m,\alpha})/\alpha^2 \to Q_m \in \mathbb{R}$ as $\alpha \to 0$.

Let $\alpha \to 0$ in (3.3) and (3.4) to derive

$$\varphi_t(x_m,t_m) + H\left(\frac{x_m}{\varepsilon_m}, D\varphi(x_m,t_m) + Q_m\right) \le 0 \tag{3.5}$$

and

$$H\left(\frac{x_m}{\varepsilon_m}, P + Q_m\right) \ge \overline{H}(P). \tag{3.6}$$

Combine (3.5) with (3.6) to get

$$\varphi_t(x_m, t_m) + \overline{H}(P) \le H\left(\frac{x_m}{\varepsilon_m}, P + Q_m\right) - H\left(\frac{x_m}{\varepsilon_m}, D\varphi(x_m, t_m) + Q_m\right)$$
$$\le \omega(|P - D\varphi(x_m, t_m)|)$$

for some modulus ω. Letting $m \to \infty$, we get the result. □

Note that the effective Hamiltonian is continuous and coercive. By a standard comparison principle in the theory of viscosity solutions, we can easily prove

Lemma 3.3. *The limit problem* (H) *has the unique viscosity solution.*

Theorem 3.1 is a straightforward result of Lemmas 3.2 and 3.3.

4. Selection Problem for (C)$_P$

In this section, we consider the discounted approximation

$$(\mathrm{D})_\delta \qquad \delta v^\delta + H(x, P + Dv^\delta) = \overline{H}(P) \quad \text{in } \mathbb{T}^n,$$

again, and investigate the selection problem, which is addressed in Introduction, appearing in this approximation procedure. Note here that we add $\overline{H}(P)$ in the right hand side for normalization.

Here is a main theorem.

Theorem 4.1. *Assume that*

$$(A4) \qquad \begin{cases} H \in C^2(\mathbb{T}^n \times \mathbb{R}^n), & D_p^2 H \ge 0, \\ |D_x H(x,p)| \le C(1 + H(x,p)) & \text{for all } (x,p) \in \mathbb{T}^n \times \mathbb{R}^n, \\ \lim_{|p| \to +\infty} \dfrac{H(x,p)}{|p|} = +\infty, & \text{uniformly for } x \in \mathbb{T}^n. \end{cases}$$

For each $\delta > 0$, let v^δ be the solution to (D)$_\delta$*. Then, we have that, as $\delta \to 0$,*

$$v^\delta(x) \to v^0(x) := \sup_{\phi \in \mathcal{E}} \phi(x) \quad \text{uniformly for } x \in \mathbb{T}^n, \tag{4.1}$$

where we denote by $\mathcal{E}$ the family of solutions v of (C)$_P$ *satisfying*

$$\iint_{\mathbb{T}^n \times \mathbb{R}^n} v \, d\mu \le 0 \quad \text{for all } \mu \in \mathcal{M}. \tag{4.2}$$

The set $\mathcal{M}$, which is a family of probability measures on $\mathbb{T}^n \times \mathbb{R}^n$, is defined in Sect. 4.1.

Remark 2.
(i) The selection problem appearing in the discount approximation procedure was proposed by Lions, Papanicolaou, Varadhan [12]. See also Bardi, Capuzzo-Dolcetta [1, Remark 1.2, page 400]. It remained unsolved for almost 30 years. Recently, there was significant progress in the case of convex Hamiltonians. First, a partial characterization of the possible limits was given by Gomes [9] in terms of the Mather measures. Davini, Fathi, Iturriaga, Zavidovique [2] and Mitake, Tran [16] gave a positive answer for this question, respectively, by using a dynamical approach and the nonlinear adjoint method.
(ii) From a view point of the existence result for (C)$_P$, it seems to be natural to consider this selection problem for non-convex Hamilton–Jacobi equations. However, it seems quite challenging to study this under a general nonconvex setting. In [10], some partial answers were obtained. We also emphasize that a rate of the convergence (quantitative results) of Theorem 4.1 remains quite open. In [15], it is demonstrated that error estimates would depend highly on dynamics of the dynamical systems in general.

