Graph structure characterized by periodic discrete-time quantum walks

A dissertation presented

by

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Abstract

A quantum walk is regarded as a quantum analog of a random walk and the dynamics is interpreted as a wave propagation on the underlying graph of which the time evolution is defined by a unitary operator. Thus, periodicity is one of the most significant characteristics of quantum walks which cannot be seen in random walks, and an important problem is to determine the classes of graphs which admit a periodic quantum walk. We mainly focus on the Grover walk, which is a particular type of quantum walks with applications to searching problems, graph-isomorphism problems and so forth. In this paper, we first characterize some classes of graphs on which a periodic Grover walk is exhibited, e.g., cycle graphs, path graphs, distance-regular graphs, and generalized Bethe trees. We next derive conditions of graph structure by means of spectral analysis which gives rise to a periodic Grover walk. By using those conditions, we construct classes of graphs which admit a periodic Grover walk. Finally, we introduce several graph-transformations, e.g., multiplex and subdivision graphs, and prove that they preserve the periodic behavior. Moreover, we consider the staggered walk on the generalized line graph induced from a Hoffman graph and study the relation between periodicity of the Grover walk and that of the staggered walk.

Contents

Chapter 1

Introduction

A quantum walk is interpreted as a quantum analogy of a random walk [23] and has been actively discussed in various fields such as quantum physics and computer sciences [47], [63] since Y. Aharonov *et al.* [2] introduced a quantum random walk as an antecedent of a quantum walk in 1993. The motivation was to detect a single atom in the excited state on some photons in a cavity as fast as possible. Slightly later, there has been an increasing interest in quantum walks on graphs in relation to searching problems [54], graph-isomorphism problems [15] and so forth. A quantum walk often shows a specific characteristic which is not observed in a random walk and the characteristic helps us to solve the problems. What we focus on in this paper is *periodicity* of quantum walks, which is one of the most important property of quantum walks [28], [41]. Especially, in order to shed light on the profound relation between quantum walks and graph structure we consider characterization of graphs which admit a periodic quantum walk. In this paper, we give not only an extended result of [28] but also a condition of structure to allow periodicity. Furthermore, we introduce some graph-transformations preserving the periodicity.

Unlike a random walk, a particle of a quantum walk has chirality and its motion is interpreted as a wave propagation on the underlying graphs [17]. More precisely, a particle has an amplitude expressed in terms of a Hilbert space associated to the underlying graph and the time evolution is given by a unitary operator. Then the quantum interference, that is, overlaps and cancellations of waves play an important role. Then the unitary time evolution ensures that the amplitude at each time always gives the probability distribution. There are two types of, quantum walks: discrete- and continuous-time quantum walks. Both types have been intensively studied with many applications. Indeed, one of the principal motivations to investigate quantum walks was to design quantum systems solving problems more efficiently than classical systems. As a strong evidence, Shor [59] provided a quantum algorithm to solve an integer factorization and a discrete logarithm problem, which gave an exponential speedup over classical algorithms. In addition, Grover [22] also provided a quantum search algorithm for an unordered database, which gives a quadratic speedup over classical search algorithms.

For discrete-time quantum walks, Meyer [49] proposed quantum cellular automata to realize a physical device for quantum computations. The cell automata gave foundation of discrete-time quantum walks on the one-dimensional lattice. Nayak and Vishwanath [52] formulated a discrete-time quantum walk called the *Hadamard walk* on the one-dimensional lattice by taking the Shorödinger's approach to obtain the asymptotic form of the probability distribution. Later, Ambainis *et al.* [4] analyzed quantum walks on the one-dimensional lattice not only by the Schrödinger's approach but also using path integrals. After that, Konno [35], [36] reformulated quantum walks on the one-dimensional lattice and obtained the weak limit theorem for quantum walks corresponding to the central limit theorem for a random walk. Moreover, it was shown that the proper time scaling for a quantum walk to obtain a reasonable weak limit is linear in construct to a classical random walk of which the proper time scale is the square root of time. It is one of the reasons why quantum algorithms are more efficient than classical ones. Another widely studied topic on quantum walk is *localization* which means that the particle stays at an initial position at a high probability. The localization is found in the one-dimensional lattice [37], the half line [40] and so forth. Furthermore, coexistence of the linear spreading and the localization of a quantum walk is also mentioned in [27].

While discrete-time quantum walks on the one-dimensional lattice were intensively studied, quantum walks on general graphs were in the limelight as an extension of the one-dimensional lattice by several researchers. D. Aharonov *et al.* [3] first gave an idea of quantum walks on the general graphs, where a quantum version of the mixing time, the filling time and the dispersion time are defined. In particular, a quadratically fast convergence of the mixing time for quantum walks on cycles are mentioned and the lower bound of the mixing time is obtained for general graphs. Szegedy [62] defined a quantum walk on bipartite graphs and formulated the quantum hitting time. The quantum hitting time is proved to be quadratically faster than the classical one. Furthermore, Childs *et al.* [13] and Kempe [33] proposed exponentially faster quantum hitting times. As is seen in those works, quantum walks on graphs suggest to realize more efficient systems to solve several problems such as searching problems. Indeed, searching algorithms to detect a target set on graphs as fast as possible by driving a quantum walk have been intensively analyzed. Aaronson *et al.* [1] and Shenvi *et al.* [58] proposed quantum search algorithms to find a marked item on a d-dimensional lattice and the hypercube. These algorithms are shown to provide a quadratical speed up. As an application of quantum searchings, Ambainis [6] formulated an algorithm to solve element distinctness more efficiently. Moreover, Magniez *et al.* [46] proposed a quantum algorithm to find triangles on a graph.

Continuous-time quantum walks are also well-studied fields, where the time evolution operator is defined by a Hamiltonian. Farhi [16] first proposed an idea of continuous-time quantum walks to solve a decision problem. It can be regarded as a quantum algorithm on a tree. A family of trees which provides an exponentially speedup to solve the problem than classical algorithms was found in the result. In particular, the *perfect state transfer* is one of the most interesting topics on continuous-time quantum walks. If there exist two vertices u, v such that the state of particle on u completely transfers to the another one v, such a phenomenon is called perfect state transfer. Bose [10] first proposed the perfect state transfer. Some classes of graphs which admit perfect state transfer were found in [8], [19], [34], and [61]. Furthermore, Godsil [19] found spectral restriction of graphs admitting perfect state transfer. However, it is still open what kind of graph structure is required to provide the perfect state transfer.

What we deal with in this paper is characterization of graphs in terms of specific characteristics of quantum walks, or in short, relation between graph structure and quantum walks. In this context it is worthwhile to discuss periodicity of discrete-time quantum walks which one of the most significant characteristics of quantum walks. The periodicity in this paper means that there exists an integer $k \in \mathbb{N}$ such that the k-th power of the time evolution operator becomes the identity operator. Thus, for a periodic quantum walk, the behavior of the particle becomes periodic. Recently, the periodicity of discrete-time quantum walks has been studied in [28], [41], [43], [66] and [67]. The periodicity is an interesting property of quantum walks but it is very flail. The periodicity often disappears even if we give a small change to the graph. Therefore, we are interested in what kind of graph structure gives or preserves the periodicity. In addition, research on quantum walks pays attention to analyze properties of quantum walks for a fixed graph. On the contrary, we aim to characterize graph structure given by a property of quantum walks. This standpoint is regarded as an inverse problem from the viewpoint of quantum walks.

We mainly treat the periodicity of the *Grover walk*, which is a quantum walk derived from Grover's search algorithm. The Grover walk is one of the most intensively studied quantum walks on graphs [5], [64]. A graph-isomorphism problem is one of the most interesting applications of the Grover walks. It is used to distinguish two cospectral strongly regular graphs in terms of the adjacency matrix. The cube of the time evolution operator of the Grover walk is believed to be an invariance of these graphs [15], [20]. Ren *et al.* [57] gave a relation between the Ihara zeta function and the Grover walk. Furthermore, Konno and Sato [39] expressed the characteristic polynomial of the time evolution operator of Grover walk in terms of the second weighted zeta function. It is remarkable that the spectrum of the time evolution operator of the Grover walk is reduced to that of the transition operator of a simple random walk on the underlying graph [27], [30], [48]. Such a spectral mapping is useful for some problems of quantum walks such as searching problems. Structural quantities of a graph such as the degree or the diameter are often retrieved from a simple random walk. If we suppose the periodicity of the Grover walk, there must be restriction to a simple random walk on the graph and the restriction inherits to the graph structure. Thus, we may see graph structure restricted by the periodicity. This is the principal reason why we choose the Grover walk.

This paper is organized as follows: As preliminaries, we give definition of graphs, graph-transformations, and associated operators in Chapter 2. In addition, we introduce the Chebyshev polynomials and their properties, which are useful tools throughout this paper.

In Chapter 3, we define the Grover walk and its periodicity. Especially, the spectral expression of the time evolution operator appeared in Lemma 3.2.2 plays an important role in this paper. By spectral analysis, we give several classes of graphs on which a periodic Grover walk is exhibited, i.e., cycle graphs and path graphs, see Theorems 3.4.1 and 3.5.1, respectively. We next treat classes of graphs having an equitable partition, i.e., Hamming graphs, Johnson graphs, and generalized Bethe trees. The main results are stated in Theorems 3.6.3, 3.6.4, and 3.7.6. Indeed, these results give an extension of the ones in [28].

In Chapter 4, we derive a condition of a graph to admit a periodic Grover walk in Theorem 4.1.2, which is the most important statement in this paper. Thereby, we construct new classes of graphs by an operator called join on which a periodic Grover walk is exhibited. We join two cycles, several cycles, a cycle and a claw, and several path graphs and show that the Grover walks on these graphs are periodic in Theorems 4.2.1, 4.2.2, 4.2.3, and 4.2.4, respectively. Moreover, we find the shape of graphs admitting an odd-periodic Grover walk in Theorem 4.3.1.

In Chapter 5, we introduce, as graph-transformations, multiplex graphs, subdivision graphs and generalized line graphs induced by a Hoffman graph. We show that multiplex and subdivision graph preserve the periodicity of Grover walks in Theorems 5.1.1, and 5.2.1, respectively. Furthermore, we study another quantum walk called staggered walk on a generalized line graph. Combining the ideas of multiplex and subdivision graphs, we show that the staggered walk on the generalized line graph becomes periodic whenever the Grover walk on the original graph is periodic, see Theorem 5.3.2.

Finally, Chapter 6 summarizes the main results. In addition, we discuss some future directions. We extend the condition of periodicity and state some works derived from the extended conditions.

Chapter 2

Preliminaries

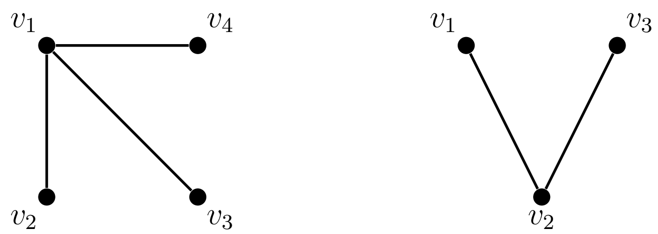
In this Chapter, we prepare basic notions and notations of graphs, and introduce several classes of graphs and graph-transformations. Moreover, we recall the Chebyshev polynomials and their useful properties.

2.1 Graphs

A graph $G = (V, E)$ is composed of a vertex set $V(G)$ and an edge set $E(G)$. The vertex set $V(G)$ is a countable set and an element of $V(G)$ is called a *vertex*. The edge set $E(G)$ is a 2-element subset of $V(G)$ and an element of $E(G)$ is called an *edge*. We say that two vertices $u, v \in V(G)$ are *adjacent* (denoted by $u \sim v$), if an unordered pair $\{u, v\}$ is a member of $E(G)$. An edge connecting u and v is denoted by uv and these vertices are called the *endpoints* of uv. Then

$$E(G) = \{uv \mid u, v \in V(G), u \sim v\}.$$

We may relax the above conditions to allow a multiple edge. In that case $E(G)$ is under-



$G = (V, E)$
$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4\}$

$H = (V, E)$
$V(H) = \{v_1, v_2, v_3\}$
$E(H) = \{v_1v_2, v_2v_3\}$

Figure 2.1: Examples of graphs

stood as a multiset. Let m_{uv} be the multiplicity of an edge uv. Then we express the edge set by

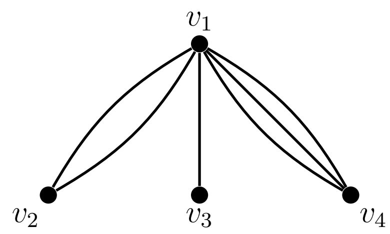
$$E(G) = \{uv(m_{uv}) \mid u \sim v\}.$$

A graph having a multiedge is called a *multigraph*. For a multigraph G, we define a simple graph called the *underlying graph* $\overline{G}$ by

$$V(\overline{G}) = V(G),$$

$$E(\overline{G}) = \{uv \mid uv \in E(G)\}.$$

In other words, the underlying graph is given by regarding a multiedge as a single edge.



$$G = (V, E)$$
$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2(2), v_1v_3(1), v_1v_4(3)\}$$

Figure 2.2: A multigraph

If a multigraph has no multiedge, it is called a *simple graph*. An edge $uv \in E(G)$ may be given two directions, i.e., from u to v and from v to u. Then an edge with a direction is called an *arc*. We denote an arc from u to v by (u, v). For a simple graph G we define the symmetric arc set by

$$\mathcal{A}(G) = \{(u, v), (v, u) \mid uv \in E(G)\}.$$

For a multigraph, the symmetric arc set is denoted by

$$\mathcal{A}(G) = \{(u, v; j), (v, u; j) \mid uv \in E(G), 1 \leq j \leq m_{uv}\},$$

where m_{uv} is the multiplicity of an edge uv.

Let $u \in V(G)$ and $e \in E(G)$. If u is one of the endpoints of e, we write $u \approx e$. If two edges e and f have a common endpoint, then we write $e \approx f$. Let G and H be graphs. If H satisfies that $V(H) \subset V(G)$ and $E(H) \subset E(G)$, H is a *subgraph* of G by definition. Let S be a subset of $V(G)$ and H a subgraph of G with $V(H) = S$. If each edge satisfying that both endpoints belong to S is in $E(H)$, H is called the *induced subgraph* spanned by S and is denoted by $G[S]$. For a vertex $u \in V(G)$, every vertex v with $v \sim u$ is called a *neighbor* of u and we define

$$N(u) := \{v \in V(G) \mid u \sim v\}.$$

The number of neighbors of u is called the *degree* of u and denoted by $\deg_G(u)$. If there is no danger of confusion, omitting the subscripts we write $\deg(u)$. A vertex of degree one is called a *leaf.* A sequence of distinct vertices $\{u_0, u_1 \ldots, u_k\} \subset V(G)$ satisfying $u_i \sim u_{i+1}$ for $0 \leq i \leq k-1$ is called a *path* from u_0 to u_k. If $u_0 \sim u_k$ in addition, the sequence $\{u_0, u_1 \ldots, u_k, u_0\}$ is called an *essential cycle.* We define the *distance* between $u, v \in V(G)$ to be the length of a shortest path from u to v and denote it by $d(u, v)$.

Furthermore, if $|V(G)| < \infty$, the graph G is called *finite.* If there exists a path between any pair of vertices, the graph G is called *connected.* Throughout this paper, we only deal with finite and connected graphs. In that case, we define the *diameter* of G by

$$\operatorname{diam}(G) := \max_{u,v \in V(G)} d(u, v).$$

We will introduce several fundamental classes of graphs in the next subsection. In particular, if the vertex set of G is decomposed into the disjoint union of two subsets such that any vertex in each subset is not adjacent each other, then G is called a *bipartite graph.* Thus, the lengths of all the essential cycles in G are even if and only if G is a bipartite graph. A graph G with unique essential cycle is called a *unicycle graph.* If the number of vertices of the unique essential cycle of a unicycle graph is odd (resp. even), then the graph is called an *odd-unicycle* graph (resp. *even-unicycle graph*). If the degrees of all the vertices of G are constant, then G is called a *regular graph.* If the degree of a regular graph is k, it is called a *k-regular graph.* Moreover, a graph G having no essential cycle is called a *tree.*

2.1.1 Several classes of graphs

Here, let us introduce several important classes of graphs.

- Cycle graphs.

  For $n \in \mathbb{N}$ with $n \geq 3$, the cycle graph on n vertices is denoted by C_n. For $V(C_n) = \{v_1, \ldots, v_n\}$, the edge set $E(C_n)$ is defined by

  $$E(C_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n\},$$

  where we tacitly understand that $v_{n+1} = v_1$.

- Path graphs.

  For $n \in \mathbb{N}$, the path graph on n vertices is denoted by P_n. For $V(P_n) = \{v_1, \ldots, v_n\}$, the edge set $E(P_n)$ is defined by

  $$E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}.$$

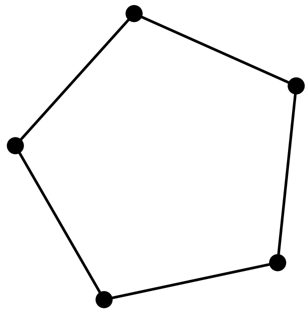


Figure 2.3: A cycle graph

Figure 2.4: A path graph

- Complete graphs.

