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論 文 内 容 の 要 旨

In spectral graph theory, various matrices are defined from graphs and properties of graphs are analyzed by investigating their spectrum. Representative matrices determined by graphs are the adjacency matrix, the Laplace matrix, the signless Laplace matrix and so on.
Such as the probability transition matrix of simple random walks, there are also matrices from graphs to appear in other areas. Recently, quantum walks on graphs are actively studied and they are defined by providing unitary matrices from graphs, so unitary matrices to define quantum walks can be said they are also matrices determined by graphs. One of them, the positive support of the Grover transfer matrix, is known as a strong tool for distinguish strongly regular graphs. There used to be the conjecture that the positive support of the cubed Grover transfer matrix is a matrix to solve the graph isomorphism problem for the strongly regular graphs, but unfortunately, this seemed to be negatively resolved.

For example, the spectrum of the Laplace matrix is related to connectivity of graphs, the second smallest eigenvalue in particular is called the algebraic connectivity and it is well-studied. In addition, the Laplace matrix has applications to engineering sides and
it is used on the control of multi-agent systems, so it can be said that the recent spectral graph theory has not only mathematical tools for analyzing graphs but also engineering applications.

In the point of view of properties of graphs, analysis of the spectrum of the adjacency matrix has plenty of good results. For example, bipartiteness and regularity are characterized by the adjacency spectrum and theorems to estimate the chromatic number and the clique number are famous. Also, it is known that spectral characterizations of certain graphs themselves. Relatively easy ones are the complete graphs and the cycle graphs, but the line graph of the complete graph K_n except $n = 8$ and the line graph of the complete bipartite graph $K_{n,n}$ except $n = 4$ can also be characterized by their spectrum. On the graphs with strong regularities such as distance-regular graphs, it was recently proven that the Hamming graphs with certain parameters and the Ivanov--Ivanov--Faradjev graph and so on are characterized by their spectrum.

As above, the spectrum of the adjacency matrix can bring out various information of graphs and sometimes characterizes themselves, so it should be a very strong graph invariant. Then, can the spectrum of the adjacency matrix distinguish all graphs? Of course, the answer is no. It is not easy to find graphs with the same spectrum by touch, but there certainly exist pairs of graphs with the same spectrum but not isomorphic to each other. The most famous pair is $K_{1,4}$ and $C_4 ¥cup K_1$, whose spectra are both $¥{ 2^{(1)}, 0^{(3)}, -2^{(1)} ¥}$. On the other hand, it is known that a method to obtain cospcetral graphs from known ones. The most famous one is the one called Godsil--McKay switching. Roughly speaking, Godsil--McKay switching can be used when the vertex set of a graph has an equitable partition equipped with a good cell called Godsil--McKay cell. Not to mention, giving such partitions is not easy. Indeed, most switching partitions so far are the partition formed by a 4-subset and its complement. Considering it, we suggest using orbit partitions of automorphism groups of graphs as a general and new method to get switching partitions.

For the symplectic graph $Sp(2¥nu,2)$, Abiad and Haemers considered a special 4-subset S and the partition $¥{S, V(Sp(2¥nu,2)) ¥setminus S¥}$ and they obtained many strongly regular graphs with the same parameters as the symplectic graph by Godsil--McKay swithcing. In Chapter~3, we also aim to construct many strongly regular graphs with the same parameters as the symplectic graph by applying Godsil--McKay switching, but partitions of the vertex set we consider are orbit partitions of groups of automorphisms. We consider the following groups:

*The automorphism group that fixes the standard basis.

*The automorphism group that fixes a special 4-subset by Abiad and Haemers.

As a result, we obtain four families of strongly regular graphs with the same parameters as the symplectic graphs. Also, we see one of them is isomorphic to the one discovered by Abiad and Haemers. More precisely, we see the edges involved with switching are the same.

Additionally, on the symplectic graph, we can regard the set of common neighbors of some vertices as the solution set of a system of linear equations. From this point of view, we investigate the number of common neighbors of three vertices as an invariant for isomorphism. As a result, we prove that the graphs in the five families, which are the four switched ones and the original one, are certainly pairwise non-isomorphic.