4.1 Construction of $\mathcal{M}$

Set $H^P(x,p) := H(x, P + p) - \overline{H}(P)$. Then, we can rewrite (C)$_P$ and (D)$_\delta$ as

$$\begin{aligned} H^P(x, Dv) &= 0 \qquad \text{in } \mathbb{T}^n, \\ \delta v^\delta + H^P(x, v^\delta) &= 0 \qquad \text{in } \mathbb{T}^n, \end{aligned} \tag{4.3}$$

respectively. We henceforth denote H^P by H for simplicity, which hopefully will not confuse readers…

Since v^δ, v are not smooth in general, in order to perform our analysis, we need a regularizing process. For each $\delta, \eta > 0$, we consider

$$(\mathrm{A})_\delta^\eta \qquad \delta v^{\delta,\eta} + H(x, Dv^{\delta,\eta}) = \eta^2 \Delta v^{\delta,\eta} \qquad \text{in } \mathbb{T}^n,$$

where $\Delta f := \sum_{i=1}^n \partial_{x_i x_i} f$ for any smooth function f, which is a standard approximation (which is called a *vanishing* *viscosity approximation*) of $(\mathrm{D})_\delta$. Due to the appearance of viscosity term $\eta^2\Delta v^{\delta,\eta}$, $(\mathrm{A})_\delta^\eta$ has a (unique) smooth solution $v^{\delta,\eta}$. The following result on the rate of convergence of $v^{\delta,\eta}$ to v^δ as $\eta \to 0$ is standard, and we omit the proof here. See [11, Theorem 4.4] and [11, Proposition 5.5] for instance.

Lemma 4.1. *Assume* (A4). *Then there exists a constant $C > 0$ independent of δ and η so that*

$$\|Dv^{\delta,\eta}\|_{L^\infty(\mathbb{T}^n)} \le C, \quad \|v^{\delta,\eta} - v^\delta\|_{L^\infty(\mathbb{T}^n)} \le \frac{C\eta}{\delta}.$$

It is time to use the nonlinear adjoint method to construct the set $\mathcal{M} \subset \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ in Theorem 4.1, where we denote by $\mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ the set of Radon probability measures on $\mathbb{T}^n \times \mathbb{R}^n$. For $x_0 \in \mathbb{T}^n$ fixed, we consider an adjoint equation of the linearized operator of $(\mathrm{A})_\delta^\eta$:

$$(\mathrm{AJ})_\delta^\eta \qquad \delta\theta^{\delta,\eta} - \mathrm{div}(D_pH(x, Dv^{\delta,\eta})\theta^{\delta,\eta}) = \eta^2\Delta\theta^{\delta,\eta} + \delta\delta_{x_0} \qquad \text{in } \mathbb{T}^n,$$

where δ_{x_0} denotes the Dirac delta measure at x_0. Note here that since the equation $(\mathrm{AJ})_\delta^\eta$ is a uniformly elliptic equation, by approximating δ_{x_0} by a smooth function and considering the limit, we can construct a smooth solution $\theta^{\delta,\eta} \in C^\infty(\mathbb{T}^n \setminus \{x_0\})$. By the maximum principle and integrating $(\mathrm{AJ})_\delta^\eta$ on $\mathbb{T}^n$, we obtain

$$\theta^{\delta,\eta} > 0 \text{ in } \mathbb{T}^n \setminus \{x_0\}, \quad \text{and} \quad \int_{\mathbb{T}^n} \theta^{\delta,\eta}(x)\,dx = 1. \tag{4.4}$$

Define the linear functional $\mathcal{L} : C_c(\mathbb{T}^n \times \mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{L}\psi := \int_{\mathbb{T}^n} \psi(x, Dv^{\delta,\eta})\theta^{\delta,\eta}(x)\,dx.$$