  For $n \in \mathbb{N}$, the complete graph on n vertices is denoted by K_n. For $V(K_n) = \{v_1, \ldots, v_n\}$, the edge set $E(K_n)$ is defined by

  $$E(K_n) = \{v_i v_j \mid 1 \leq i < j \leq n\}.$$

  Then all the vertices are adjacent each other. Let G be a graph and S be a subset of $V(G)$. If $G[S]$ becomes a complete graph, S is called a *clique*.

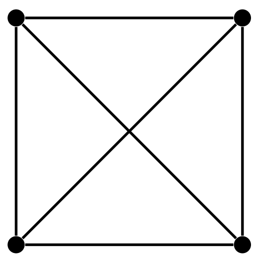


Figure 2.5: A complete graph

- Complete bipartite graphs.

  For r and $s \in \mathbb{N}$, the complete bipartite graph, denoted by $K_{r,s}$, is defined as a bipartite graph with two disjoint subsets A and B with $|A| = r, |B| = s$ such that

  $$E(K_{r,s}) = \{uv \mid u \in A, v \in B\}.$$

  Then any vertex in A is adjacent to every vertex in B. For $r \geq 2$, the graph $K_{1,r}$ is called a *claw*.

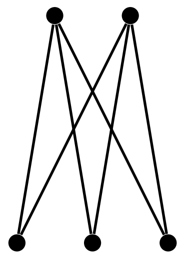


Figure 2.6: A complete bipartite graph

- Strongly regular graphs.

  For $n, k, \lambda, \mu \in \mathbb{N}$ a strongly regular graph $\mathrm{SRG}(n, k, \lambda, \mu)$ is defined as a k-regular graph on n vertices such that

$$|N(u) \cap N(v)| = \lambda$$

  if $u \sim v$ and

$$|N(u) \cap N(v)| = \mu$$

  otherwise.

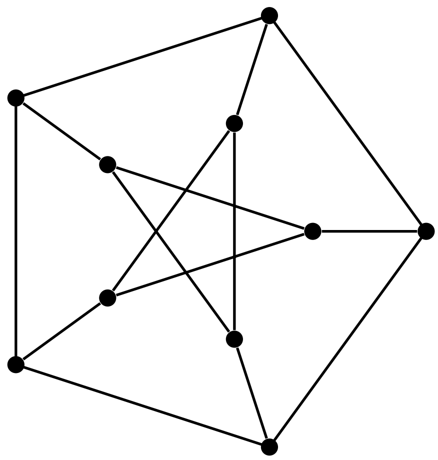


Figure 2.7: A strongly regular graph $\mathrm{SRG}(10, 3, 0, 1)$

2.1.2 Graph-transformations

Let $\mathcal{G}$ and $\mathcal{G}'$ be families of graphs. Then mappings from $\mathcal{G}$ to $\mathcal{G}'$ are called *Graph-transformations*. We introduce useful graph-transformations.

- Line graph.

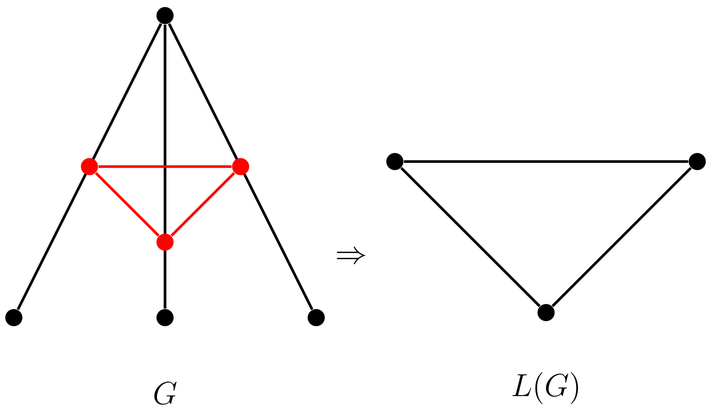


Figure 2.8: A line graph

For a simple graph G the *line graph* $L(G)$ is defined by

$$V(L(G)) := E(G),$$

$$E(L(G)) := \{e, f \in E(G), ef \mid e \approx f \text{ and } e \neq f\}.$$

In other words, each edge in G is regarded as a vertex of $L(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G has a common neighbor.

- Multiplex graph.

  For $n \in \mathbb{N}$, a multigraph given by replacing each edge of G by a multiedge with multiplicity n is called the *multiplex graph* $MP_n(G)$ in this paper. To be precise, $MP_n(G)$ is defined by

$$V(MP_n(G)) := V(G),$$

$$E(MP_n(G)) := \{e(n) \mid e \in E(G)\}.$$

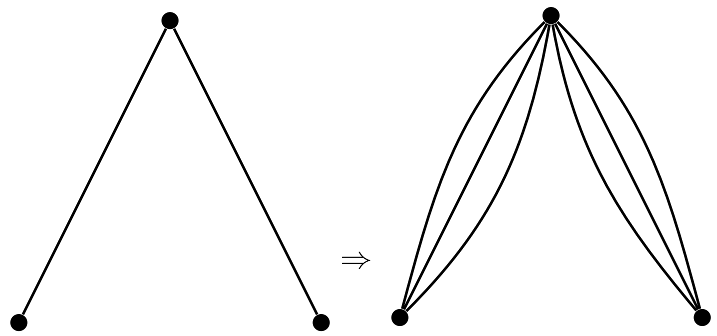
- Subdivision graph.

  Set $V(G) = \{v_i \mid 1 \leq i \leq n\}$. For $l \in \mathbb{N}$, the *l-subdivision graph* $S_l(G)$ is given by embedding the path graph with length l in each edge of G. For $e = v_i v_j \in E(G)$, let us denote the path graph embedded to e by $P^{(e)}$. We define

$$V(P^{(e)}) := \{w_{i,j}^{(r)} \mid 0 \leq r \leq l\},$$

and

$$E(P^{(e)}) := \{w_{i,j}^{(r-1)} w_{i,j}^{(r)} \mid 1 \leq r \leq l\},$$



G $\qquad$ $MP_3(G)$

Figure 2.9: A multiplex graph

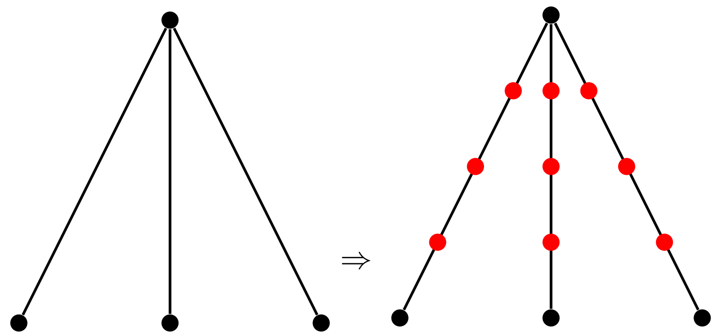
where we tacitly understand that $w_{i,j}^{(0)} = v_i, w_{i,j}^{(l)} = v_j$ and $w_{i,j}^{(r)} = w_{j,i}^{(l-r)}$ for $0 \le r \le l$. Then we have

$$V(S_l(G)) = V(G) \cup \bigsqcup_{e \in E(G)} V(P^{(e)})$$

and

$$E(S_l(G)) = \bigcup_{e \in E(G)} E(P^{(e)}).$$

In other words, $S_l(G)$ is given by adding $l-1$ vertices to each edge of G.



G $\qquad$ $S_4(G)$

Figure 2.10: A subdivision graph

2.1.3 Operators associated to graphs

For a countable set Λ, we define the Hilbert space of square summable functions on Λ by

$$\ell^2(\Lambda) := \{f : \Lambda \to \mathbb{C} \mid \|f\|^2 < \infty\},$$

where the inner product and the norm are given by

$$\langle f, g \rangle := \sum_{x \in \Lambda} \overline{f(x)} g(x)$$

and

$$\|f\| := \sqrt{\langle f, f \rangle},$$

respectively. The standard basis of $\ell^2(\Lambda)$ is given by $\{\delta_x \mid x \in \Lambda\}$, where $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ otherwise. Let us define the identity operator on $\ell^2(\Lambda)$ by I_Λ, that is, $I_\Lambda f = f$ for any $f \in \ell^2(\Lambda)$. For a finite Λ, let $\mathbf{j}_\Lambda$ and $\mathbf{0}_\Lambda$ be the all-one and zero function in $\ell^2(\Lambda)$, respectively. That is, $\mathbf{j}_\Lambda(x) = 1, \mathbf{0}_\Lambda(x) = 0$ for any $x \in \Lambda$. Throughout this paper, for an operator X on $\ell^2(\Lambda)$ we denote the set of eigenvalues of X by $\sigma(X)$.The *support* of $f \in \ell^2(\Lambda)$ is defined by

$$\mathrm{supp}(f) := \{x \in \Lambda \mid f(x) \neq 0\}.$$

For a simple graph $G = (V, E)$, we define the *adjacency operator* $A = A(G)$ as follows:

$$Af(u) = \sum_{v \sim u} f(v), \quad f \in \ell^2(V(G)).$$

Define the *transition operator* $T = T(G)$ by

$$Tf(u) = \frac{1}{\deg(u)} \sum_{v \sim u} f(v), \quad f \in \ell^2(V(G)),$$

and the *degree operator* $M = M(G)$ by

$$Mf(u) = \deg(u) f(u), \quad f \in \ell^2(V(G)).$$

Then it follows immediately that

$$T = M^{-1} A.$$

Then the representation matrices of A and T with respect to the standard basis $\{\delta_u \mid u \in V(G)\}$ are written as

$$A_{u,v} = \begin{cases} 1, & \text{if } u \sim v, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$T_{u,v} = \begin{cases} \frac{1}{\deg(u)}, & \text{if } u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

We sometimes consider the symmetric transition operator

$$\tilde{T} = M^{\frac{1}{2}} T M^{-\frac{1}{2}} = M^{-\frac{1}{2}} A M^{-\frac{1}{2}}$$

instead of T since their spectra are in coincidence. If G is a multigraph, the transition operator is defined as follows:

$$Tf(u) = \sum_{v \sim u} \frac{m_{uv}}{M_u} f(v), \quad f \in \ell^2(V(G)), \tag{2.1}$$

where m_{uv} is the multiplicity of an edge uv and $M_u = \sum_{v \sim u} m_{uv}$. The transition operator expresses a random walk on G. If G is simple, it expresses a simple random walk on G. In that case, it is easily shown that $\mathbf{j}_{V(G)}$ is an eigenfunction of T for 1, which is the maximum eigenvalue. Thus, $1 \in \sigma(T)$ and the absolute values of all the eigenvalues of T is at most one. It is known that $-1 \in \sigma(T)$ if G is a bipartite graph [12].

2.2 Chebyshev polynomials

Let $\{T_j\}_{j=0}^{\infty}$ and $\{U_j\}_{j=0}^{\infty}$ be sequences of polynomials satisfying

$$T_0(\lambda) = 1, T_1(\lambda) = \lambda,$$

$$U_0(\lambda) = 1, U_1(\lambda) = 2\lambda$$

and the same recurrence relation;

$$L_j(\lambda) = 2\lambda L_{j-1}(\lambda) - L_{j-2}(\lambda), j \geq 2.$$

Then $\{T_j(\lambda)\}$ and $\{U_j(\lambda)\}$ are called the *Chebyshev polynomials* of the first and of the second kind, respectively. For the Chebyshev polynomials of the second kind, it is convenient to set $U_{-1}(\lambda) = 0$. The coefficients of the term of the maximum degree of $T_j(\lambda)$ and $U_j(\lambda)$ become 2^{j-1} and 2^j, respectively. From the above relations, it follows that

$$T_j(\cos\theta) = \cos(j\theta),$$

$$U_j(\cos\theta) = \frac{\sin((j+1)\theta)}{\sin\theta},$$

for $j \in \mathbb{N}$ and $\theta \in \mathbb{R}$. We will provide some key properties of the Chebyshev polynomials.

Lemma 2.2.1. *If a complex sequence $\{a_i\}_{i=0}^{\infty}$ satisfies*

$$\lambda a_i = \frac{1}{2}(a_{i-1} + a_{i+1})$$

for $\lambda \in \mathbb{C}$, then we have

$$a_i = U_{i-1}(\lambda) a_1 - U_{i-2}(\lambda) a_0.$$

Proof. Straightforward by induction. □

Lemma 2.2.2. *Let $j \geq 0$ and $\lambda \in \mathbb{C}$.*

(i) $\lambda U_j(\lambda) - U_{j-1}(\lambda) = T_{j+1}(\lambda)$.

(ii) $\lambda T_j(\lambda) - T_{j-1}(\lambda) = (\lambda^2 - 1)U_{j-1}(\lambda)$.

(iii) $U_j^2(\lambda) - U_{j-1}(\lambda)U_{j+1}(\lambda) = 1$.

Proof. Straightforward by induction. □

Chapter 3

Periodicity of Grover walks

In this Chapter, we define the Grover walk and touch its *periodicity* which is our main topic in this paper. Preparing some useful tools to analyze the periodicity, we characterize several classes of graphs which admit periodic Grover walks.

3.1 Grover walks

Let G be a multigraph. Here, a particle moves around arcs of the underlying graph rather than vertices. In addition, the motion of the particle can be interpreted as a dynamics on the arcs. Now, we give a unitary operator on $\ell^2(\mathcal{A}(G))$ defined by

$$U\varphi(f) = \sum_{o(f)=t(e)} \left(\frac{2}{\deg(t(f))} - \delta_{e,f^{-1}} \right) \varphi(e), \quad \varphi \in \ell^2(\mathcal{A}(G)),$$

where $\delta_{e,f}$ is the Kronecker delta symbol. Then U is called the *Grover transfer operator* and the quantum walk defined by the above U is called the *Grover walk*. Let $\varphi_0 \in \ell^2(\mathcal{A}(G))$ be a function with

$$\sum_{e\in\mathcal{A}(G)} |\varphi_0(e)|^2 = 1.$$

Then we define a sequence $\{\varphi_t\}_{t=0}^{\infty}$ as

$$\varphi_t = U^t \varphi_0. \tag{3.1}$$

Due to the unitarity of U, we have

$$\sum_{e\in\mathcal{A}(G)} |\varphi_t(e)|^2 = 1$$

at each time t. Thus, we understand that the value $|\varphi_t(e)|^2$ is the finding probability of the particle on e at time t. Now, the equation (3.1) gives the dynamics of the Grover walk. Then the function φ_t and the value $\varphi_t(e)$ are called the *quantum state* and *amplitude*, respectively.

3.2 Periodic Grover walk

Let G be a multigraph and U the Grover transfer operator. If there exists a positive integer $k \in \mathbb{N}$ such that $U^k = I_{\mathcal{A}(G)}$, the Grover walk on G is *periodic*. In that case, the smallest $k \in \mathbb{N}$ such that $U^k = I_{\mathcal{A}(G)}$ is called the *period* of the Grover walk. For simplicity, we call a graph satisfying the above condition a *Grover-periodic graph*. Especially, if the period is specified as k, we call the graph a *Grover-k-periodic graph*.

Proposition 3.2.1. *A graph G is Grover-k-periodic if and only if $\lambda^k = 1$ for every $\lambda \in \sigma(U)$ and there exists $\lambda \in \sigma(U)$ such that $\lambda^j \neq 1$ for $0 < j < k$.*

Proof. Immediate by diagonalization of U. $\square$

Let U be the Grover transfer operator with $\sigma(U) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$. We set

$$\mathcal{N}_i := \min\{k \in \mathbb{N} \mid \lambda_i^k = 1, \quad 1 \leq i \leq d.\}$$

Then the period of the Grover walk is given by

$$\mathcal{N} = \mathrm{lcm}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_d),$$

where the right-side stands for the least common multiple of $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_d$.

Lemma 3.2.2 (Higuchi, Konno, Sato, Segawa [28])**.** *Let G be a multigraph. The spectrum of the Grover transfer operator U is given by*

$$\sigma(U) = \{e^{\pm i \cos^{-1}(\sigma(T(G)))}\} \cup \{1\}^{b_1(G)} \cup \{-1\}^{b_1(G)-1+\mathbf{1_B}(G)},$$

where $b_1(G) = |E(G)| - |V(G)| + 1$ and $\mathbf{1_B}(G) = 1$ if G is bipartite, $\mathbf{1_B}(G) = 0$ otherwise.

Throughout this paper, the branch of $\cos^{-1}$ is specified as $[0, \pi]$. If G is simple, $b_1(G)$ coincides with the number of the essential cycles in G, and called the *first Betti number*.

Proposition 3.2.3. *A graph G is Grover-periodic if and only if every eigenvalue of T is the real part of a root of unity, that is,*

$$\cos^{-1}(\sigma(T(G))) \subset \pi\mathbb{Q}.$$

Proof. By Proposition 3.2.1, every eigenvalue of U is the root of unity if G is Grover-periodic. Hence, it holds $\cos^{-1}(\sigma(T(G))) \subset \pi\mathbb{Q}$ by Lemma 3.2.2. $\square$

Let us define the *Joukowski transformation* $\mathcal{J} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ as follows: For $z \in \mathbb{C}$,

$$\mathcal{J}(z) := \frac{z + z^{-1}}{2}. \tag{3.2}$$

Lemma 3.2.4. *For $z \in \mathbb{C}$ with $|z| = 1$ and $j \geq 0$ the following relations hold:*

(i) $(2z)^j T_j(\mathcal{J}(z)) = 2^{j-1}(z^{2j} + 1)$.