Also, there are problems related to the switching whether we can obtain aimed graphs by Godsil--McKay switching. The representative result is the paper by Munemasa, and it is related to the twisted Grassmann graphs. The twisted Grassmann graphs are the ones introduced by Van Dam and Koolen and these graphs were studied by many researchers as the first family of non-vertex-transitive distance-regular graphs with unbounded diameter. These graphs were originally constructed by converting a part of lines of a point-line incidence structure, but recently, Munemasa proved that the twisted Grassmann graphs are actually obtained %from the ordinary Grassmann graphs by Godsil--McKay switching, too.

Similarly to the ordinary Grassmann graphs and the twisted Grassmann graphs, there are many pairs of distance-regular graphs that have the same intersection array but they are not isomorphic to each other. Can we obtain aimed distance-regular graphs by switching like the twisted Grassmann graphs? Answering this question is one of the goals in Chapter~4. We show that the Doob graphs can be obtained from the Hamming graphs by switching many times. We use compatibility with switching and the Cartesian product. Indeed, we can find a partition for switching on the graph after taking product and show the isomorphism between the graph taking product after switching and the switched graph after taking product. Actually, this compatibility holds for not only the Cartesian product but also many other graph products. To prove this, we consider unified graph products which are written as the sum of tensor products of the identity matrix and the adjacency matrices of the original graph and its complement. These products enable us to treat many graph products in a unified manner and we show that compatibility with switching holds on these products. Furthermore, this compatibility suggests the possibility that some other pair of distance-regular graphs that have the same intersection array can be mapped to each other by switching. If the dual polar graphs $B_d(q)$ and $C_d(q)$ can be done so, then we simultaneously see that $D_d(q)$ and ${¥rm Hem}_{d}(q)$ can also be mapped to each other by switching.

論文審査結果の要旨

グラフのスペクトル、すなわち固有値の性質がグラフを完全に特徴付けることはできないということも、スペクトルによって特徴付けられるグラフの例もよく知られている。同じスペクトルを持つが非同型なグラフの組を、コスペクトラルなグラフと言う。このようなグラフの一般的構成法として知られているのがゴドシル・マッカイのスイッチングと呼ばれる構成法であり、ある条件を満たす頂点集合の分割が存在すれば、辺の付け替えによってコスペクトラルなグラフが構成できるというものである。著者は、この構成法の特別な場合として、自己同型群の部分群による軌道分解として得られる頂点集合の分割の中から、ゴドシル・マッカイの仮定を満たす実例を見つけ出し、さらにそのようにして得られたコスペクトラルなグラフが非同型であることを示した。また、距離正則なコスペクトラルグラフとして知られるハミンググラフとドゥーブグラフについても、ゴドシル・マッカイのスイッチングを複数回適用することで一方が他方から得られることを示した。本論文はその成果をまとめたもので、本編４章からなる。

第１章では、これまでのコスペクトラルグラフの構成理論に関する研究を概観し、本論文で扱う強正則グラフ、特にシンプレクティックグラフについての先行研究について述べている。

第２章では、ゴドシル・マッカイのスイッチングの定義とその適用方法、特に自己同型群の軌道として頂点集合を分割した時に確認すべき条件を述べている。

第３章では、シンプレクティックグラフと呼ばれる強正則グラフについて、その構成法と自己同型群について詳しく述べ、自己同型群のいくつかの部分群について頂点集合の軌道分解がゴドシル・マッカイのスイッチングの条件を満たす頂点集合の分割を与えることを確認している。続いて、実際にゴドシル・マッカイのスイッチングを適用した後、得られたグラフが元のグラフと非同型となることを、３頂点の共通の近傍の個数の分布という、一般には計算が困難な不変量を利用して示している。

第４章では、グラフの積を行列を用いて統一的に扱う方法を述べ、その方法によりゴドシル・マッカイのスイッチングがグラフの積と整合性があることを示している。このことにより、ドゥーブグラフがそれとコスペクトラルであるハミンググラフから、ゴドシル・マッカイのスイッチングを複数回適用することで得られることを示している。

以上要するに本論文は、コスペクトラルグラフ構成問題を、具体例と積という一般的な状況の両方で適用可能であること、実際に非同型なグラフができていることを示すことでゴドシル・マッカイのスイッチングの有用性を示すものである。従来の理論的成果を応用する新しい方法を示すことによって新たな具体的な成果を導いたことから、情報基礎科学ならびに代数的組合せ論の発展に寄与するところが少なくない。

よって、本論文は博士（情報科学）の学位論文として合格と認める。