Due to (4.4), $\mathcal{L}$ is bounded. Thus, in light of the Riesz theorem (see [17, Theorem 6.19] for instance), for every $\delta, \eta > 0$, there exists a probability measure $\nu^{\delta,\eta} \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{T}^n} \psi(x, Dv^{\delta,\eta})\theta^{\delta,\eta}(x)\,dx = \iint_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\,d\nu^{\delta,\eta}(x,p) \tag{4.5}$$

for all $\psi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$. It is clear that $\mathrm{supp}(\nu^{\delta,\eta}) \subset \mathbb{T}^n \times \overline{B}(0, C)$ for some $C > 0$ due to Lemma 4.2. Since

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\,d\nu^{\delta,\eta}(x,p) = 1 \quad \text{for all } \delta > 0, \eta > 0,$$

due to the compactness of weak convergence of measures, there exist two subsequences $\eta_k \to 0$ and $\delta_j \to 0$ as $k \to \infty$, $j \to \infty$, respectively, and probability measures $\nu^{\delta_j}, \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ (see [3, Theorem 4] for instance) so that

$$\begin{aligned} &\nu^{\delta_j,\eta_k} \rightharpoonup \nu^{\delta_j} && \text{as } k \to \infty,\\ &\nu^{\delta_j} \rightharpoonup \nu && \text{as } j \to \infty, \end{aligned} \tag{4.6}$$

in the sense of measures. We also have that $\mathrm{supp}(\nu^{\delta_j}), \mathrm{supp}(\nu) \subset \mathbb{T}^n \times \overline{B}(0, C)$. For each such ν, set $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ so that the pushforward measure of μ associated with $\Phi(x, q) = (x, D_qL(x, q))$ is ν, that is, for all $\psi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$,

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x, D_qL(x,q))\,d\mu(x,q). \tag{4.7}$$

Here, the function $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Legendre transform of H, i.e.,

$$L(x,q) := \sup_{p\in\mathbb{R}^n}(p\cdot q - H(x,p)) \quad \text{for all } (x,q) \in \mathbb{T}^n \times \mathbb{R}^n.$$

By (A4), L is well-defined, that is, $L(x, q)$ is finite for each $(x, q) \in \mathbb{T}^n \times \mathbb{R}^n$. Furthermore, L is of class C^1, convex with respect to q, and superlinear.

Notice that the measure μ constructed by the above process depends on the choice of $x_0, \{\eta_k\}_k, \{\delta_j\}_j$, and when needed, we write $\mu = \mu(x_0, \{\eta_k\}_k, \{\delta_j\}_j)$ to demonstrate the clear dependence. In general, there could be many such limit μ for different choices of x_0, $\{\eta_k\}_k$ or $\{\delta_j\}_j$. We define the set $\mathcal{M} \subset \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ by

$$\mathcal{M} := \bigcup_{x_0\in\mathbb{T}^n,\{\eta_k\}_k,\{\delta_j\}_j} \{\mu(x_0, \{\eta_k\}_k, \{\delta_j\}_j)\}.$$

The following simple proposition records important properties of ν and μ.

Proposition 4.2. *Assume that* (A4) *holds. Let ν and μ be probability measures given by* (4.6) *and* (4.7). *Then,*

(i) $\displaystyle\iint_{\mathbb{T}^n\times\mathbb{R}^n} (D_pH(x,p)\cdot p - H(x,p))\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,q)\,d\mu(x,q) = 0,$

(ii) $\displaystyle\iint_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x,p)\cdot D\varphi\,d\nu(x,p) = \iint_{\mathbb{T}^n\times\mathbb{R}^n} q\cdot D\varphi\,d\mu(x,q) = 0$ *for any* $\varphi \in C^1(\mathbb{T}^n)$.