(ii) $(2z)^j U_j(\mathcal{J}(z)) = 2^j \sum_{k=0}^{i} z^{2j-2k}$,

where T_j and U_j are the Chebyshev polynomials of the first and second kind, respectively.

Proof. Straightforward by direct calculation. □

Lemma 3.2.5 (Higuchi, Konno, Sato, Segawa [28])**.** *Let f be a monic and rational polynomial of degree n. Then the zeros of f are the real parts of a root of unity if and only if the polynomial $(2z)^n f(\mathcal{J}(z))$ is a product of some cyclotomic polynomials for $z \in \mathbb{C}$ with $|z| = 1$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the distinct zeros of f, that is,

$$f(x) = \prod_{k=1}^{d} (x - \lambda_k)^{m_k},$$

where m_k is the multiplicity of λ_k. Suppose that they are the real parts of a root of unity. Then we have

$$\begin{aligned}(2z)^n f(\mathcal{J}(z)) &= (2z)^n \prod_{k=1}^{d} \left(\frac{z + z^{-1}}{2} - \lambda_k \right)^{m_k} \\ &= \prod_{k=1}^{d} (z^2 - 2\lambda_k z + 1)^{m_k} \\ &= \prod_{k=1}^{d} \{(z - e^{i\theta_k})(z - e^{-i\theta_k})\}^{m_k}, \end{aligned} \tag{3.3}$$

where $\theta_k = \cos^{-1}(\lambda_k)$. Since $\theta_k \in \pi\mathbb{Q}$ and f is rational, the right-hand side of (3.3) is a product of cyclotomic polynomials.

□

3.3 Related works

Finite graphs that admit periodic quantum walks have been investigated. It is shown [41] that the Hadamard walk on the cycle C_n is periodic if and only if $n = 2, 4$ or 8, whose periods are $2, 8$, or 24, respectively. In [28], periodic Szegedy walks on finite graphs are analyzed. The Szegedy walk is induced from a random walk on the underlying graph and the Grover walk is a special case of the Szegedy walk induced from a simple random walk. In [28], Szegedy walks induced from not only a simple random walk but also from a lazy random walk are considered. The results are summarized as follows:

- Complete graphs.

  The Szegedy walk on a complete graph K_n with $n \geq 2$ induced from a lazy random walk with a laziness l is periodic if and only if $(n, l) = (2, 0), (3, 0), \left(n, \frac{1}{n}\right), \left(2, \frac{1}{4}\right)$ and $\left(n, \frac{n+1}{2n}\right)$, whose periods are $2, 3, 4, 6$ and 6, respectively.

- Complete bipartite graphs.

  The Szegedy walk on a complete bipartite graph $K_{r,s}$ with $r + s \geq 3$ induced from a lazy random walk with a laziness l is periodic if and only if $l = 0$ or $\frac{1}{2}$, whose periods are 4 or 12, respectively.

- Strongly regular graphs.

  The Szegedy walk on a strongly regular graph $\mathrm{SRG}(n, k, \lambda, \mu)$ induced from a simple random walk is periodic if and only if

  $$(n, k, \lambda, \mu) = (5, 2, 0, 1), (2k, k, 0, k), (3\lambda, 2\lambda, \lambda, 2\lambda),$$

  whose periods are $5, 4$, or 12, respectively. Those graphs are nothing but $C_5, K_{k,k}$, and $K_{\lambda,\lambda,\lambda}$, respectively.

- Cycle graphs.

  Here, we consider the cycle graph C_n ($n \geq 3$), and set $C_2 = P_2$. For $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ with $v_1 = v_{n+1}$, define the transition probability from v_i to v_{i+1} and from v_{i+1} to v_i for $1 \leq i \leq n$ by p and $1 - p$ with $p \neq \frac{1}{2}$, respectively. Then the Szegedy walk on a cycle graph C_n induced from the non-reversible random walk with the above transition probabilities is periodic if and only if

  (i) $n = 2$ and $p = \frac{2-\sqrt{3}}{4}, \frac{2-\sqrt{2}}{4}$ or $\frac{1}{4}$, whose periods are $6, 8$ or 12, respectively.

  (ii) $n = 4$ and $p = \frac{2-\sqrt{3}}{4}, \frac{2-\sqrt{2}}{4}$ or $\frac{1}{4}$, whose periods are $12, 8$ or 12, respectively.

  (iii) $n = 8$ and $p = \frac{2-\sqrt{2}}{4}$ whose period is 24.

3.4 Cycle graphs

Theorem 3.4.1. *The cycle graph C_n is Grover-periodic for any $n \geq 3$, and its period is n.*

Proof. Let $A = A(C_n)$ and $T = T(C_n)$ be the adjacency and transition operators, respectively. It is well-known (e.g., [12]) that

$$\sigma(A) = \left\{ 2\cos\frac{2\pi}{n} j \mid 0 \leq j \leq n-1 \right\}.$$

Since C_n is a 2-regular graph, it follows immediately that

$$T = \frac{1}{2}A$$

and

$$\sigma(T) = \left\{ \cos \frac{2\pi}{n} j \mid 0 \leq j \leq n-1 \right\}.$$

Thus, it follows that $\cos^{-1}(\sigma(T)) \subset \pi\mathbb{Q}$. By Proposition 3.2.3, we assert that the graph C_n is Grover-periodic and its period is n for $n \in \mathbb{N}$. $\square$

3.5 Path graphs

Theorem 3.5.1. *The path graph P_n is Grover-periodic for any $n \geq 2$, and its period is $2(n-1)$.*

Proof. Set $G = P_n$ and let $T = T(G)$ be the transition operator. Suppose that $\lambda f = Tf$ for $f \in \ell^2(V(G))$, that is,

$$\lambda f(u) = \frac{1}{\deg(u)} \sum_{v \sim u} f(v), \quad u \in V(G).$$

Then we have

$$\lambda f(v_1) = f(v_2), \tag{3.4}$$
$$\lambda f(v_n) = f(v_{n-1}), \tag{3.5}$$
$$\lambda f(v_i) = \frac{1}{2}(f(v_{i-1}) + f(v_{i+1})), \quad 2 \leq i \leq n-1 \tag{3.6}$$

From (3.6) we see that

$$f(v_{i+1}) = 2\lambda f(v_i) - f(v_{i-1}), \quad 2 \leq i \leq n-1.$$

In view of Lemma 2.2.1, we obtain

$$f(v_{i+1}) = U_{i-1}(\lambda) f(v_2) - U_{i-2}(\lambda) f(v_1),$$

where $\{U_i(\lambda)\}$ is the Chebyshev polynomial of the second kind. By (3.4) and (i) of Lemma 2.2.2, we have

$$\begin{aligned} f(v_{i+1}) &= \lambda U_{i-1}(\lambda) - U_{i-2}(\lambda) f(v_1) \\ &= T_i(\lambda) f(v_1), \end{aligned} \tag{3.7}$$

where $\{T_i(\lambda)\}$ is the Chebyshev polynomial of the first kind. In particular, setting $i = n-1$, we get

$$f(v_n) = T_{n-1}(\lambda) f(v_1).$$

On the other hand, taking (3.5) into account, we have

$$\begin{aligned} \lambda T_{n-1}(\lambda) &= \lambda f(v_n) \\ &= f(v_{n-1}) \\ &= T_{n-2}(\lambda) f(v_1). \end{aligned}$$

If $f(v_1) = 0$, we see from (3.7) that $f(v) = 0$ for $v \in V(G)$, which is not an eigenfunction. Hence, $f(v_1) \neq 0$ and we obtain

$$\begin{aligned} \lambda T_{n-1}(\lambda) - T_{n-2}(\lambda) &= 0 \\ (\lambda^2 - 1)U_{n-2}(\lambda) &= 0, \end{aligned}$$

where we applied (ii) of Lemma 2.2.2 to the first equation. We set $\lambda = \cos\theta$ for $0 < \theta < \pi$. Then $U_{n-2}(\lambda) = 0$ if and only if

$$\frac{\sin{(n-1)\theta}}{\sin\theta} = 0,$$

from which we obtain

$$\lambda = \cos\frac{\pi}{n-1}j, \quad 0 < j < n-1.$$

Thus, we find $n-2$ distinct eigenvalues together with linearly independent eigenfunctions of T. The remaining eigenvalues are ± 1 since G is a bipartite graph. Since $2 + (n-2) = n$ coincides with $|V(G)|$, we obtain

$$\sigma(T) = \left\{\cos\frac{\pi}{n-1}j \mid 0 < j < n-1\right\} \cup \{\pm 1\}.$$

Obviously, $\cos^{-1}(\sigma(T)) \subset \pi\mathbb{Q}$, which implies that G is Grover-periodic and its period is $2(n-1)$ by Proposition 3.2.3.

□

3.6 Distance-regular graphs

Let $G = (V, E)$ be a simple graph. If $V(G)$ is decomposed into a disjoint union of non-empty subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_r \subset V(G)$, then $\pi = \{\Gamma_1, \Gamma_2, \ldots \Gamma_r\}$ is called a *partition* of G. If Γ_i is a clique for $1 \leq i \leq r$, we call π a *clique partition*. A partition π is called an *equitable partition*, if there exists a non-negative integer $b_{i,j}$ such that each vertex $u \in \Gamma_i$ has $b_{i,j}$ neighbors in Γ_j for every pair of indices i, j, regardless the choice of u, that is,

$$|N(u) \cap \Gamma_j| = b_{i,j}$$

for $1 \leq i, j \leq r$ and $u \in \Gamma_i$.

3.6.1 Definition of distance-regular graphs

Let G be a simple, k-regular graph. Let $d = \mathrm{diam}(G)$. For a vertex $x \in V(G)$, we define

$$\Gamma_j(x) := \{y \in V(G) \,|\, d(x, y) = j\}, \quad 0 \leq j \leq d. \tag{3.8}$$

Then G is a *distance-regular graph* if it holds that

$$|N(y) \cap \Gamma_{j-1}(x)| = c_j, \tag{3.9}$$

$$|N(y) \cap \Gamma_{j}(x)| = a_j, \tag{3.10}$$

$$|N(y) \cap \Gamma_{j+1}(x)| = b_j, \tag{3.11}$$

for any $x \in V(G)$, $1 \le j \le d$ and $y \in \Gamma_j(x)$. Then $\{\Gamma_0(x), \Gamma_1(x), \cdots, \Gamma_d(x)\}$ becomes an equitable partition of G. The above non-negative parameters $\{a_j, b_j, c_j\}$ are called the *intersection numbers* [11]. In addition, it holds that for $0 \le j \le d$,

$$c_j + a_j + b_j = k, \tag{3.12}$$

where we set $c_0 = b_d = 0$. From the connectivity of G it follows immediately that $b_i \neq 0, c_j \neq 0$ for $0 \le i \le d-1$ and $1 \le j \le d$.

3.6.2 Restriction on diameter

For an operator X, let us denote the number of distinct eigenvalues of X is denoted by $n(X)$. For any graph G and its adjacency operator $A = A(G)$, it is well-known that $\mathrm{diam}(G) < n(A)$ [12]. This relation also holds for the transition operator, that is,

$$\mathrm{diam}(G) < n(T), \tag{3.13}$$

which is proved in a similar way.

Theorem 3.6.1. *Let r be a rational number with $|r| \le 1$. Then it holds that $\cos^{-1}(r) \in \pi\mathbb{Q}$ if and only if*

$$r \in \left\{0, \pm 1, \pm\frac{1}{2}\right\}.$$

Proof. We only show the necessity since the sufficiency is clear. Put $h(x) = x - r$. Then we have

$$2z \cdot h(\mathcal{J}(z)) = z^2 - 2rz + 1. \tag{3.14}$$

By Lemma 3.2.5 and Theorem 3.6.1, (3.14) is a product of cyclotomic polynomials. According as $r = 0, \pm 1$, or $\pm\frac{1}{2}$, (3.14) becomes

$$z^2 + 1,$$
$$z^2 \pm 2z + 1,$$
$$z^2 \pm z + 1,$$

which are products of cyclotomic polynomials. □

Theorem 3.6.2. *Let G be a Grover-periodic graph. If all the eigenvalues of $T(G)$ are rational, it holds necessarily that*

$$\mathrm{diam}(G) < 5.$$

Proof. Let λ be an eigenvalue of $T(G)$. By Proposition 3.2.3, we have $\lambda \in \left\{0, \pm 1, \pm\frac{1}{2}\right\}$. Then it follows $n(T) \le 5$. In view of (3.13), we obtain $\mathrm{diam}(G) < 5$. □

3.6.3 Hamming graphs

For positive integers $d \geq 1$ and $q \geq 2$ the Hamming graph $H(d,q)$ is defined as follows: Let F be a finite set of q elements. The vertex set of $H(d,q)$ is F^d and two vertices $x = (x_1, x_2, \cdots, x_d), y = (y_1, y_2, \cdots, y_d) \in F^d$ are adjacent if $|\{i \,|\, x_i \neq y_i, 1 \leq i \leq d\}| = 1$. It follows that the graph is $d(q-1)$-regular and $\mathrm{diam}(H(d,q)) = d$. Let A and T be the adjacency and transition operators of $H(d,q)$, respectively. It is known [11] that the distinct eigenvalues of the adjacency operator A are given by

$$\sigma(A) = \{d(q-1) - qi \,|\, 0 \leq i \leq d\}.$$

Then it follows immediately that

$$\sigma(T) = \left\{1 - \frac{qi}{d(q-1)} \,\middle|\, 0 \leq i \leq d\right\}.$$

Theorem 3.6.3. *The only Grover-periodic Hamming graphs $H(d,q)$ are*

$$H(1,2), H(1,3), H(2,2), H(3,3), H(4,2)$$

whose periods are $2, 3, 12$*, and* 12*, respectively.*

Proof. For $d = 1$, the Hamming graph is nothing but else the complete graph K_q. It is known [28] that the complete graph K_q is Grover-periodic if and only if $q = 2$ or 3. Then the Grover-periodic Hamming graphs $H(1,q)$ are the cases of $q = 2$ and 3, whose periods are 2 and 3, respectively. Now, we suppose $d \geq 2$ and every eigenvalue of T is rational. From the proof of Theorems 3.6.1 and 3.6.2, it follows that $\mathrm{diam}(H(d,q)) = d < 5$ and

$$\sigma(T) \subset \left\{0, \pm 1, \pm\frac{1}{2}\right\}. \tag{3.15}$$

For $d = 2$, we have

$$\sigma(T) = \left\{1 - \frac{2q}{2(q-1)}, 1 - \frac{q}{2(q-1)}, 1\right\}.$$

Then it is easily seen that $q = 2$ only fulfills (3.15) and we have

$$\sigma(T) = \{-1, 0, 1\}.$$

Thus, we have $d = q = 2$. The only Grover-periodic Hamming graph among $H(2,q)$ is $H(2,2) \simeq C_4$ and the period is 4. For $d = 3$ we have

$$\sigma(T) = \left\{1 - \frac{q}{q-1}, 1 - \frac{2q}{3(q-1)}, 1 - \frac{q}{3(q-1)}, 1\right\}.$$

Then $q = 3$ only fulfills (3.15) and we have

$$\sigma(T) = \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1\right\}.$$

Thus, the only Grover-periodic Hamming graph among $H(3,q)$ is $H(3,3)$ and the period is 12. For $d = 4$, we have

$$\sigma(T) = \left\{1 - \frac{q}{q-1}, 1 - \frac{3q}{4(q-1)}, 1 - \frac{q}{2(q-1)}, 1 - \frac{q}{4(q-1)}, 1\right\}.$$

Similarly, we obtain $q = 2$ and

$$\sigma(T) = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}.$$

Thus, the only Grover-periodic Hamming graph among $H(4,q)$ is $H(4,2)$, and the period is 12. $\square$

3.6.4 Johnson graphs

For two positive integers n and k with $n \geq k$, the Johnson graph $J(n,k)$ is defined as follows: The vertices of $J(n,k)$ are the k-element subsets of a fixed n-element set. Two vertices $X, Y \in V(J(n,k))$ are adjacent if $|X \cap Y| = k - 1$. It follows that the graph is $k(n-k)$-regular and $\mathrm{diam}(J(n,k)) = \min\{k, n-k\}$. Let A and T be the adjacency and transition operators of $J(n,k)$, respectively. It is known that the $d+1$ distinct eigenvalues of A are given by

$$\sigma(A) = \{(d-j)(n-d-j) - j \mid 0 \leq i \leq d\},$$

where $d = \min\{k, n-k\}$ [11]. Hence, we obtain

$$\sigma(T) = \left\{\frac{(d-j)(n-d-j) - j}{d(n-d)} \mid 0 \leq j \leq d\right\}.$$

