氏名	CAVALLINA LORENZO (カヴァッリーナ ロレンツォ)
学位の種類	博　士（情報科学）
学位記番号	情　博　第　668　号
学位授与年月日	平成３０年　９月２５日
学位授与の要件	学位規則第４条第１項該当
研究科、専攻	東北大学大学院情報科学研究科（博士課程）システム情報科学専攻
学位論文題目	Analysis of two-phase shape optimization problem by means of shape derivatives (形状微分を用いた二相最適化問題の解析)
論文審査委員	（主査）東北大学教　授　坂口　茂 東北大学教　授　尾畑　伸明　　東北大学教授　瀬野　裕美 東北大学准教授　舩野　敬

論文内容の要旨

第１章　Introduction and main results

We consider the abstract mathematical model of a (two-phase) composite medium containing a core made of a different material. The aim of this work is twofold.

First, we study the so-called torsional rigidity of this medium. The torsional rigidity (from now on we will refer to it as E) of a composite medium is defined as the integral of some function u (the so-called *stress function*) defined as the solution of some boundary value problem. Our aim is to investigate how rotational symmetry is related to optimality (in the precise sense of "maximizing the torsional rigidity functional under volume constraints"). The one-phase analogue of this problem (i.e. the case of a homogeneous medium with no core) was first studied by Pólya in 1948: he proved that ball maximizes the torsional rigidity among all domains of a fixed volume. Indeed, such a result does not hold true when a core is present. We will use shape derivatives (see Chapter 3) to show that the radial configuration given by two concentric balls is a stationary point for the torsional rigidity functional. Moreover, such radially symmetric configuration can either be a local maximum or a saddle point depending on the relative hardness of the materials used (see Chapter 4 for the details).

The second original result of this PhD Thesis (see Chapter 5) concerns overdetermined elliptic problems and the resulting symmetry of its solutions. In 1971 Serrin proved that the ball is the only domain for which the solution of some elliptic boundary value problem (involving the Laplace operator) simultaneously satisfies both Dirichlet and Neumann constant boundary condition. Our result states that, when the Laplace operator is replaced by a two-phase operator (i.e. an operator in divergence form, but with only piecewise constant coefficients), then resulting overdetermined problem admits infinitely many non radially symmetric solutions.

第２章　Classical results in the one-phase setting

This chapter introduces the original proofs of Pólya and Serrin for the one-phase torsion problem and the classical (one-phase) Serrin overdetermined problem. We recall that these proofs are based on spherical rearrangement inequalities and the moving plane method respectively, which fully exploit the invariance properties of the Laplace operator, and hence cannot be easily adapted to the case of two-phase differential operators of divergence form.

第３章　Shape derivatives

This chapter introduces the fundamental tool used in this PhD Thesis, namely shape derivatives. They can be thought as an analogue of classical derivatives for shape functionals i.e. functionals that map domains in the Euclidean space (or tuples of them) to the set of real numbers (or more generally, any Banach space).
Although far from a complete introduction of the subject, this chapter aims to lay the mathematical foundation to deal with functionals like *torsional rigidity* (defined in the introduction) that depend on shapes indirectly, by means of the solutions to some partial differential equations.

The first step consists in giving a formula (called *Hadamard's formula*) for differentiating simple shape functionals in integral form. We actually give to versions for this formula: one for volume (N dimensional) integrals and one for surface (N-1) dimensional one.

Another deep result introduced in this chapter is the so-called *structure theorem*, which roughly states that the first order shape derivative of **any** shape functional is "concentrated at the boundary". More specifically we say that it just depend on the normal component of the perturbation field on the boundary of the domain in question. A similar (but more involved) result holds true for second order shape derivatives as well.

Finally we present some ways of computing the shape derivative of particular Banach space valued shape functionals, namely those that associate to any domain the solution u of some boundary value problem defined thereon (such a function u is usually referred to as *state function* in the literature). Under mild regularity assumptions, the structure theorem holds true and we are able to characterize the shape derivative of u as the solution of some other boundary value problem.

第４章　Two phase torsional rigidity

The computations here are quite straightforward, although tedious at some time. As a first step we define the class of perturbations that we will be working with, i.e. sufficiently smooth perturbations that fix the barycenter of our composite medium and do not alter either its total volume, nor the ratio of the volumes of the two phases. By an application of the formulas of Chapter 3 we obtain first and second order volume and barycenter preserving conditions (which will be useful later on).

We recall that the torsional rigidity of our composite medium can be written as the integral of the stress function over it. Now, by combining the computation of the shape derivative of integral functionals (Section 3.2) and the derivative of the state function u (Section 3.4) we obtain the first order shape derivative of the torsional rigidity functional E. As predicted by the general theory, this shape derivative can be expressed as a boundary integral that does not depend on the shape derivative of the state function u. Finally, by the volume preserving condition at first order, we have that the shape derivative of E vanishes when the composite medium is radially symmetric.