Proof. Equation $(\mathrm{A})^{\eta}_{\delta}$ can be rewritten as

$$\delta v^{\delta,\eta} + D_pH(x, Dv^{\delta,\eta}) \cdot Dv^{\delta,\eta} - \eta^2 \Delta v^{\delta,\eta} = D_pH(x, Dv^{\delta,\eta}) \cdot Dv^{\delta,\eta} - H(x, Dv^{\delta,\eta}).$$

Multiply this by $\theta^{\delta,\eta}$ and integrate on $\mathbb{T}^n$ to get

$$\begin{aligned}
&\int_{\mathbb{T}^n} (\delta v^{\delta,\eta} + D_pH(x, Dv^{\delta,\eta}) \cdot Dv^{\delta,\eta} - \eta^2 \Delta v^{\delta,\eta})\theta^{\delta,\eta}\, dx \\
&\quad = \int_{\mathbb{T}^n} (\delta\theta^{\delta,\eta} - \mathrm{div}(D_pH(x, Dv^{\delta,\eta})\theta^{\delta,\eta}) - \eta^2 \Delta\theta^{\delta,\eta}) v^{\delta,\eta}\, dx \\
&\quad = \int_{\mathbb{T}^n} \delta\delta_{x_0} v^{\delta,\eta}\, dx = \delta v^{\delta,\eta}(x_0).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{T}^n} (D_pH(x, Dv^{\delta,\eta}) \cdot Dv^{\delta,\eta} - H(x, Dv^{\delta,\eta}))\theta^{\delta,\eta}\, dx \\
&\quad = \iint_{\mathbb{T}^n\times\mathbb{R}^n} (D_pH(x,p) \cdot p - H(x,p))\, d\nu^{\delta,\eta}(x,p).
\end{aligned}$$

Set $\eta = \eta_k$, $\delta = \delta_j$, and let $k \to \infty$, $j \to \infty$ in this order to yield

$$\begin{aligned}
0 &= \iint_{\mathbb{T}^n\times\mathbb{R}^n} (D_pH(x,p) \cdot p - H(x,p))\, d\nu(x,p) \\
&= \iint_{\mathbb{T}^n\times\mathbb{R}^n} (D_pH(x, D_qL(x,q)) \cdot D_qL(x,q) - H(x, D_qL(x,q)))\, d\mu(x,q) \\
&= \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,q)\, d\mu(x,q),
\end{aligned}$$

by (4.7) and the duality of convex functions. Note in the above computation that we have $\lim_{j\to\infty} \delta_j v^{\delta_j}(x_0) = 0$ because v^{δ} is the viscosity solution to $(\mathrm{D})_{\delta}$.

We now proceed to prove the second part. Fix $\varphi \in C^2(\mathbb{T}^n)$. Multiply $(\mathrm{AJ})^{\eta}_{\delta}$ by φ and integrate on $\mathbb{T}^n$ to get

$$\int_{\mathbb{T}^n} (D_pH(x, Du^{\delta,\eta}) \cdot D\varphi)\theta^{\delta,\eta}\, dx = \eta^2 \int_{\mathbb{T}^n} \Delta\varphi\theta^{\delta,\eta}\, dx + \delta\varphi(x_0) - \delta \int_{\mathbb{T}^n} \varphi\theta^{\delta,\eta}\, dx.$$

We use (4.5) for $\delta = \delta_j$, $\eta = \eta_k$, and let $k \to \infty$ to obtain

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x,p) \cdot D\varphi\, d\nu^{\delta_j}(x,p) = \delta_j\varphi(x_0) - \delta_j \iint_{\mathbb{T}^n\times\mathbb{R}^n} \varphi(x)\, d\nu^{\delta_j}(x,p).$$

Finally, let $j \to \infty$ to complete the proof. □

4.2 Mather measures

We are concerned with the following minimization problem

$$\min_{\mu\in\mathcal{F}} \iint_{\mathbb{T}^n\times\mathbb{R}^n} L(x,q)\, d\mu(x,q), \tag{4.8}$$

where

$$\mathcal{F} := \left\{ \mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n) : \iint_{\mathbb{T}^n\times\mathbb{R}^n} q \cdot D\phi\, d\mu(x,q) = 0 \quad \text{for all } \phi \in C^1(\mathbb{T}^n) \right\}.$$

Measures belonging to $\mathcal{F}$ are called *holonomic measures* or *closing measures* associated with (4.3). By (ii) of Proposition 4.3, $\mathcal{M} \subset \mathcal{F}$.