Theorem 3.6.4. *The only Grover-periodic Johnson graphs $J(n,k)$ are*

$$J(2,1), J(3,1), J(4,2)$$

whose periods are 2, 3 *and* 12*, respectively.*

Proof. Note that every eigenvalue of the transition operator of $J(n,k)$ is rational. Hence, we have $d < 5$ by Theorem 3.6.2. For $d = 1$, we have

$$\sigma(T) = \left\{-\frac{1}{n-1}, 1\right\}.$$

In order to achieve (3.15), it necessarily holds that $n = 2$ or 3. If $d = k$, that is, $2k \leq n$, then we obtain $(n,k) = (2,1)$ and $(3,1)$. If $d = n - k$, that is, $2k \geq n$, then we obtain $(n,k) = (2,1)$ and $(3,2)$. Indeed, it is easily checked that $J(3,2) \simeq J(3,1)$. Thus the Grover-periodic Johnson graphs $J(n,1)$ are $J(2,1)$ or $J(3,1)$, which are nothing but else K_2 or K_3, respectively. For $d = 2$ we have

$$\sigma(T) = \left\{-\frac{1}{n-2}, \frac{n-4}{2n-4}, 1\right\}.$$

Hence, it is necessary that $n = 3$ or 4. If $n = 3$, we have $k = 2$ or 1 according as the diameter is $d = k$ or $d = n - k$. However, these pairs satisfy neither the assumption $2k \leq n$ for $d = k$ nor $2k \geq n$ for $d = n - k$. If $n = 4$, we obtain $k = 2$ for both of the cases $d = k$ and $d = n - k$ and we have

$$\sigma(T) = \left\{ -\frac{1}{2}, 0, 1 \right\}.$$

Thus, the graph is $J(4, 2)$ and the period is 12. For the cases of $d = 3$ and $d = 4$, we obtain

$$\sigma(T) = \left\{ -\frac{1}{n-3}, \frac{n-7}{3n-9}, \frac{2n-9}{3n-9}, 1 \right\}$$

and

$$\sigma(T) = \left\{ -\frac{1}{n-4}, \frac{n-10}{4n-16}, \frac{2n-14}{4n-16}, \frac{3n-16}{4n-16}, 1 \right\},$$

respectively. However, any n does not fulfill (3.15) for both of the two cases. Thus, there are no Grover-periodic Johnson graphs for $d = 3$ and $d = 4$. $\square$

3.7 Generalized Bethe trees

3.7.1 Definition

Let G be a tree and fix an arbitrary $x \in V(G)$. For $j \geq 0$, let $\Gamma_j(x)$ be defined as in (3.8) and n be defined as $\Gamma_n(x) \neq \phi$ and $\Gamma_{n+1}(x) = \phi$. The rooted tree G is called a *generalized Bethe tree* if

$$|N(u) \cap \Gamma_{i+1}(x)| = d(i), \quad 0 \leq i \leq n-1,$$

is independent of $u \in \Gamma_i$. In that case, the graph is denoted by $B(d(0), d(1), \ldots, d(n-1))$. Note that $\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_n(x)\}$ becomes an equitable partition of G. For simplicity, we put $C_i := \Gamma_i(x)$. Examples of generalized Bethe trees are shown in Figure 3.1. For $u \in C_i$ with $i \neq 0, n$, the *child* and *parent* of u are defined to be vertices in $N(u) \cap C_{i+1}$ and $N(u) \cap C_{i-1}$, respectively.

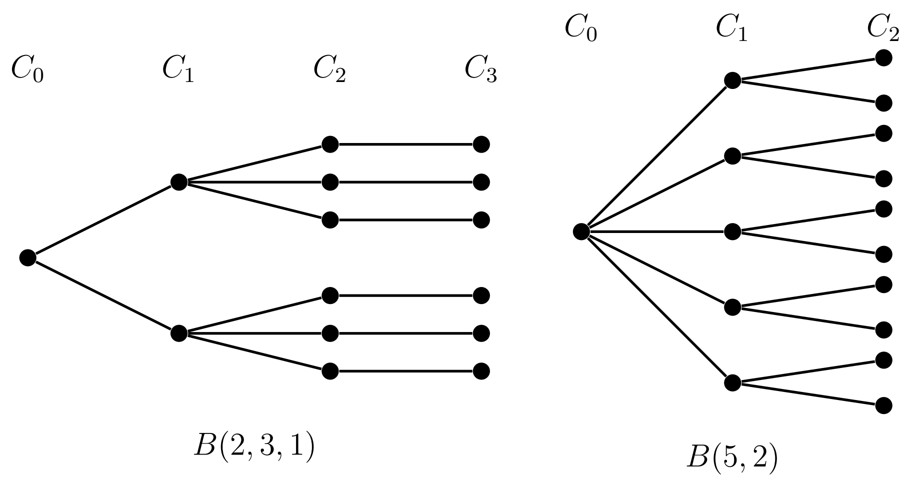


Figure 3.1: Examples of generalized Bethe trees

Define

$$D_0 = \frac{1}{d(1)+1}, \quad D_i = \frac{d(i)}{(d(i)+1)(d(i+1)+1)}, \quad 1 \leq i \leq n.$$

For $0 \leq i \leq n$ let us define

$$\Psi_i = \frac{1}{\sqrt{C_i}} \sum_{v \in C_i} \delta_v$$

and let $\mathcal{A}$ be the subspace of $\ell^2(V(G))$ spanned by $\Psi_0, \Psi_1, \ldots, \Psi_n$. Then it follows that

$$\mathcal{A}^{\perp} = \left\{ f \in \ell^2(V(G)) \middle| \sum_{v \in C_i} f(v) = 0,\ 0 \le i \le n \right\}.$$

It is easily checked that T is invariant on $\mathcal{A}^{\perp}$. By direct calculation, the following holds.

Lemma 3.7.1. *For a generalized Bethe tree $B(d(0), d(1), \cdots, d(n-1))$, it holds that*

$$\tilde{T}\Psi_i = \begin{cases} \sqrt{D_0}\Psi_1, & i = 0, \\ \sqrt{D_{n-1}}\Psi_{n-1}, & i = n, \\ \sqrt{D_i}\Psi_{i+1} + \sqrt{D_{i-1}}\Psi_{i-1}, & \textit{otherwise.} \end{cases}$$

The representation matrix of $\tilde{T}$ with respect to $\{\Psi_0, \Psi_1, \cdots \Psi_n\}$ becomes

$$\tilde{T}|_{\mathcal{A}} = \begin{pmatrix} 0 & \sqrt{D_0} & & & & & \\ \sqrt{D_0} & 0 & \sqrt{D_1} & & & & \\ & \sqrt{D_1} & 0 & \sqrt{D_2} & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & \sqrt{D_{n-2}} & 0 & \sqrt{D_{n-1}} \\ & & & & & \sqrt{D_{n-1}} & 0 \end{pmatrix}.$$

3.7.2 Spectral analysis of transition operator

Let $G = B(d(0), d(1), \cdots, d(n-1))$ be a generalized Bethe tree. For $v \in V(G) \backslash C_0$ let $P(v)$ be the parent of v. For $1 \le i \le n-1$ we define

$$P^i(v) = \underbrace{P(P(\cdots P(P}_{i}(v))),$$

and set $P^0(v) = v$. For $v \in V$, we define $N_n(v) \subset V(G)$ by

$$N_n(v) = \{w \in C_n \,|\, {}^{\exists}j, P^j(w) = v\}.$$

We define a sequence of polynomials in $\lambda \in \mathbb{C}$ inductively as follows:

$$\begin{cases} g_0(\lambda) = 1, \\ g_1(\lambda) = \lambda, \\ g_i(\lambda) = (d(n-i+1)+1)\,\lambda g_{i-1}(\lambda) - d(n-i+1) g_{i-2}(\lambda), \quad 2 \le i \le n, \\ g_{n+1}(\lambda) = d(0)(\lambda g_n(\lambda) - g_{n-1}(\lambda)). \end{cases}$$

Then it is easily checked that the coefficient of the maximum degree of g_i is

$$\prod_{j=1}^{i-1}(d(n-j)+1), \quad 2 \leq i \leq n,$$

and that of g_{n+1} is $d(0)\prod_{j=1}^{n-1}(d(n-j)-1)$.

Moreover, let us define

$$\Omega := \{1 \leq i \leq n \,|\, d(n-i) \geq 2\}.$$

For $i \in \Omega$ and $v^* \in C_{n-i}$, let $N(v^*) \cup C_{n-i+1} = \{v_1, v_2, \ldots, v_l\}$.

Lemma 3.7.2. *For every $i \in \Omega$ the zeros of g_i are eigenvalues of T. Let λ be a zero of g_i. Then the following function $f \in \ell^2(V(G))$ is an eigenfunction of T for λ and $f \in \mathcal{A}^\perp$. For $w \in N_n(v_1)$, $0 \leq j \leq i-1$,*

$$f(P^j(w)) = g_j(\lambda);$$

for $w \in N_n(v_2)$, $0 \leq j \leq i-1$,

$$f(P^j(w)) = -g_j(\lambda);$$

for $w \in C_n \setminus (N_n(v_1) \cup N_n(v_2))$, $0 \leq j \leq i-1$,

$$f(P^j(w)) = 0;$$

for $w \in \cup_{j=0}^{n-i} C_j$,

$$f(w) = 0.$$

Proof. For any $u \in \cup_{j=0}^{n-i} C_j \backslash \{v^*\}$ it holds clearly that $(Tf)(u) = \lambda f(u)$. For v^*, we have

$$\begin{aligned}
(Tf)(v^*) &= \frac{1}{\deg(v^*)} \sum_{x \sim v^*} f(x) \\
&= \frac{1}{\deg(v^*)} (f(v_1) + f(v_2)) \\
&= \frac{1}{\deg(v^*)} \left(g_{i-1}(\lambda) - g_{i-1}(\lambda)\right) \\
&= 0 \\
&= \lambda f(v^*).
\end{aligned}$$

For any $w \in N_n(v_1)$ and $0 \le j \le i-1$ it follows that

$$\begin{aligned}(Tf)(P^j(w)) &= \frac{1}{\deg(P^j(w))} \sum_{u \sim P^j(w)} f(u) \\ &= \frac{1}{d(n-j)+1} \left(f(P^{j+1}(w)) + \sum_{\substack{u \sim P^j(w) \\ u \in C_{n-j+1}}} f(u) \right) \\ &= \frac{1}{d(n-j)+1} \left(g_{j+1}(\lambda) + d(n-j) g_{j-1}(\lambda) \right) \\ &= \lambda g_j(\lambda) \\ &= \lambda f(P^j(w)).\end{aligned}$$

For any $w \in N_n(v_2)$ and every $0 \le j \le i-1$ it similarly holds that

$$(Tf)(P^j(w)) = -\lambda g_j(\lambda) = \lambda f(P^j(w)).$$

Furthermore, it is clear that

$$(Tf)(P^j(w)) = 0 = \lambda f(P^j(w))$$

for any $w \in C_n \setminus (N_n(v_1) \cup N_n(v_2))$ and $0 \le j \le i-1$. Thus, $(Tf)(x) = \lambda f(x)$ for $x \in \cup_{j=n-i+1}^{n} C_j$. Hence, it holds that $(Tf)(x) = \lambda f(x)$ for any $x \in V(G)$.

Since $\sum_{u \in C_i} f(u) = 0$ for any $0 \le i \le n$ by the definition of f, we have $f \in \mathcal{A}^\perp$. $\square$

It follows from Lemma 3.7.2 that $\sigma(\tilde{T}|_{\mathcal{A}^\perp}) \supset \bigcup_{i \in \Omega} \{\lambda \in \mathbb{C} \mid g_i(\lambda) = 0\}$.

Lemma 3.7.3. *It holds that*

$$\sigma(\tilde{T}|_{\mathcal{A}^\perp}) = \bigcup_{i \in \Omega} \{\lambda \in \mathbb{C} \mid g_i(\lambda) = 0\}.$$

Proof. For the assertion it is sufficient to show that the number of the linearly independent eigenfunctions obtained during the above argument coincides with $\dim \mathcal{A}^\perp = |V(G)| - (n+1)$. By definition of a generalized Bethe tree it follows that $|C_0| = 1$ and

$$|C_i| = \prod_{j=0}^{i-1} d(j), \quad 1 \le i \le n-1. \tag{3.16}$$

Thus,

$$|V(G)| = \sum_{i=0}^{n} |C_i| = |C_0| + \sum_{i=0}^{n-1} |C_{n-i}| = 1 + \sum_{i=0}^{n-1} \prod_{j=0}^{n-i-1} d(j).$$

Then it holds that $\dim \mathcal{A}^{\perp} = \sum_{i=0}^{n-1} \prod_{j=0}^{n-i-1} d(j) - n$. For $i \in \Omega$, let λ be a zero of g_i. By Lemma 3.7.2 there are $d(n-i)-1$ linearly independent eigenfunctions for λ for every $v \in C_{n-i}$. Hence, we find $|C_{n-i}|\,(d(n-i)-1)$ linearly independent eigenfunctions for λ. The zeros of g_i are the eigenvalues of a tri-diagonal matrix and all the eigenvalues of the tri-diagonal matrices are simple [53]. Hence, the number of the linearly independent eigenfunctions for to the zeros of g_i is $i|C_{n-i}|\,(d(n-i)-1)$. Here, we put $F_i := i|C_{n-i}|\,(d(n-i)-1)$ for $1 \le i \le n$. By (3.16) it follows that

$$F_i = i\left(\prod_{j=0}^{n-i} d(j) - \prod_{j=0}^{n-i-1} d(j)\right),$$

for $1 \le i \le n-1$ and $F_n = n(d(0)-1)$. Then the number of the linearly independent eigenfunctions for the zeros of every g_i for $i \in \Omega$ is $\sum_{i=1}^{n} F_i$ because $F_i = 0$ for i with $d(n-i) = 1$. Thus, we get

$$\begin{aligned}
\sum_{i=1}^{n} F_i &= \sum_{i=1}^{n-1} i\left(\prod_{j=0}^{n-i} d(j) - \prod_{j=0}^{n-i-1} d(j)\right) + F_n \\
&= \sum_{i=1}^{n-1} \left(i\prod_{j=0}^{n-i} d(j) - i\prod_{j=0}^{n-i-1} d(j)\right) + F_n \\
&= \sum_{i=1}^{n-1} \prod_{j=0}^{n-i} d(j) - (n-1)d(0) + n(d(0)-1) \\
&= \sum_{i=1}^{n-1} \prod_{j=0}^{n-i} d(j) + d(0) - n \\
&= \sum_{i=1}^{n} \prod_{j=0}^{n-i} d(j) - n,
\end{aligned}$$

which equals $\dim \mathcal{A}^{\perp}$. $\square$

Here, let us put $\tilde{g}_0(\lambda) = 1$ and

$$\tilde{g}_0 = 1, \quad \tilde{g}_i = \frac{1}{\prod_{j=1}^{i-1}(d(n-j)+1)} g_i, \quad 1 \le i \le n-1.$$

We define

$$\begin{cases} p_0(\lambda) = 1, \\ p_i(\lambda) = \det(\lambda I_i - \tilde{T}^{(i)}), \quad 1 \le i \le n+1, \end{cases} \tag{3.17}$$

where I_i is the $i \times i$ identity matrix and $\tilde{T}^{(i)}$ is the principal submatrix of $\tilde{T}|_{\mathcal{A}}$ given by removing the first row up to the $(n+1-i)$-th row and the first column up to the $(n+1-i)$-th column.

Lemma 3.7.4. *It holds that*

$$p_i(\lambda) = \tilde{g}_i(\lambda)$$

for any $\lambda \in \mathbb{C}$ *and* $0 \leq i \leq n+1$.

Proof. Since $\tilde{g}_0(\lambda) = p_0(\lambda) = 1$ and $\tilde{g}_1(\lambda) = p_1(\lambda) = \lambda$, it is sufficient to show that $\{\tilde{g}_i\}$ and $\{p_i\}$ satisfy the same recurrence relation. First, by the cofactor expansion, we have

$$p_i(\lambda) = \lambda p_{i-1}(\lambda) - D_{n-i+1} p_{i-2}(\lambda) \tag{3.18}$$

for $0 \leq i \leq n+1$. If $i = 2$ or 3, it is easily seen that $\tilde{g}_i(\lambda) = \lambda \tilde{g}_{i-1}(\lambda) - D_{n-i+1}\tilde{g}_{i-2}(\lambda)$ since $d(n) = 0$. For $4 \leq i \leq n$,

$$\begin{aligned}
\tilde{g}_i(\lambda) &= \frac{(d(n-i+1)+1)\lambda g_{i-1}(\lambda)}{\prod_{j=1}^{i-1}(d(n-j)+1)} - \frac{d(n-i+1)g_{i-2}(\lambda)}{\prod_{j=1}^{i-1}(d(n-j)+1)} \\
&= \frac{\lambda g_{i-1}(\lambda)}{\prod_{j=1}^{i-2}(d(n-j)+1)} - \frac{d(n-i+1)g_{i-2}(\lambda)}{(d(n-i+1)+1)(d(n-i+2)+1)\prod_{j=1}^{i-3}(d(n-j)+1)} \\
&= \lambda \tilde{g}_{i-1}(\lambda) - D_{n-i+1}\tilde{g}_{i-2}(\lambda).
\end{aligned}$$

We have $\tilde{g}_{n+1}(\lambda) = \lambda \tilde{g}_n(\lambda) - D_0 \tilde{g}_{n-1}(\lambda)$ since $g_{n+1}(\lambda) = d(0)(\lambda g_n(\lambda) - g_{n-1}(\lambda))$ and $D_0 = \frac{1}{d(1)+1}$, so we see that $p_i(\lambda) = \tilde{g}_i(\lambda)$ for any $\lambda \in \mathbb{C}$ and $0 \leq i \leq n+1$. $\square$

Then it follows immediately that the zeros of $p_{n+1}(\lambda)$ are the eigenvalues of $\tilde{T}|_{\mathcal{A}}$. Indeed, the polynomials $\{p_i\}_{i=0}^n$ are the monic orthogonal polynomials with Jacobi coefficients $\{\sqrt{D_0}, \sqrt{D_1}, \cdots, \sqrt{D_{n-1}}\}$. Then together with Lemma 3.7.3, the above discussion is summarized as follows:

Theorem 3.7.5. *It holds that*

$$\sigma(\tilde{T}|_{\mathcal{A}^\perp}) = \bigcup_{i \in \Omega} \{\lambda \in \mathbb{C} \mid p_i(\lambda) = 0\},$$

$$\sigma(\tilde{T}|_{\mathcal{A}}) = \{\lambda \in \mathbb{C} \mid p_{n+1}(\lambda) = 0\}.$$

3.7.3 Grover-periodic generalized Bethe trees

Theorem 3.7.6. *The only Grover-periodic generalized Bethe trees are* $S_k(B(1,2,3))$ *and* $S_k(B(s,3))$ *for* $k, s \in \mathbb{N}$, *and both of their periods are* $12k$.