The computation of the second order shape derivative of E at the radially symmetric configuration given by two concentric balls is radically more involved and is the core of this chapter. Expressing the second order derivative of E in terms of u and its shape derivative u' is essentially just given by a further application of the Hadamard formula. The laborious part consists in the accurate analysis of the terms containing u' and its gradient: these compu tations can be carried out explicitly by means of spherical harmonics (see Appendix B).

第５章　A two-phase overdetermined problem of Serrin type

In this chapter we show the existence of nontrivial (i.e. non radially symmetric) solutions to the two-phase overdetermined problem described in Chapter 1. This result is proven by means of a perturbation argument based on the implicit function theorem for Banach spaces applied in a neighborhood of a trivial solution. In the end we give a refined version of this result, taking into account volume or surface area constraints.

付録A　Elements of tangential calculus

In this Appendix we present the basic definitions of tangential differential operators (tangential gradient, tangential divergence and so on), which occur so often in calculations involving shape derivatives (see Chapter 3 for some applications). Essential properties and theorems (e.g. the tangential analogue of integration by parts) are also discussed.

付録B　Spherical harmonics

In the last Appendix of this work, we introduce an essential tool for the computations performed in Chapter 4: *spherical harmonics*. They can be defined as the restriction of uniform harmonic polynomials to the unit sphere, although the equivalent characterization as eigenfunctions of the Laplace-Beltrami operator turns out to be more useful in some calculations. We also provide a classical proof of the following fact: the family of (appropriately normalized) spherical harmonics forms an orthonormal basis of the vector space of square summable functions on the unit sphere.

論文審査結果の要旨

２種以上の物質からなる複合媒質上の偏微分方程式を伴う数理モデルは，複合材料を用いて工学的に最適な構造を設計する形状最適化の問題，医学や工学において既知の媒質に含まれる未知の介在物を探索する逆問題，複合媒質のミクロ構造とマクロ構造の関係を調べる均質化の問題等様々な実世界の問題に現れる。一方，単一媒質に比べて，複合媒質上の偏微分方程式の解析には，複合媒質のもつ物理量の不連続性のために新たな技術的困難さが伴う。本論文では，特に２種の物質からなる複合媒質に対するねじり弾性の形状最適化問題を扱い，球対称な形状の局所最適性が２種の物質の伝導率の大小に依存して異なることを示し(本論文の定理 II)，単一媒質の場合には球対称な形状が最適性を与えるという有名な 1948 年の Pólya の定理との決定的な違いを明らかにした。さらに，２種の物質からなる複合媒質上の楕円型優決定問題において球対称でない形状の存在を示し(本論文の定理 III)，単一媒質の場合には球対称な形状に限られるという有名な 1971 年の Serrin の定理との決定的な違いも明らかにした。本論文の目的はその複合媒質に関する成果を単一媒質の場合と対比して述べることにある。本論文は全５章と付録からなる。

第１章は序であり，定理 II と定理 III を含む主要定理が述べられている。

第２章は単一媒質の場合の Pólya の定理と Serrin の定理とそれらの証明を与える。前者の証明には関数の球面再配分の理論が紹介され，後者の証明には楕円型方程式に対する最大値原理に基づく平面移動法が紹介されている。

第３章では，定理 II の証明に用いる形状微分の理論を本論文の参考文献(Henrot-Pierre[HP] と Delfour-Zolésio[DZ])の助けを借りて，本論文の問題設定に対応させて詳細に紹介している。

第４章では第３章で準備した形状微分の理論と球面調和関数の理論を駆使して定理 II を証明する。

第５章ではバナッハ空間における陰関数定理と球面調和関数の理論を巧みに用いて定理 III を証明する。

付録では前章までの計算で用いられた超曲面上の接微分公式と球面調和関数の基本性質が述べられている。

以上，本論文は，単一媒質の場合の有名な Pólya の定理と Serrin の定理に対比させて，単一媒質と複合媒質の場合の決定的な違いを明らかにしたものである。その過程で，単一媒質の場合の証明法が複合媒質には適用できないことが明らかになり，主定理の証明には形状微分の理論，球面調和関数の理論およびバナッハ空間における陰関数定理が巧みに用いられている。本論文は，様々な実世界の問題に現れる複合媒質上の偏微分方程式を伴う数理モデルの研究における新しい視点や方向性を与えるものであり，システム情報科学の発展に寄与するところが少なくない。

よって，本論文は博士（情報科学）の学位論文として合格と認める。