Definition 2. *We let $\widetilde{\mathcal{M}}$ to be the set of all minimizers of (4.8). Each measure in $\widetilde{\mathcal{M}}$ is called a stochastic Mather measure.*

It is worth mentioning that the holonomic condition is equivalent to the invariance condition under the *Euler–Lagrange flow*

$$\frac{d}{ds} D_qL(\gamma(s), \dot\gamma(s)) = D_xL(\gamma(s), \dot\gamma(s)).$$

This idea was first discovered by Mañé [13], who relaxed the original idea of Mather [14]. Minimizers of the minimizing problem (4.8) are precisely Mather measures for first-order Hamilton–Jacobi equations.

Proposition 4.3. *Fix $\mu \in \mathcal{M}$. Then μ is a minimizer of (4.8).*

This proposition clearly implies that $\mathcal{M} \subset \widetilde{\mathcal{M}}$.

Lemma 4.4. *Assume that* (A4) *holds. We have*

$$\iint_{\mathbb{T}^n \times \mathbb{R}^n} L(x,q)\, d\mu(x,q) \ge 0 \quad \textit{for all } \mu \in \mathcal{F}. \tag{4.9}$$

Furthermore,

$$\min_{\mu \in \mathcal{F}} \iint_{\mathbb{T}^n \times \mathbb{R}^n} L(x,q)\, d\mu(x,q) = 0.$$

Proof of Lemma 4.5 and Proposition 4.4. Let w be a solution of cell problem (4.3). Since a solution w of (4.3) is not smooth in general, in order to use the holonomic condition in $\mathcal{F}$, we need to find a family of smooth approximations of w, which are approximate subsolutions to (4.3). A natural way to perform this task is to use the usual convolution technique. More precisely, for each $\eta > 0$, let

$$w^\eta(x) := \gamma^\eta * w(x) = \int_{\mathbb{R}^n} \gamma^\eta(y) w(x+y)\, dy, \tag{4.10}$$

where $\gamma^\eta(y) = \eta^{-n}\gamma(\eta^{-1}y)$ (here $\gamma \in C_c^\infty(\mathbb{R}^n)$ is a standard symmetric mollifier such that $\gamma \ge 0$, $\operatorname{supp}\gamma \subset \overline{B}(0,1)$ and $\|\gamma\|_{L^1(\mathbb{R}^n)} = 1$).

By the Jensen inequality,

$$H(x, Dw^\eta) \le C\eta \quad \text{in } \mathbb{T}^n.$$

For any $\mu \in \mathcal{F}$, one has

$$\begin{aligned} C\eta &\ge \iint_{\mathbb{T}^n \times \mathbb{R}^n} H(x, Dw^\eta)\, d\mu(x,q) \ge \iint_{\mathbb{T}^n \times \mathbb{R}^n} (-L(x,q) + q \cdot Dw^\eta)\, d\mu(x,q) \\ &= -\iint_{\mathbb{T}^n \times \mathbb{R}^n} L(x,q)\, d\mu(x,q), \end{aligned}$$

where we use the admissible condition of $\mu \in \mathcal{F}$ to go from the second line to the last line. Let $\eta \to 0$ to deduce that

$$\iint_{\mathbb{T}^n \times \mathbb{R}^n} L(x,q)\, d\mu(x,q) \ge 0.$$

Thus, item (i) in Proposition 4.3 confirms that any measure $\mu \in \mathcal{M}$ minimizes the action (4.8). This is equivalent to the fact that $\mathcal{M} \subset \widetilde{\mathcal{M}}$. □

4.3 Key estimates

In this section, we give two important estimates.