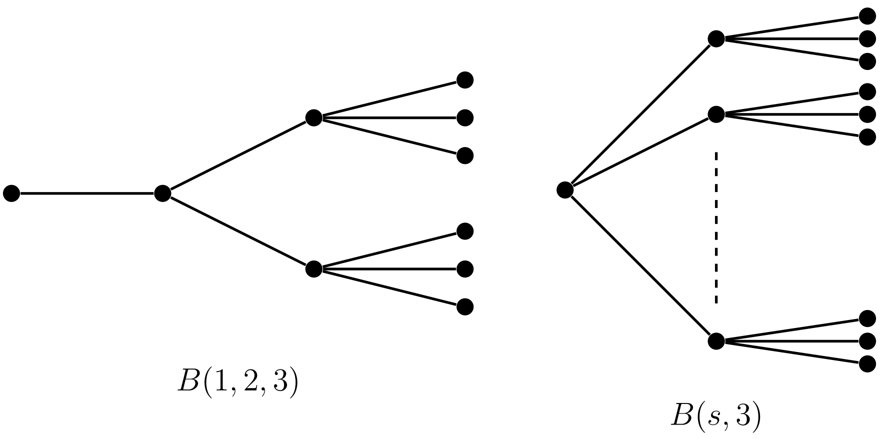


Figure 3.2: Grover-periodic generalized Bethe trees

Proof. Let $G = B(d(0), d(1), \cdots, d(n))$ be a generalized Bethe tree and put $l = |\Omega|$. We define positive integers $k_1, k_2, \ldots, k_l$ in such a way that $n - K_i \in \Omega$ for $1 \le i \le l$, where $K_i = \sum_{j=1}^{i} k_j$ (See Figure 3.3). Then we define $k_{l+1} = n - K_l$ and $K_{l+1} = k_{l+1} + K_l = n$. Moreover, we put $d_i = d(n - K_i)$. By Lemmas 3.7.4 and 3.7.2, the zeros of p_{K_i} are the eigenvalue of $\tilde{T}|_{\mathcal{A}^\perp}$ for $1 \le i \le l$, and those of $p_{K_{(l+1)}+1}$ are the eigenvalues of $\tilde{T}|_{\mathcal{A}}$. We will find the polynomial p_{K_i} and check whether $(2z)^{K_i} p_{K_i}(\mathcal{J}(z))$ is a product of cyclotomic polynomials for every $1 \le i \le l+1$. The argument will be divided into a few steps.

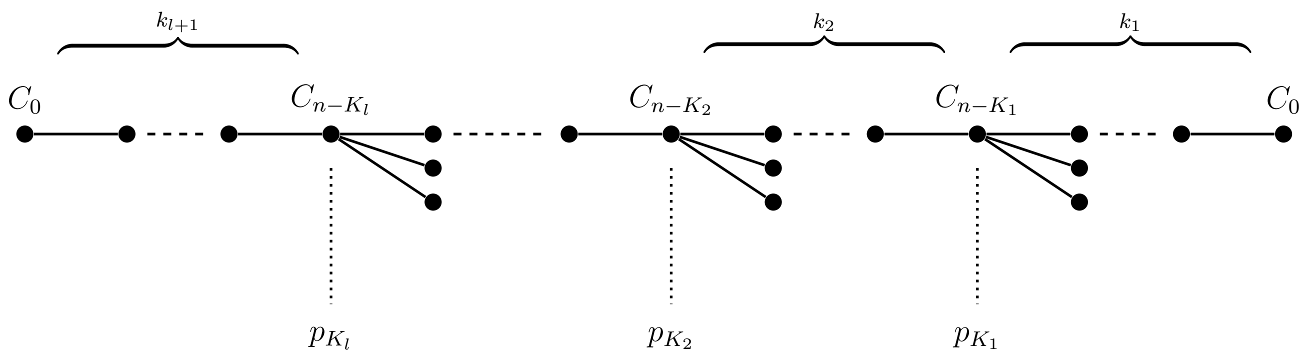


Figure 3.3: Setting

Lemma 3.7.7. *Let p_i be defined as in (3.17). For $1 \le i \le K_1$, it holds that*

$$p_i = \frac{1}{2^{i-1}} T_i. \tag{3.19}$$

Proof. If $K_1 = 1, 2$, then it clearly holds because $p_1 = \lambda = T_1$, and $p_2 = \lambda p_1 - D_{n-1}p_0 = \lambda^2 - \frac{1}{2} = \frac{1}{2}T_2$. Here, we assume that $K_1 \geq 3$. We prove it by induction on i. Indeed, $D_{n-i+1} = \frac{1}{4}$ for $3 \leq i \leq K_1$ and it holds that

$$\begin{aligned} p_i &= \lambda p_{i-1} - D_{n-i+1}p_{i-2} \\ &= \lambda \left\{ \frac{1}{2^{i-2}} T_{i-1} \right\} - \frac{1}{4} \left\{ \frac{1}{2^{i-3}} T_{i-2} \right\} \\ &= \frac{1}{2^{i-1}} (2\lambda T_{i-1} - T_{i-2}) \\ &= \frac{1}{2^{i-1}} T_i. \end{aligned}$$

□

Lemma 3.7.8. *For $2 \leq j \leq l+1$ with $k_j \geq 2$ and $2 \leq i \leq k_j$, it holds that*

$$p_{K_{(j-1)}+i} = \frac{1}{2^{i-2}} U_{i-2} p_{K_{(j-1)}+2} - \frac{1}{2^{i-1}} U_{i-3} p_{K_{(j-1)}+1}. \tag{3.20}$$

Proof. It is easy to confirm that (3.7.8) holds for $i = 2$ and 3. Then we assume that $i \geq 4$ and we have

$$\begin{aligned} p_{K_{(j-1)}+i} &= \lambda p_{K_{(j-1)}+i-1} - D_{n-K_{(j-1)}-i+1} p_{K_{(j-1)}+i-2} \\ &= \lambda p_{K_{(j-1)}+i-1} - \frac{1}{4} p_{K_{(j-1)}+i-2} \\ &= \lambda \left\{ \frac{1}{2^{i-3}} U_{i-3} p_{K_{(j-1)}+2} - \frac{1}{2^{i-2}} U_{i-4} p_{K_{(j-1)}+1} \right\} \\ &\qquad - \frac{1}{4} \left\{ \frac{1}{2^{i-4}} U_{i-4} p_{K_{(j-1)}+2} - \frac{1}{2^{i-3}} U_{i-5} p_{K_{(j-1)}+1} \right\} \\ &= \frac{1}{2^{i-2}} (2\lambda U_{i-3} - U_{i-4}) p_{K_{(j-1)}+2} - \frac{1}{2^{i-1}} (2\lambda U_{i-4} - U_{i-5}) p_{K_{(j-1)}+1} \\ &= \frac{1}{2^{i-2}} U_{i-2} p_{K_{(j-1)}+2} - \frac{1}{2^{i-1}} U_{i-3} p_{K_{(j-1)}+1}. \end{aligned}$$

□

If $l = 0$, the graph is nothing but else the path graph P_{n+1}, which is the Grover-$2n$-periodic graph. If $l = 1$, the graph is expressed as

$$B(\underbrace{1, 1, \cdots, 1}_{k_2}, d_1, \underbrace{1, 1, \cdots, 1}_{k_1 - 1}).$$

If $k_2 = 0$ and $k_1 = 1$, or $k_2 = 1$ and $k_1 = 1$, then these graphs are nothing but else claws, which are Grover-4-periodic graphs. Hence, we omit these cases. By Lemma 3.7.7, we have

$$p_{K_1} = \frac{1}{2^{K_1 - 1}} T_{K_1}.$$

Furthermore, it follows by (i) on Lemma 3.2.4 that

$$(2z)^{K_1} p_{K_1}(\mathcal{J}(z)) = z^{2K_1} + 1, \tag{3.21}$$

which is a cyclotomic polynomial. The zeros of (3.21) are the eigenvalues of $\tilde{T}|_{\mathcal{A}^\perp}$ and satisfy

$$z^{4k_1} = 1. \tag{3.22}$$

These are $4k_1$-th roots of unity. Next, we find the polynomial $p_{n+1} = p_{K_2+1}$ to analyze the eigenvalues of $\tilde{T}|_{\mathcal{A}}$. Then we have

$$p_{K_1+1} = \begin{cases} \lambda p_1 - \frac{d_1}{d_1+1} p_0 & \text{if } k_1 = 1, \\ \lambda p_{K_1} - \frac{d_1}{2(d_1+1)} p_{K_1-1} & \text{if } k_1 \geq 2, \end{cases}$$

and

$$p_{K_1+2} = \begin{cases} \lambda p_{K_1+1} - \frac{1}{d_1+1} p_{K_1} & \text{if } k_2 = 1, \\ \lambda p_{K_1+1} - \frac{1}{2(d_1+1)} p_{K_1} & \text{if } k_2 \geq 2. \end{cases}$$

Indeed, for any $k_1 \in \mathbb{N}$, we have

$$(2z)^{K_1+1} p_{K_1+1}(\mathcal{J}(z)) = z^{2K_1+2} + \frac{1-d_1}{d_1+1} z^{2K_1} + \frac{1-d_1}{d_1+1} z^2 + 1. \tag{3.23}$$

Furthermore, if $k_2 = 1$, then

$$(2z)^{K_1+2} p_{K_1+2}(\mathcal{J}(z)) = z^{2K_1+4} - \frac{2}{d_1+1} z^{2K_1+2} + \frac{1-d_1}{d_1+1} z^{2K_1} + \frac{1-d_1}{d_1+1} z^4 - \frac{2}{d_1+1} z^2 + 1 \tag{3.24}$$

and if $k_2 \geq 2$, then

$$(2z)^{K_1+2} p_{K_1+2}(\mathcal{J}(z)) = z^{2K_1+4} + \frac{1-d_1}{d_1+1} z^{2K_1} + \frac{1-d_1}{d_1+1} z^4 + 1 \tag{3.25}$$

by (i) on Lemma 3.2.4. If $k_2 = 1$, then $K_2 + 1 = K_1 + 2$. Furthermore, if $k_2 = 2$, then $K_2 + 1 = K_1 + 3$, i.e., $p_{K_2+1} = p_{K_1+3} = \lambda p_{K_2+2} - \frac{1}{2} p_{K_1+1}$. Therefore, it follows that

$$p_{K_2+1} = \begin{cases} p_{K_1+2} & \text{if } k_2 = 1, \\ \lambda p_{K_1+2} - \frac{1}{2} p_{K_1+1} & \text{if } k_2 = 2, \\ \frac{1}{2^{k_2-2}} \left(\lambda U_{k_2-2} - U_{k_2-3}\right) p_{K_1+2} - \frac{1}{2^{k_2-1}} \left(\lambda U_{k_2-3} - U_{k_2-4}\right) p_{K_1+1}. & \text{if } k_2 \geq 3, \end{cases}$$

by Lemma 3.7.8. Using (3.23), (3.24), (3.25) and (ii) on Lemma 3.2.4, we get

$$(2z)^{K_2+1} p_{K_2+1}(\mathcal{J}(z)) = z^{2K_2+2} - z^{2K_2} + \frac{1-d_1}{d_1+1} \left(z^{2k_2+2} - z^{2k_2} - z^{2K_1+2} + z^{2K_1}\right) - z^2 + 1.$$

Unless $k_1 = K_1 = k_2$, the above polynomial is not an integer polynomial, which implies that it is not a product of cyclotomic polynomials. Thus, the Bethe tree is nothing k_1-subdivision of $S_{k_2}(K_{1,d_1+1})$. Then the above polynomial becomes

$$z^{4k_1+2} - z^{4k_1} - z^2 + 1 = (z^2 - 1)(z^{4k_1} - 1)$$

and the zeros satisfy $z^{4k_1} = 1$. Therefore, the period is $4k_1$.

Lemma 3.7.9. *If G is Grover-periodic and $l \geq 2$, it holds that $k_1 = k_2$ and $d_1 = 3$.*

Proof. By Lemma 3.7.8, we obtain

$$p_{K_2} = \begin{cases} p_{K_1+1} & \text{if } k_2 = 1, \\ \frac{1}{2^{k_2-2}} U_{k_2-2} p_{K_1+2} - \frac{1}{2^{k_2-1}} U_{k_2-3} p_{K_1+1} & \text{if } k_2 \geq 2. \end{cases}$$

Indeed, for any $k_1, k_2 \in \mathbb{N}$ it follows that

$$(2z)^{K_2} p_{K_2}(\mathcal{J}(z)) = z^{2K_2} + \frac{1-d_1}{d_1+1} z^{2k_1} + \frac{1-d_1}{d_1+1} z^{2k_2} + 1 \tag{3.26}$$

by (3.23), (3.24), (3.25), and (ii) of Lemma 3.2.4. Thus, (3.26) is not an integer polynomial unless $k_1 = k_2$ and $d_1 = 3$. □

Then (3.26) turns to

$$z^{4k_1} - z^{2k_1} + 1, \tag{3.27}$$

and the zeros satisfy

$$z^{12k_1} = 1. \tag{3.28}$$

Lemma 3.7.10. *If $l \geq 3$, then G is not Grover-periodic.*

Proof. We suppose $l \geq 3$ and derive a contradiction. Then the zeros of p_{K_3} are the eigenvalue of $\tilde{T}$ on $\mathcal{A}^{\perp}$ by Theorem 3.7.5. Then we also have

$$p_{K_3} = \begin{cases} p_{K_2+1} & \text{if } k_3 = 1, \\ \frac{1}{2^{k_3-2}} U_{k_3-2} p_{K_2+2} - \frac{1}{2^{k_3-1}} U_{k_3-3} p_{K_2+1} & \text{if } k_3 \geq 2. \end{cases}$$

by Lemma 3.7.8. We set $k_1 = k_2$, and $d_1 = 3$ from Lemma 3.7.9. Therefore, it follows that $K_2 = 2k_1$ and

$$p_{K_2+2} = \lambda p_{K_2+1} - \frac{1}{2(d_2+1)} p_{K_2}$$

$$p_{K_2+1} = \begin{cases} \lambda p_{K_2} - \frac{d_2}{4(d_2+1)} p_{K_2-1} & \text{if } k_1 = k_2 = 1, \\ \lambda p_{K_2} - \frac{d_2}{2(d_2+1)} p_{K_2-1} & \text{if } k_1 = k_2 \geq 2. \end{cases}$$

$$p_{K_2-1} = \begin{cases} p_{K_1} & \text{if } k_1 = k_2 = 1, \\ p_{K_1+1} & \text{if } k_1 = k_2 = 2, \\ \frac{1}{2^{k_1-3}} U_{k_1-3} p_{K_1+2} - \frac{1}{2^{k_1-2}} U_{k_1-4} p_{K_1+1} & \text{if } k_1 = k_2 \geq 3. \end{cases}$$

Combining (3.21), (3.23), (3.24), (3.25), and (3.27), we have

$$(2z)^{K_2-1} p_{K_2-1}(\mathcal{J}(z)) = z^{4k_1-2} - \frac{1}{2} z^{2k_1} - \frac{1}{2} z^{2k_1-2} + 1, \tag{3.29}$$

$$(2z)^{K_2+1}p_{K_2+1}(\mathcal{J}(z)) = z^{4k_1+2} + \frac{1-d_2}{d_2+1}z^{4k_1} - \frac{1}{d_2+1}z^{2k_1+2} - \frac{1}{d_2+1}z^{2k_1} + \frac{1-d_2}{d_2+1}z^2 + 1, \tag{3.30}$$

$$(2z)^{K_2+2}p_{K_2+2}(\mathcal{J}(z)) = z^{4k_1+4} + \frac{1-d_2}{d_2+1}z^{4k_1} - \frac{1}{d_2+1}z^{2k_1+4} - \frac{1}{d_2+1}z^{2k_1} + \frac{1-d_2}{d_2+1}z^4 + 1. \tag{3.31}$$

Thus, for any $k_1, k_3 \in \mathbb{N}$ we have

$$(2z)^{K_3}p_{K_3}(\mathcal{J}(z)) = z^{2K_3} + \frac{1-d_2}{d_2+1}z^{4k_1} - \frac{1}{d_2+1}z^{2k_1} - \frac{1}{d_2+1}z^{2k_1+2k_3} + \frac{1-d_2}{d_2+1}z^{2k_3} + 1. \tag{3.32}$$

The above polynomial is not an integer polynomial for any $d_2 \in \mathbb{N}_{\geq 2}$, and $k_1, k_3 \in \mathbb{N}$. In other words, the zeros of p_{K_3} are not roots of the real part of the unity for any $d_2 \in \mathbb{N}_{\geq 2}$, and $k_1, k_3 \in \mathbb{N}$. Therefore, we have to set $l < 3$ so that the zeros of $p_{K_3}(\lambda)$ are not the eigenvalues of $\tilde{T}$. □

Therefore, the candidates of Grover-periodic generalized Bethe trees are written as

$$B(\underbrace{1, 1, \cdots, 1}_{k_3}, d_2, \underbrace{1, 1, \cdots, 1}_{k_1-1}, 3, \underbrace{1, 1, \cdots, 1}_{k_1-1}), \tag{3.33}$$

or,

$$B(d_2, \underbrace{1, 1, \cdots, 1}_{k_1-1}, 3, \underbrace{1, 1, \cdots, 1}_{k_1-1}). \tag{3.34}$$

For (3.33), we consider the eigenvalues of $T|_{\mathcal{A}}$ to determine the parameters $k_3 \in \mathbb{N}$ and $d_2 \in \mathbb{N}_{\geq 2}$. We find $p_{n+1} = p_{K_3+1}$ and analyze its zeros since these coincide with the eigenvalues of $\tilde{T}|_{\mathcal{A}}$. Then we have

$$p_{K_3+1} = \begin{cases} \lambda p_{K_3} - \frac{1}{d_2+1}p_{K_2} & \text{if } k_3 = 1, \\ \lambda p_{K_3} - \frac{1}{2}p_{K_3-1} & \text{if } k_3 \geq 2. \end{cases}$$

For $k_1, k_3 \in \mathbb{N}$ the polynomials $(2z)^{K_3}p_{K_3}(\mathcal{J}(z))$ and $(2z)^{K_2}p_{K_2}(\mathcal{J}(z))$ are of the forms (3.32) and (3.27), respectively. By (3.27), (3.30), (3.31) and

$$p_{K_3-1} = \begin{cases} p_{K_2} & \text{if } k_3 = 1, \\ p_{K_2+1} & \text{if } k_3 = 2, \\ \frac{1}{2^{k_3-3}}U_{k_3-3}p_{K_2+2} - \frac{1}{2^{k_3-2}}U_{k_3-4}p_{K_2+1} & \text{if } k_3 \geq 3, \end{cases}$$

we have

$$\begin{aligned} (2z)^{K_3-1}p_{K_3-1}(\mathcal{J}(z)) = z^{2(K_3-1)} &+ \frac{1-d_2}{d_2+1}z^{4k_1} - \frac{1}{d_2+1}z^{2k_1} \\ &- \frac{1}{d_2+1}z^{2k_1+2(k_3-1)} + \frac{1-d_2}{d_2+1}z^{2(k_3-1)} + 1. \end{aligned} \tag{3.35}$$

Indeed, by (3.32) and (3.35), we obtain

$$\begin{aligned}(2z)^{K_3+1}p_{K_3+1}\left(\mathcal{J}(z)\right) =& z^{2K_3+2} - \frac{1}{d_2+1}\left(z^{2k_1+2k_3+2} - z^{2k_1+2k_3} - z^{2k_1+2} + z^{2k_1}\right) \\ &- \frac{1-d_2}{d_2+1}\left(z^{4k_1+2} - z^{4k_1} - z^{2k_3+2} + z^{2k_3}\right) - z^{2K_3} - z^2 + 1.\end{aligned}$$

for any $k_3 \in \mathbb{N}$. The above polynomial is not an integer polynomial unless $k_1 = k_3$ and $d_2 = 2$. Then the right hand side becomes

$$z^{6k_1+2} - z^{6k_1} - z^2 + 1 = (z+1)(z-1)(z^{6k_1} - 1),$$

whose zeros satisfy

$$z^{6k_1} = 1. \tag{3.36}$$

Thus, it is necessary that $k_3 = k_1$ and $d_2 = 2$. Put $k_1 = k$. Then the generalized Bethe tree is $B(\underbrace{1,1,\cdots,1}_{k},2,\underbrace{1,1,\cdots,1}_{k-1},3,\underbrace{1,1,\cdots,1}_{k-1})$, which is nothing but else $S_k(B(1,2,3))$. Then it follows from (3.22), (3.28), and (3.36) that the period is $12k$.

For (3.34), we find $p_{n+1} = p_{K_2+1}$ and analyze its zeros. We have

$$p_{K_2+1} = \begin{cases} \lambda p_{K_2} - \frac{1}{4}p_{K_1} & \text{if } k_1 = 1, \\ \lambda p_{K_2} - \frac{1}{2}p_{K_2-1} & \text{if } k_1 \geq 2. \end{cases}$$

The polynomials p_{K_1}, p_{K_2} and p_{K_2-1} are of the forms (3.21), (3.27) and (3.29), respectively. Thus, for any $k_1 \in \mathbb{N}$ we obtain

$$(2z)^{K_2+1}p_{K_2+1}(\mathcal{J}(z)) = z^{4k_1+2} - z^{4k_1} - z^2 + 1 = (z+1)(z-1)(z^{4k_1} - 1),$$

whose zeros satisfy

$$z^{4k_1} = 1. \tag{3.37}$$

Put $k_1 = k$ and $d_2 = s$. Then the generalized Bethe tree is $B(s,\underbrace{1,1,\cdots,1}_{k-1},3,\underbrace{1,1,\cdots,1}_{k-1})$, which is nothing but else $S_k(B(s,3))$. The period is $12k$ by (3.22), (3.28) and (3.37). Therefore, the only Grover-periodic Bethe trees are $S_k(B(1,2,3))$ and $S_k(B(s,3))$ for $k, s \in \mathbb{N}$ and we complete the proof.

□

Chapter 4

Restricted structure on Grover-periodic graphs

In the previous Chapter, we characterized Grover-periodic graphs in some special classes of graphs, in particular, distance-regular graphs and generalized Bethe trees. In this Chapter, we will find a tight condition for general Grover-periodic graphs. What we would like to state most in this paper is the condition. The tightness of the condition enables us to determine several classes of Grover-periodic graphs. After that, we construct a new class of graphs by a method called *join*. Then the condition plays an important role to construct a new Grover-periodic graphs obtained by the join. Furthermore, we reach our conjecture for Grover-odd-periodic graphs by the condition.

4.1 Necessary condition for Grover-periodic graphs

Let G be a simple graph on n vertices and T be the transition operator. We define

$$\varphi(x) := \det(xI_n - T) = \sum_{i=0}^{n} \rho_i x^i, \tag{4.1}$$

where I_n is the $n \times n$-identity matrix. Note that the determinant in (4.1) is determined uniquely independent of the representation matrix of T. Moreover, we define

$$\Psi(z) := (2z)^n \varphi\left(\mathcal{J}(z)\right) = \sum_{j=0}^{2n} \alpha_j z^j. \tag{4.2}$$

Lemma 4.1.1. *Let α_j be defined as in (4.2). Then it holds that*

(i) $\alpha_{2n} = 1$,

(ii) $\alpha_{2n-j} = \alpha_j$ *for every* $0 \le j \le n$.

Proof. Since $\varphi(x)$ is a monic polynomial, so is $\Psi(z)$ and (i) follows. By definition, we obtain

$$\Psi(z) = (2z)^n \left\{ \sum_{i=0}^{n} \rho_i \left(\frac{z+z^{-1}}{2} \right)^i \right\} \tag{4.3}$$

$$= \sum_{i=0}^{n} \left\{ 2^{n-i} \rho_i \sum_{r=0}^{i} \binom{i}{r} z^{n-2r+i} \right\}. \tag{4.4}$$

The coefficient of z^{2n-j} in (4.4) is obtained by the terms corresponding to $i = n-(j-2k)$, $r = k$ with $0 \le k \le \lfloor \frac{j}{2} \rfloor$. Thus, it follows that

$$\alpha_{2n-j} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} 2^{j-2k} \rho_{n-(j-2k)} \binom{n-(j-2k)}{k}. \tag{4.5}$$

On the other hand, the coefficient of z^j of (4.4) is obtained by the terms corresponding to $i = n-(j-2k)$, $r = n-(j-k)$ with $0 \le k \le \lfloor \frac{j}{2} \rfloor$. Thus, we obtain

$$\alpha_j = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} 2^{j-2k} \rho_{n-(j-2k)} \binom{n-(j-2k)}{n-(j-k)}. \tag{4.6}$$

Then (ii) follows from $\binom{n-(j-2k)}{k} = \binom{n-(j-2k)}{n-(j-k)}$. $\square$

By Lemma 3.2.5, in order to induce periodic Grover walks, it is necessary that $\Psi(z)$ is a product of cyclotomic polynomials. Then it follows that $\alpha_j \in \mathbb{Z}$ for every $0 \le j \le 2n$.

Theorem 4.1.2. *For a simple graph $G = (V, E)$ with $n = |V|$, let ρ_i be defined as in (4.1). If G is Grover-periodic, it holds that*

$$2^j \rho_{n-j} \in \mathbb{Z}, \quad 1 \le j \le n$$

Proof. We prove by contradiction. Take j in such a way that $2^j \rho_{n-j} \notin \mathbb{Z}$ and $2^l \rho_{n-l} \in \mathbb{Z}$ for every $1 \le l \le j-1$. By (4.5), we have

$$\alpha_{2n-j} = 2^j \rho_{n-j} + \sum_{k=1}^{\lfloor \frac{j}{2} \rfloor} 2^{j-2k} \rho_{n-(j-2k)} \binom{n-(j-2k)}{k}. \tag{4.7}$$

By the assumption, the second term is an integer while the first term is not. Therefore $\alpha_{2n-j} \notin \mathbb{Z}$, which implies that $\Psi(z)$ is not represented by a product of cyclotomic polynomials. Then G is not Grover-periodic by Lemma 3.2.5. $\square$

For a simple graph G with $|V(G)| = n$, let F_i be a subset of $V(G)^i$ defined by

$$F_i := \{(v_1, v_2, \ldots, v_i) \mid v_1 \sim v_2 \sim \cdots \sim v_i \sim v_1\}$$

for $2 \leq i \leq n$. Two elements $v = (v_1, v_2, \ldots v_i)$ and $v' = (v'_1, v'_2, \ldots, v'_i)$ in F_i are *cyclicly equivalent* if there exists $k \in \mathbb{N}$ such that $v_j = v'_{j+k}$ for $1 \leq j \leq i$, where the index is understood in modulo i. Then we denote $v \overset{c_i}{\sim} v'$. In addition, we define

$$E_i := F_i / c_i.$$

For $v \in F_i$, let $[v]$ be the equivalent class of v with respect to $\overset{c_i}{\sim}$. For $w = [(v_1, v_2, \ldots, v_i)] \in E_i$, we define a map $M_i : E_i \to \mathbb{Q}$ by

$$M_i(w) = \prod_{j=1}^{i} \frac{1}{\deg v_j}, \quad 2 \leq i \leq n.$$

By definition, we see that M_i is a well-defined map. Let $Y = xI_n - T$. By definition, we have

$$\det Y = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{v \in V(G)} Y_{v,\sigma(v)}, \tag{4.8}$$

where S_n is the set of the permutations on $V(G)$ and $\operatorname{sgn}(\sigma)$ is 1 if σ is an even permutation, and -1 otherwise. A permutation of the from

$$\sigma = \begin{pmatrix} v_1 & v_2 & \ldots & v_i \\ v_2 & v_3 & \ldots & v_1 \end{pmatrix}, \ldots, \begin{pmatrix} v_1 & v_2 & \ldots & v_i \\ v_i & v_1 & \ldots & v_{i-1} \end{pmatrix}$$

is called a *cyclic permutations*. For a cyclic permutation $\sigma = \begin{pmatrix} x_1 & x_2 & \ldots & x_l \\ x_{i_1} & x_{i_2} & \ldots & x_{i_l} \end{pmatrix}$, we define the *length* by $|\sigma| = l$. Note that a cyclic permutation with length 2 is called a *transposition*.

Proposition 4.1.3. *For a graph G with $|V(G)| = n$, let ρ_j be the same as in (4.1). Then we have $\rho_n = 1, \rho_{n-1} = 0$ and*

$$\rho_{n-j} = \sum_{\substack{i_1 + i_2 + \cdots + i_\gamma = j \\ 2 \leq i_k \leq n}} (-1)^\gamma \left(\sum_{\substack{[V^{(k)}] \in E_{i_k} \\ V^{(k)} \cap V^{(k')} = \phi}} \prod_{k=1}^{\gamma} M_{i_k}([V^{(k)}]) \right)$$

for $2 \leq j \leq n$.

Proof. It is clear that $\rho_n = 1$ since $\varphi(x)$ is a monic polynomial. Now, ρ_{n-j} consists of j constants and $n - j$ variables. Here, there is no permutation which contributes to x^{n-1} since $T_{u,u} = 0$ for $u \in V(G)$. Hence, we have $\rho_{n-1} = 0$. For $2 \leq j \leq n$, the term x^{n-j} is contributed by combinations of distinct permutations such that the summation over their length is j. Then the permutation σ is expressed by

$$\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_\gamma,$$

where σ_k is a permutation on $V^{(k)} = \{v_1^{(k)}, v_2^{(k)}, \dots, v_{i_k}^{(k)}\}$ with $V^{(k)} \cap V^{(k')} = \phi$ for $1 \leq k, k' \leq \gamma$ and $\sum_{k=1}^{\gamma} |\sigma_k| = j$. Since it holds that $T_{v,\sigma_k(v)} \neq 0$ for every $1 \leq k \leq \gamma$ and $v \in V^{(k)}$, we have $v_1^{(k)} \sim \sigma(v_1^{(k)}) \sim \dots, \sim \sigma^{i_k-1}(v_1^{(k)}) \sim v_1^{(k)}$. Thus, σ_k is a cyclic permutation on $V^{(k)}$. For $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_\gamma$, we have

$$\rho_{n-j} = \sum_{\substack{i_1+i_2+\cdots+i_\gamma=j \\ 2 \leq i_k \leq n}} \mathrm{sgn}(\sigma) \sum_{\substack{[V^{(k)}] \in E_{i_k} \\ V^{(k)} \cap V^{(k')}}} \prod_{k=1}^{\gamma} \prod_{v \in V^{(k)}} Y_{v,\sigma_k(v)}$$

by (4.8). Here, it follows that

$$\prod_{v \in V^{(k)}} Y_{v,\sigma_k(v)} = \prod_{v \in V^{(k)}} (-1)^{i_k} \frac{1}{\deg(\sigma_k(v))} = (-1)^{i_k} M_{i_k}([V^{(k)}])$$

since σ_k is a cyclic permutation. Moreover, we have

$$\mathrm{sgn}(\sigma) = \prod_{k=1}^{\gamma} \mathrm{sgn}(\sigma_k) = \prod_{k=1}^{\gamma} (-1)^{i_k+1} = (-1)^{j+\gamma}.$$

Therefore, we have

$$\begin{aligned}
\rho_{n-j} &= \sum_{\substack{i_1+i_2+\cdots+i_\gamma=j \\ 2 \leq i_k \leq n}} (-1)^{j+\gamma} \sum_{\substack{[V^{(k)}] \in E_{i_k} \\ V^{(k)} \cap V^{(k')} = \phi}} \prod_{k=1}^{\gamma} (-1)^{i_k} M_{i_k}([V^{(k)}]) \\
&= \sum_{\substack{i_1+i_2+\cdots+i_\gamma=j \\ 2 \leq i_k \leq n}} (-1)^{j+\gamma} \cdot (-1)^{j} \sum_{\substack{[V^{(k)}] \in E_{i_k} \\ V^{(k)} \cap V^{(k')} = \phi}} \prod_{k=1}^{\gamma} M_{i_k}([V^{(k)}]) \\
&= \sum_{\substack{i_1+i_2+\cdots+i_\gamma=j \\ 2 \leq i_k \leq n}} (-1)^{\gamma} \left(\sum_{\substack{[V^{(k)}] \in E_{i_k} \\ V^{(k)} \cap V^{(k')} = \phi}} \prod_{k=1}^{\gamma} M_{i_k}([V^{(k)}]) \right)
\end{aligned}$$

and we complete the proof. □

4.2 Join of Grover-periodic graphs

For two graphs G and H, we construct a new graph by the *join* of G and H with respect to two vertices $u \in V(G)$ and $v \in V(H)$. The join in this paper is defined by identifying

u and v. Then the graph is denoted by $G^{(u)} * H^{(v)}$, that is,

$$V(G^{(v)} * H^{(v')}) := (V(G)\backslash\{v\}) \cup (V(H)\backslash\{v'\}) \cup \{u\},$$
$$E(G^{(v)} * H^{(v')}) := E(G) \cup E(H).$$

If there is no danger of confusion, omitting the superscripts we write $G * H$.