Lemma 4.5. *Assume that* (A4) *holds. Let $w \in C(\mathbb{T}^n)$ be any viscosity solution of (4.3). For $\delta, \eta > 0$, let $v^{\delta,\eta}$, w^η and $\theta^{\delta,\eta}$ be, respectively, the solution to* $(\mathrm{A})_\delta^\eta$*, the function given by (4.10) and the solution to* $(\mathrm{AJ})_\delta^\eta$ *for some $x_0 \in \mathbb{T}^n$. Then,*

$$v^{\delta,\eta}(x_0) \ge w^\eta(x_0) - \int_{\mathbb{T}^n} w^\eta \theta^{\delta,\eta}\, dx - \frac{C\eta}{\delta}. \tag{4.11}$$

Proof. We first calculate, for every $x \in \mathbb{T}^n$,

$$\begin{aligned} |\Delta w^\eta(x)| &\le \int_{\mathbb{R}^n} |D\gamma^\eta(y) \cdot Dw(x+y)|\, dy \\ &\le \frac{C}{\eta^{n+1}} \int_{\mathbb{R}^n} |D\gamma(\eta^{-1}y)|\, dy = \frac{C}{\eta} \int_{\mathbb{R}^n} |D\gamma(z)|\, dz \le \frac{C}{\eta}, \end{aligned}$$

which immediately implies $\eta^2|\Delta w^\eta| \le C\eta$. Thus,

$$H(x, Dw^\eta) \le \eta^2 \Delta w^\eta + C\eta \qquad \text{in } \mathbb{T}^n.$$

Subtract $(\mathrm{A})_\delta^\eta$ from the above inequality to yield

$$\begin{aligned} &\delta w^\eta + C\eta \\ &\quad \ge \delta(w^\eta - v^{\delta,\eta}) + H(x, Dw^\eta) - H(x, Dv^{\delta,\eta}) - \eta^2 \Delta(w^\eta - v^{\delta,\eta}) \\ &\quad \ge \delta(w^\eta - v^{\delta,\eta}) + D_pH(x, Dv^{\delta,\eta}) \cdot D(w^\eta - v^{\delta,\eta}) - \eta^2 \Delta(w^\eta - v^{\delta,\eta}), \end{aligned}$$

where we use the convexity of H in the last inequality.

Then, multiplying this by $\theta^{\delta,\eta}$, integrating on $\mathbb{T}^n$, and using the integration by parts, we get

$$
\begin{aligned}
&\int_{\mathbb{T}^n} (\delta w^\eta + C\eta)\theta^{\delta,\eta}\,dx \\
&\geq \delta \int_{\mathbb{T}^n} (w^\eta - v^{\delta,\eta})\theta^{\delta,\eta}\,dx + \int_{\mathbb{T}^n} (D_pH(x,Dv^{\delta,\eta}) \cdot D(w^\eta - v^{\delta,\eta}) - \eta^2\Delta(w^\eta - v^{\delta,\eta}))\theta^{\delta,\eta}\,dx \\
&= \delta \int_{\mathbb{T}^n} (w^\eta - v^{\delta,\eta})\theta^{\delta,\eta}\,dx - \int_{\mathbb{T}^n} (\mathrm{div}(D_pH(x,Dv^{\delta,\eta})\theta^{\delta,\eta}) + \eta^2\Delta\theta^{\delta,\eta})(w^\eta - v^{\delta,\eta})\,dx \\
&= \delta \int_{\mathbb{T}^n} (w^\eta - v^{\delta,\eta})\theta^{\delta,\eta}\,dx - \int_{\mathbb{T}^n} (\delta\theta^{\delta,\eta} - \delta\delta_{x_0})(w^\eta - v^{\delta,\eta})\,dx \\
&= \delta(w^\eta - v^{\delta,\eta})(x_0),
\end{aligned}
$$

which implies (4.11) after a rearrangement. □

Proposition 4.6. *Assume that* (A4) *holds. Let v^δ be the viscosity solution of* $(\mathrm{D})_\delta$*, and $\mu \in \mathcal{M}$. Then, for any $\delta > 0$,*