4.2.1 Join of two cycles

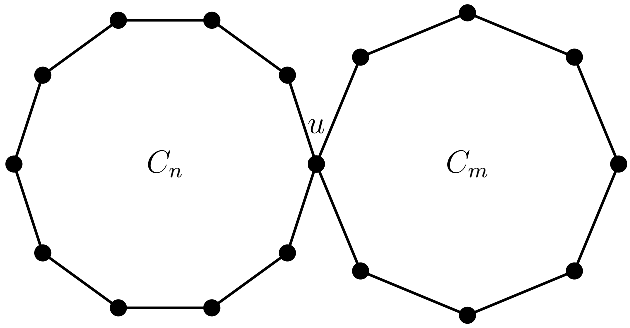


Figure 4.1: The graph $C_n * C_m$

Theorem 4.2.1. *For any $n, m \in \mathbb{N}$ with $n, m \geq 3$, the graph $C_n * C_m$ is Grover-periodic and its period is $lcm(n, m, n+m)$.*

Proof. We first consider the case where $n \neq m$. Let

$$V(C_n) = \{v_0, v_1, v_2, \cdots, v_{n-1}\}, \quad V(C_m) = \{w_0, w_1, w_2, \cdots, w_{m-1}\},$$

and identify $v_0 = w_0 = u$. Let $G = C_n^{(v_0)} * C_m^{(w_0)}$. We will find all the linearly independent eigenfunctions of $T = T(G)$ and show that their eigenvalues are the real parts of a root of unity. We need to find $|V(G)| = m + n - 1$ linearly independent eigenfunctions. Let $f \in \ell^2(V(G))$ be an eigenfunction of $T = T(G)$ associated to an eigenvalue λ. We put $\lambda = \cos\theta$ for $0 < \theta < \pi$. Then $Tf = \lambda f$ reduces to the following system of equations:

$$\lambda f(v_i) = \frac{1}{2}\left(f(v_{i-1}) + f(v_{i+1})\right), \qquad 1 \leq i \leq n \tag{4.9}$$
$$\lambda f(w_j) = \frac{1}{2}\left(f(w_{j-1}) + f(w_{j+1})\right), \qquad 1 \leq j \leq m \tag{4.10}$$
$$\lambda f(u) = \frac{1}{4}\left(f(v_1) + f(v_{n-1}) + f(w_1) + f(w_{m-1})\right), \tag{4.11}$$

where $v_n = w_m = u$. Then it follows from Lemma 2.2.1 that

$$f(v_i) = U_{i-1}(\lambda)f(v_1) - U_{i-2}(\lambda)f(u), \qquad 1 \leq i \leq n \tag{4.12}$$
$$f(w_j) = U_{j-1}(\lambda)f(w_1) - U_{j-2}(\lambda)f(u), \qquad 1 \leq j \leq m. \tag{4.13}$$

Setting $i = n$ and $j = m$ in the above equations, we have $f(u) = U_{n-1}f(v_1) - U_{n-2}f(u)$ and $f(u) = U_{m-1}f(w_1) - U_{m-2}f(u)$. Thus, we have

$$f(v_1) = \frac{1 + U_{n-2}}{U_{n-1}} f(u), \tag{4.14}$$

and

$$f(w_1) = \frac{1 + U_{m-2}}{U_{m-1}} f(u), \tag{4.15}$$

if $U_{n-1} \neq 0$ and $U_{m-1} \neq 0$, that is,

$$\theta \neq \frac{\pi}{n} r, \quad 0 \leq r \leq n,$$

and

$$\theta \neq \frac{\pi}{m} s, \quad 0 \leq s \leq m,$$

Now, $f(x)$ is expressed in terms of $f(u)$ by (4.12), (4.13), (4.14) and (4.15) for $x \in V(G)$. Without loss of generality, we set $f(u) = 1$. We obtain

$$\begin{aligned} f(v_{n-1}) &= U_{n-2}f(v_1) - U_{n-3}f(u) \\ &= \frac{U_{n-2}(1 + U_{n-2})}{U_{n-1}} - U_{n-3} \\ &= \frac{U_{n-2} + U_{n-2}^2 - U_{n-3}U_{n-1}}{U_{n-1}} \\ &= \frac{U_{n-2} + 1}{U_{n-1}}, \end{aligned} \tag{4.16}$$

where we applied (iii) of Lemma 2.2.2 to the third equation. Similarly, we obtain

$$f(w_{m-1}) = \frac{U_{m-2} + 1}{U_{m-1}}. \tag{4.17}$$

Inserting (4.14), (4.15), (4.16) and (4.17) to (4.11), we have

$$\begin{aligned} & 4\lambda = \left(\frac{1 + U_{n-2}}{U_{n-1}} + \frac{1 + U_{n-2}}{U_{n-1}} + \frac{1 + U_{m-1}}{U_{m-1}} + \frac{1 + U_{m-1}}{U_{m-1}} \right) \\ & 2\lambda = \frac{1 + U_{n-2}}{U_{n-1}} + \frac{1 + U_{m-2}}{U_{m-1}} \\ & \lambda - \frac{1 + U_{n-2}}{U_{n-1}} + \lambda - \frac{1 + U_{m-2}}{U_{m-1}} = 0 \\ & \frac{\lambda U_{n-1} - U_{n-2} - 1}{U_{n-1}} + \frac{\lambda U_{m-1} - U_{m-2} - 1}{U_{m-1}} = 0 \\ & \frac{T_n - 1}{U_{n-1}} + \frac{T_m - 1}{U_{m-1}} = 0, && (4.18) \\ & U_{m-1}(T_n - 1) + U_{n-1}(T_m - 1) = 0, && (4.19) \end{aligned}$$

where we applied (i) of Lemma 2.2.2 to the fourth equation. Put $\lambda = \cos\theta$. Then it follows that

$$\begin{aligned}
&\frac{\sin m\theta}{\sin\theta}(\cos n\theta - 1) + \frac{\sin n\theta}{\sin\theta}(\cos m\theta - 1) = 0,\\
&2\sin\frac{(m+n)\theta}{2}\cos\frac{(m+n)\theta}{2} - 2\sin\frac{(m+n)\theta}{2}\cos\frac{(m-n)\theta}{2} = 0,\\
&-2\sin\frac{(m+n)\theta}{2}\sin\frac{n\theta}{2}\sin\frac{m\theta}{2} = 0.
\end{aligned}$$

Now, we suppose that $\theta \neq \frac{\pi}{n}r$ and $\frac{\pi}{m}s$ for $0 \leq r \leq n$ and $0 \leq s \leq m$. Then it follows that

$$\sin\frac{(m+n)\theta}{2} = 0,$$

that is,

$$\theta = \frac{2\pi}{m+n}k, \quad 1 \leq k \leq \left\lfloor \tfrac{m+n-1}{2} \right\rfloor.$$

Therefore, we have found $\lfloor \frac{m+n-1}{2} \rfloor$ eigenvalues and associated linearly independent eigenfunctions of T.

Now, we suppose that $U_{n-1}(\lambda) = 0$, or $U_{m-1}(\lambda) = 0$. A zero of U_{n-1} is written as

$$\lambda_r = \cos\frac{\pi}{n}r, \quad 0 < r < n. \tag{4.20}$$

Remark that $U_{n-2}(\lambda_r) = (-1)^{r+1}$. For an even r as in (4.20), let us define $g_r \in \ell^2(V(G))$ as follows:

$$g_r(v) = \begin{cases} U_{i-1}(\lambda_r), & \text{if } v = v_i,\\ 0 & v = u \end{cases} \tag{4.21}$$

for $1 \leq i \leq n-1$. We show that g_r is an eigenfunction of T associated to λ_r for an even r. It is enough to show that $(Tg_r)(u) = \lambda_r g(u)$ since it clearly holds that $(Tg_r)(x) = \lambda_r g_r(x)$ for $x \in V(G)\backslash\{u\}$ by the definition of the Chebyshev polynomials. In fact, we have

$$\begin{aligned}
(Tg_r)(u) &= \frac{1}{4}(f(v_1) + f(v_{n-1}) + f(w_1) + f(w_{m-1}))\\
&= \frac{1}{4}(U_0(\lambda_r) + U_{n-2}(\lambda_r))\\
&= \frac{1}{4}(1-1)\\
&= 0\\
&= \lambda_r g_r(u).
\end{aligned}$$

Then g_r is an eigenfunction of T associated to λ_r for an even r and it satisfies that $\mathrm{supp}(g_r) \subset V(C_r)$. In this paper, we call an eigenfunction in the form as in (4.21) a *persistent eigenfunction* on C_n. Hence, we have found $\lfloor \frac{n-1}{2} \rfloor$ other linearly independent eigenfunctions. By a similar procedure, we also find $\lfloor \frac{m-1}{2} \rfloor$ linearly independent eigenfunctions of T associated to $\cos \frac{\pi}{m} s$ for an even s with $0 < s < m$, which are persistent eigenfunctions on C_m. Note that G is a bipartite graph if both of m, n are even. Then we have $1, -1 \in \sigma(T)$ if both of m and n are even, and $1 \in \sigma(T), -1 \notin \sigma(T)$ otherwise. Hence, the number of the linearly independent eigenfunctions that we have found so far is

$$\left\lfloor \frac{m+n-1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 + \mathbf{1}_{\mathbf{B}}(G) = m + n - 1,$$

which coincides with $|V(G)|$. Therefore, we have found all the linearly independent eigenfunctions of T. Moreover, the eigenvalues of T are

$$\lambda = \cos \frac{2\pi}{m+n} k, \quad \cos \frac{\pi}{n} r, \quad \cos \frac{\pi}{m} s,$$

where the ranges of k and even r, s are as mentioned in above. Obviously, these are the real parts of the $m+n, n$ and m-th root of unity, respectively. Therefore, $C_n * C_m$ is Grover-periodic by Proposition 3.2.3 and its period is $\mathrm{lcm}(n, m, m+n)$.

We next consider the case where $n = m$. By a similar procedure, g_r as in (4.21) becomes a persistent eigenfunction on each cycle associated to $\lambda_r = \cos \frac{\pi}{n} r$. We have thus found $2\lfloor \frac{n-1}{2} \rfloor$ linearly independent eigenfunctions. For an even r with $0 < r < n$ let us define $\hat{g}_r \in \ell^2(V(G))$ as follows:

$$\hat{g}_r(v) = \begin{cases} T_i(\lambda_r), & \text{if } v = v_i \text{ or } v = w_i, \\ 1 & v = u \end{cases} \tag{4.22}$$

for $1 \le i \le n-1$. We show that $\hat{g}_r$ is another eigenfunction of $T(G)$ associated to λ_r for an even r. Remark that $T_{n-1}(\lambda_r) = (-1)^r \lambda_r$. Similarly, it is enough to show that $(T\hat{g}_r)(u) = \lambda_r \hat{g}_r$. In fact, we have

$$\begin{aligned} (T\hat{g}_r)(u) &= \frac{1}{4}(f(v_1) + f(v_{n-1}) + f(w_1) + f(w_{n-1})) \\ &= \frac{1}{4}(T_1(\lambda_r) + T_{n-1}(\lambda_r) + T_1(\lambda_r) + T_{n-1}(\lambda_r)) \\ &= \lambda_r \\ &= \lambda_r \hat{g}_r(u). \end{aligned}$$

Then $\hat{g}_r$ is an eigenfunction of $T(G)$ associated to λ_r for an even r and we have thus found $\lfloor \frac{n-1}{2} \rfloor$ another linearly independent eigenfunctions. Furthermore, for an odd r with $0 < r < n$ let us define $h_r \in \ell^2(V(G))$ as follows:

$$h_r(v) = \begin{cases} U_{i-1}(\lambda_r), & \text{if } v = v_i, \\ -U_{i-1}(\lambda_r), & \text{if } v = w_i, \\ 0 & v = u \end{cases} \tag{4.23}$$

for $1 \leq i \leq n-1$. Similarly, it is enough to show that $(Th_r)(u) = \lambda_r h_r(u)$. By $U_{n-2}(\lambda_r) = (-1)^{r+1}$, we have

$$\begin{aligned}(Th_r)(u) &= \frac{1}{4}(f(v_1) + f(v_{n-1}) + f(w_1) + f(w_{n-1})) \\ &= \frac{1}{4}(U_0(\lambda_r) + U_{n-2}(\lambda_r) - U_0(\lambda_r) - U_{n-2}(\lambda_r)) \\ &= 0 \\ &= \lambda_r h_r(u).\end{aligned}$$

Then we have thus found $\lfloor \frac{n}{2} \rfloor$ another linearly independent eigenfunctions of $T(G)$ associated to λ_r for an odd r. Therefore, the number of linearly independent eigenfunctions that we have obtained is

$$2\left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1 + \mathbf{1}_B(G) = 2n-1,$$

which coincides with $|V(G)|$. Thus, we have exhausted all the linearly independent eigenfunctions of $T(G)$ and the eigenvalues are

$$\lambda = \cos \frac{\pi}{n} r, \quad 0 \leq r \leq n.$$

Hence, G is Grover-periodic by Proposition 3.2.3 and the period is $2n = \mathrm{lcm}(n, m, n+m)$, which completes the proof. $\square$

4.2.2 Join of several cycles

In this subsection, we consider the join of cycle graphs $C_{N_1}, C_{N_2}, \ldots, C_{N_n}$ with $n \geq 3$ by identifying a single vertex. Let $V(C_{N_i}) = \{v_j^{(i)} \mid 0 \leq j \leq N_i - 1\}$ with $v_0 = v_{N_i}$ for $1 \leq i \leq n$. Then we consider the periodicity of the Grover walk on $G = C_{N_1}^{v_0^{(1)}} * C_{N_2}^{v_0^{(2)}} * \cdots * C_{N_n}^{v_0^{(n)}}$.

Theorem 4.2.2. *Let $n \geq 3$ and $3 \leq N_1 \leq N_2 \leq \cdots \leq N_n$. Then $G = C_{N_1}^{(v_0^{(1)})} * C_{N_2)}^{(v_0^{(2)}} * \cdots * C_{N_n)}^{(v_0^{(n)}}$ is Grover-periodic if and only if $N_1 = N_2 = \cdots = N_n = N$, or n is even, $N_1 = N_2 = \cdots = N_{\frac{n}{2}} = N$ and $N_{\frac{n}{2}+1} = N_{\frac{n}{2}+2} = \cdots = N_n = M$ whose periods are $2N$ and $\mathrm{lcm}(N, M, N+M)$, respectively.*

Proof. We first consider the case where $N_1 = N_2 = \cdots = N_n = N$. Now, we find persistent eigenfunctions as in (4.21) on each cycle associated to $\lambda_r = \cos \frac{\pi}{N} r$ for an even r with $0 < r < N$. We have thus found $n \lfloor \frac{N-1}{2} \rfloor$ linearly independent eigenfunctions of $T(G)$. For λ_r, let us define $\hat{g}_r, h_r^{(i)} \in \ell^2(V(G))$ as follows:

$$\hat{g}_r(v_j^{(i)}) = T_j(\lambda_r), \quad 1 \leq i \leq n, \quad 1 \leq j \leq N$$

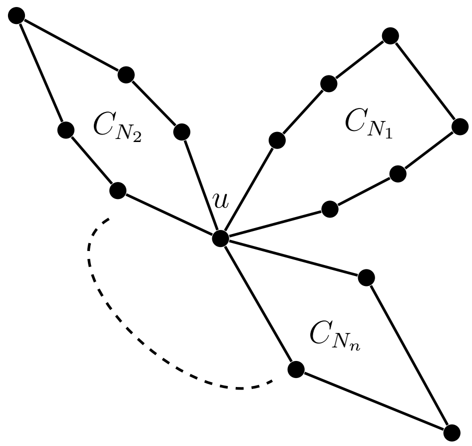


Figure 4.2: A join of cycles

and

$$h_r^{(i)}(v) = \begin{cases} U_{j-1}(\lambda_r) & \text{if } v = v_j^{(1)}, \\ -U_{j-1}(\lambda_r) & \text{if } v = v_j^{(i)}, \\ 0 & v = u \end{cases}$$

for $2 \leq i \leq n$ and $0 \leq j \leq N_i - 1$. By a similar way as in the proof of Theorem 4.2.1, it is checked that $\hat{g}_r$ and $h_r^{(i)}$ are eigenfunctions of $T(G)$ associated to λ_r for an even and an odd r, respectively. Then we have thus found $\lfloor \frac{N-1}{2} \rfloor + (n-1)\lfloor \frac{N}{2} \rfloor$ another linearly independent eigenfunctions. The remaining eigenfunctions are ones associated to 1 if G is a non-bipartite graph, and ± 1 otherwise. After all, the number of linearly independent eigenfunctions that we have found is

$$n \left\lfloor \frac{N-1}{2} \right\rfloor + \left\lfloor \frac{N-1}{2} \right\rfloor + (n-1) \left\lfloor \frac{N}{2} \right\rfloor + 1 + \mathbf{1}_{\mathbf{B}}(G) = n(N-1) + 1,$$

which coincides with $|V(G)|$. Moreover, the eigenvalues of T are

$$\lambda = \cos \frac{\pi}{N} k, \quad 0 \leq k \leq N.$$

Thus, G is Grover-periodic and the period is $2N$ by Proposition 3.2.3.