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} v^\delta(x)\,d\mu(x,q) \leq 0.$$

Proof. Setting

$$\psi^\eta(x) := \int_{\mathbb{R}^n} \gamma^\eta(y)v^\delta(x+y)\,dy,$$

we have

$$\delta v^\delta + H(x, D\psi^\eta) \leq C\eta.$$

For any $q \in \mathbb{R}^n$, we use the convexity of H that $H(x, D\psi^\eta(x)) + L(x,q) \geq q \cdot D\psi^\eta(x)$ to obtain

$$\delta v^\delta + q \cdot D\psi^\eta - L(x,q) \leq C\eta.$$

Thus, in light of properties (i), (ii) in Proposition 4.3 of μ, we integrate the above inequality with respect to $d\mu(x,q)$ on $\mathbb{T}^n \times \mathbb{R}^n$ to imply

$$\iint_{\mathbb{T}^n\times\mathbb{R}^n} \delta v^\delta\,d\mu(x,q) \leq C\eta.$$

Let $\eta \to 0$ to complete the proof. □

We remark that the key idea of Proposition 4.7 was first observed in [9, Corollary 4].

4.4 Proof of Theorem 4.1

Theorem 4.1 is a straightforward consequence of the following two propositions.

Proposition 4.7. *Assume that* (A4) *holds. Let v^δ be the viscosity solution to* $(\mathrm{D})_\delta$*. Then,*

$$\liminf_{\delta\to 0} v^\delta(x) \geq v^0(x),$$

where v^0 is the function defined in Theorem 4.1.

Proof. Let $\phi \in \mathcal{E}$, that is, ϕ is a solution of (4.3) satisfying (4.2). Let $\phi^\eta = \gamma^\eta * \phi$ for $\eta > 0$.

Fix $x_0 \in \mathbb{T}^n$. Take two subsequences $\eta_k \to 0$ and $\delta_j \to 0$ so that (4.6) holds, and $\lim_{j\to\infty} v^{\delta_j}(x_0) = \liminf_{\delta\to 0} v^\delta(x_0)$. Let μ be the corresponding measure satisfying $\nu = \Phi_\#\mu$. In view of Lemma 4.6,

$$v^{\delta_j,\eta_k}(x_0) \geq \phi^{\eta_k}(x_0) - \int_{\mathbb{T}^n} \phi^{\eta_k}\theta^{\delta_j,\eta_k}\,dx - \frac{C\eta_k}{\delta_j}.$$

Let $k \to \infty$ to imply

$$v^{\delta_j}(x_0) \geq \phi(x_0) - \iint_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x)d\nu^{\delta_j}(x,p).$$

Let $j \to \infty$ in the above inequality to deduce further that

$$
\begin{aligned}
\liminf_{\delta\to 0} v^\delta(x_0) = \lim_{j\to\infty} v^{\delta_j}(x_0) &\geq \phi(x_0) - \iint_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x)d\nu(x,p) \\
&= \phi(x_0) - \iint_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x)d\mu(x,q) \geq \phi(x_0),
\end{aligned}
$$

which implies the conclusion. □

Proposition 4.8. *Assume that* (A4) *holds. Let $\{\delta_j\}_{j\in\mathbb{N}}$ be any subsequence converging to 0 such that v^{δ_j} uniformly* *converges to a solution v of (4.3) as $j \to \infty$. Then the limit v belongs to $\mathcal{E}$. In particular,*

$$\limsup_{\delta \to 0} v^{\delta}(x) \le v^0(x),$$

where v^0 is the function defined in Theorem 4.1.

Proof. In view of Proposition 4.7, it is clear that any uniform limit along subsequences belongs to $\mathcal{E}$. By the definition of the function v^0, it is also obvious that $\lim_{j\to\infty} v^{\delta_j}(x) \le v^0(x)$. □

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