We next consider the case where $N_1 = N_2 = \cdots = N_n$ does not hold. Let M_1 be the minimum length of the cycles and define

$$l_1 := |\{1 \leq i \leq n \mid N_i = M_1\}|.$$

Note that $1 \leq l_1 < n$. Recall that

$$\det(xI_{|V(G)|} - T) = \sum_{i=0}^{|V(G)|} \rho_i x^i.$$

Here, we consider $\rho_{|V(G)|-M_1}$. If M_1 is odd, the permutations which contribute to $\rho_{|V(G)|-M_1}$ are the cyclic permutations with length M_1 on each cycle with length M_1 because of the minimality of M_1. Since $\deg u = 2n$, we have

$$\begin{aligned}\rho_{|V(G)|-M_1} &= -2l_1 \cdot \frac{1}{2^{M_1-1}} \cdot \frac{1}{\deg u} \\ &= -\frac{1}{2^{M_1-1}n} l_1\end{aligned}$$

by Proposition 4.1.3. In order that G is Grover-periodic, it necessarily holds that

$$2^{M_1}\rho_{|V(G)|-M_1} = -\frac{2}{n} l_1 \in \mathbb{Z}$$

by Theorem 4.1.2. Since $l_1 < n$, we have $l_1 = \frac{n}{2}$.

If M_1 is even, the permutations which contribute to $\rho_{|V(G)|-M_1}$ are cyclic permutations with length M_1 or products of $\frac{M_1}{2}$ transpositions because of the minimality of M_1. Note that the product of transpositions corresponds to a combination of disjoint $\frac{M_1}{2}$ edges on G. Define $B_u := \{e \in E(G) \mid e \approx u\}$. If a combination of disjoint edges does not involve an edge in B_u, this permutation yields

$$\underbrace{\frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{\frac{M_1}{2}} = \frac{1}{2^{M_1}}.$$

If a combination of disjoint edges involves $e \in B_u$, this permutation yields

$$\underbrace{\frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{\frac{M_1}{2}-1} \cdot \frac{1}{2\deg u} = \frac{1}{2^{M_1}n}.$$

From the symmetry of G, the number of combinations of disjoint edges involving an edge in B_u is constant for every $e \in B_u$. Let t be the number. Then the number of such combinations is $t|B_u| = 2nt$. Therefore, we have

$$\begin{aligned}\rho_{|V(G)|-M_1} &= 2l_1 \cdot \frac{1}{2^{M_1-1}} \cdot \frac{1}{\deg u} + (-1)^{\frac{M_1}{2}} \frac{1}{2^{M_1}} s + (-1)^{\frac{M_1}{2}} \frac{1}{2^{M_1}n} t|B_u| \\ &= \frac{1}{2^{M_1-1}n} l + (-1)^{\frac{M_1}{2}} \left(\frac{1}{2^{M_1}} s + \frac{1}{2^{M_1-1}} t \right) \end{aligned} \tag{4.24}$$

by Proposition 4.1.3, where s is the number of combinations of disjoint edges which does not involve the edges in B_u. Then the condition $2^{M_1}\rho_{|V(G)|-M_1} \in \mathbb{Z}$ implies that

$$\frac{2}{n} l_1 \in \mathbb{Z}.$$

Thus, we have $l_1 = \frac{n}{2}$.

By the above arguments, it holds that $l_1 = \frac{n}{2}$ for the two cases. Let M_2 be the second minimum length of the cycles, i.e., $M_2 = N_{\frac{N}{2}+1}$ and define

$$l_2 := |\{1 \leq i \leq n \mid N_i = M_2\}|.$$

Here, we only give the proof for an odd M_2 as the one for an odd M_2 is similar. We consider $\rho_{|V(G)|-M_2}$. When M_1 is even, the permutations which contribute to $\rho_{|V(G)|-M_2}$ are the cyclic permutations with length M_2 on each cycle with length M_2. Then we have

$$\begin{aligned}\rho_{|V(G)|-M_2} &= -2l_2 \cdot \frac{1}{2^{M_2-1}} \cdot \frac{1}{\deg u} \\ &= -\frac{1}{2^{M_2-1}n} l_2.\end{aligned}$$

Similarly, it holds that $l_2 = \frac{n}{2}$. When M_1 is odd, the permutations which contribute to $\rho_{|V(G)|-M_2}$ are cyclic permutations with length M_2 or a product of a cyclic permutation with lengths M_1 and $M_2 - M_1$ transpositions. Such a permutation corresponds to a combination of a cycle with length M_1 and $\frac{M_2-M_1}{2}$ disjoint edges on G. Then this permutation yields

$$\underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{M_1-1} \cdot \frac{1}{2\deg u} \cdot \underbrace{\frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{\frac{M_1-M_2}{2}} = \frac{1}{2^{M_2}n}.$$

Due to the symmetry of G, the number of such combinations is constant for each cycle with length M_1. Let s' be the number. Then the total of such combinations is $2s'l_1 = s'n$. Thus, we have

$$\begin{aligned}\rho_{|V(G)|-M_2} &= -2l_2 \cdot \frac{1}{2^{M_2-1}} \cdot \frac{1}{\deg u} + (-1)^{\frac{M_2-M_1}{2}+1} \frac{1}{2^{M_2}n} 2s'l_1 \\ &= -\frac{1}{2^{M_2-1}n} l_2 + (-1)^{\frac{M_2-M_1}{2}+1} \frac{1}{2^{M_2}} s'\end{aligned}$$

Since the last term becomes an integer after multiplying 2^{M_2}, it follows that $l_2 = \frac{n}{2}$. Therefore, we have n is even and

$$l_1 = l_2 = \frac{n}{2}.$$

Now, we set $l_1 = l_2 = \frac{n}{2}$. Let $\lambda = \cos\theta$ with $0 < \theta < \pi$ be an eigenvalue of $T = T(G)$. By a similar way as the proof of Theorem 4.2.1, we have

$$U_{M_2-1}(\lambda)(T_{M_1}(\lambda) - 1) + U_{M_1-1}(\lambda)(T_{M_2}(\lambda) - 1) = 0,$$

that is

$$\theta = \frac{2\pi}{M_1 + M_2} k, \quad 0 < k < M_1 + M_2$$

if $U_{M_1-1} \neq 0$ and $U_{M_2-1} \neq 0$. Similarly, we have thus found $\lfloor \frac{M_1+M_2}{2} \rfloor$ linearly independent eigenfunctions. I addition, we find persistent eigenfunctions as in (4.21) on each cycle associated to $\cos \frac{\pi}{M_1} r$ and $\cos \frac{\pi}{M_2} r'$, where r and r' are even with $0 < r < M_1$ and $0 < r' < M_2$. Hence, the number of the persistent eigenfunctions is

$$\sum_{i=1}^{2} l_i \left\lfloor \frac{M_i - 1}{2} \right\rfloor.$$

Let $\lambda_r = \cos \frac{\pi}{M_1}$ and $\mu_{r'} = \cos \frac{\pi}{M_2} r'$ for $0 < r < M_1$ and $0 < r' < M_2$, respectively. Put $\hat{n} = \frac{n}{2}$. We define $h_r^{(i)}, \tilde{h}_{r'}^{(i)} \in \ell^2(V(G))$ as follows:

$$h_r^{(i)}(v) = \begin{cases} U_{j-1}(\lambda_r) & \text{if } v = v_j^{(1)}, \\ -U_{j-1}(\lambda_r) & \text{if } v = v_j^{(i)}, \\ 0 & \text{otherwise.} \end{cases}$$

for $2 \leq i \leq \hat{n}$ and $0 \leq j \leq N_i - 1$ and

$$\tilde{h}_{r'}^{(i)}(v) = \begin{cases} U_{j-1}(\mu_{r'}) & \text{if } v = v_j^{(\hat{n}+1)}, \\ -U_{j-1}(\mu_{r'}) & \text{if } v = v_j^{(i)}, \\ 0 & \text{otherwise.} \end{cases}$$

for $\hat{n}+2 \leq i \leq n$ and $0 \leq j \leq N_i - 1$. Similarly, it is checked that these are eigenfunctions of T associated to λ_r with odd r and $\mu_{r'}$ with odd r', respectively. We have thus found

$$\sum_{i=1}^{2} (l_i - 1) \left\lfloor \frac{M_i}{2} \right\rfloor$$

linearly independent eigenfunctions. Accordingly, the number of the linearly independent eigenfunctions of T is

$$\left\lfloor \frac{M_1 + M_2}{2} \right\rfloor + \sum_{i=1}^{2} \left(l_i \left\lfloor \frac{M_i - 1}{2} \right\rfloor + (l_i - 1) \left\lfloor \frac{M_i}{2} \right\rfloor \right) + 1 + \mathbf{1}_{\mathbf{B}}(G) = l_1(M_1 - 1) + l_2(M_2 - 1) + 1,$$

which coincides with $|V(G)|$. Moreover, the eigenvalues that we have obtained so far are given by

$$\lambda = \cos \frac{2\pi}{M_1 + M_2} k, \quad \cos \frac{\pi}{M_1} r, \quad \cos \frac{\pi}{M_2} r'$$

for $1 \leq k \leq \lfloor \frac{M_1+M_2}{2} \rfloor$, $1 \leq r \leq M_1 - 1$, and $1 \leq r' \leq M_2 - 1$.

Therefore, G is Grover-periodic if and only if $N_1 = N_2 = \ldots N_n = N$ or $N_1 = N_2 = \cdots = N_{\frac{n}{2}} = M_1$ and $N_{\frac{n}{2}+1} = N_{\frac{n}{2}+2} = \cdots = N_n = M_2$ for an even n whose periods are $2N$ and $\mathrm{lcm}(M_1 + M_2, 2M_1, 2M_2)$, respectively.

$\square$

4.2.3 Join of a cycle and a claw

Let $r \geq 2$ and consider a claw $K_{1,r}$ with $V(K_{1,r}) = \{u\} \cup \{w_i \mid 1 \leq i \leq r\}$ and $E(K_{1,r}) = \{uw_i \mid 1 \leq i \leq r\}$. We call u the *center* of a claw. We will consider the join $C_n * K_{1,r}^{(u)}$.

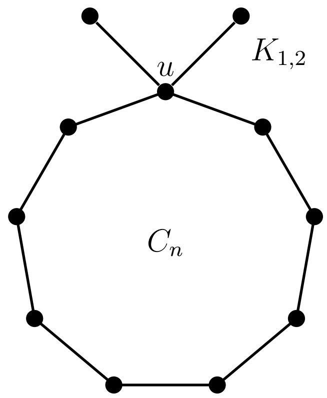


Figure 4.3: The graph $C_n * K_{1,2}^{(u)}$

Theorem 4.2.3. *Let $n \in \mathbb{N}$ and $r \geq 2$. The join $C_n * K_{1,r}^{(u)}$ is Grover periodic if and only if $r = 2$ and the period is* $\operatorname{lcm}(n, n+2, 4)$.

Proof. Let $G = C_n * K_{1,r}^{(u)}$ and ρ_j be defined as in (4.1) for $1 \leq j \leq |V(G)|$. Now the cyclic permutations with length n only contribute to $\rho_{|V(G)|-n}$, which implies that

$$\begin{aligned}\rho_{|V(G)|-n} &= -2 \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n-1} \cdot \frac{1}{\deg u} \\ &= -\frac{1}{2^{n-2}(r+2)}\end{aligned}$$

by Proposition 4.1.3. Since it holds that $2^n \rho_{|V(G)|-n} \in \mathbb{Z}$ by Theorem 4.1.2, we have $r = 2$.

We set $r = 2$. Let $V(C_n) = \{u, v_1, \ldots, v_{n-1}\}$ where we set $v_n = u$. Then we have $|V(G)| = n + 2$. Let $f \in \ell^2(V(G))$ be an eigenfunction of $T = T(G)$ for λ. By a similar procedure as in the proof of Theorem 4.2.2, we have

$$\lambda f(v_i) = \frac{1}{2}\left(f(v_{i-1}) + f(v_{i+1})\right), \tag{4.25}$$

$$\lambda f(u) = \frac{1}{4}\left(f(v_1) + f(v_{n-1}) + f(w_1) + f(w_2)\right), \tag{4.26}$$

$$\lambda f(w_1) = f(u),$$

$$\lambda f(w_2) = f(u),$$

for $1 \leq i \leq n$. If $\lambda \neq \pm 1$, we similarly obtain

$$f(v_i) = U_{i-1} f(v_1) - U_{i-2} f(u), \quad 1 \leq i \leq n$$

by (4.25). Under the assumption of $U_{n-1} \neq 0$, $f(v_1)$ is expressed in terms of $f(u)$ as (4.14):

$$f(v_1) = \frac{1 + U_{n-2}}{U_{n-1}}.$$

In addition, it follows that $f(v_{n-1}) = f(v_1)$ by (4.16). Furthermore, we suppose that $\lambda \neq 0$. Then it follows that

$$f(w_1) = f(w_2) = \frac{1}{\lambda} f(u). \tag{4.27}$$

Inserting (4.14), (4.16) and (4.27) to (4.26) and putting $f(u) = 1$, we have

$$\begin{aligned}
&4\lambda = \left(\frac{1 + U_{n-2}}{U_{n-1}} + \frac{1 + U_{n-2}}{U_{n-1}} + \frac{1}{\lambda} + \frac{1}{\lambda} \right) \\
&2\lambda = \frac{1 + U_{n-2}}{U_{n-1}} + \frac{1}{\lambda} \\
&2\lambda^2 U_{n-1} = \lambda(1 + U_{n-1}) + U_{n-1} \\
&\lambda(2\lambda U_{n-1} - U_{n-2}) = U_{n-1} + \lambda \\
&\lambda U_n - U_{n-1} = \lambda. \\
&T_{n+1} = \lambda,
\end{aligned}$$

where we applied (ii) of Lemma 2.2.2 to the fifth equation. Putting $\lambda = \cos\theta$ with $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$, it follows that

$$\begin{aligned}
&\cos{(n+1)\theta} - \cos\theta = 0, \\
&-2\sin\frac{(n+2)\theta}{2}\sin\frac{n\theta}{2} = 0.
\end{aligned}$$

Since $U_{n-1} \neq 0$, it follows that

$$\sin\frac{(n+2)\theta}{2} = 0, \tag{4.28}$$

that is,

$$\theta = \frac{2\pi}{n+2}s, \quad 0 < s < \tfrac{n+2}{2}.$$

and $s \neq \frac{n+2}{4}$. When $n \in 4\mathbb{N} - 2$, θ in the above may take 0. Now, we consider the following three cases (i) n is odd, (ii) n is even but $n \notin 4\mathbb{N} - 2$, (iii) $n \in 4\mathbb{N} - 2$.

For the case of (i), we find $\frac{n+1}{2}$ linearly independent eigenfunctions of T associated to the eigenvalues as in (4.28). In addition, we find $\frac{n-1}{2}$ persistent eigenfunctions on C_n given as in (4.21). Moreover, we define $g \in \ell^2(V(G))$ as follows:

$$g(v) = \begin{cases} 1 & \text{if } v = w_1, \\ -1 & \text{if } v = w_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

Indeed, it is an eigenfunction of $T(G)$ associated to 0. We call the above eigenfunction a persistent eigenfunction on $K_{1,2}$. Then the remaining eigenfunctions are ones associated to 1 since G now is a non-bipartite graph. Therefore, the number of the linearly independent eigenfunctions that we have found so far is

$$\frac{n+1}{2} + \frac{n-1}{2} + 1 + 1 = n + 2,$$

which coincides with $|V(G)|$.

For the case of (ii), we similarly find $\frac{n}{2}$ linearly independent eigenfunctions of T associated to the eigenvalues as in (4.28). In addition, we find the persistent eigenfunctions on C_n and $K_{1,2}$ as in (4.21) and (4.29), respectively. Note that G is a bipartite graph. Then the remaining eigenfunctions are ones associated to ± 1. Therefore, the number of the linearly independent eigenfunctions that we have found is

$$\frac{n}{2} + 2 + \frac{n-2}{2} + 1 = n + 2,$$

which coincides with $|V(G)|$.

For the case of (iii), we find $\frac{n}{2} - 1$ linearly independent eigenfunctions of T for the eigenvalues as in (4.28) and the persistent ones on C_n and $K_{1,r}$. In addition, we set the following $\tilde{g} \in \ell^2(V(G))$: $\tilde{g}(w_1) = -1, \tilde{g}(w_2) = -1$ and

$$\tilde{g}(v_i) = \begin{cases} 0 & \text{if } i \text{ is even}, \\ 1 & \text{if } i \in 4\mathbb{N} + 1, \\ -1 & \text{if } i \in 4\mathbb{N} + 3. \end{cases}$$

Indeed, $\tilde{g}$ becomes an eigenfunction of $T(G)$ associated to 0. Since G is a bipartite graph, the remaining eigenfunctions are ones for ± 1. Therefore, the number of the linearly independent eigenfunctions of T that we have found is

$$\left(\frac{n}{2} - 1\right) + 2 + \frac{n-2}{2} + 1 + 1 = n + 2,$$

which coincides with $|V(G)|$.

Therefore, the eigenvalues of T are exhausted as

$$\lambda = \cos \frac{2\pi}{n+2} s, \quad \cos \frac{\pi}{n} r, \quad 0,$$

for all the cases, where the ranges of r and s are as mentioned above. Obviously, these are the real parts of the $n+2, n$ and 4-th root of unity, respectively. Consequently, G is Grover-periodic with the period $\operatorname{lcm}(n, n+2, 4)$ by Proposition 3.2.3. □

4.2.4 Join of several paths

For $n, m \in \mathbb{N}$, we join m path graphs on n vertices by identifying their leaves. Let us denote the graph by $P_{n,m}$. Define the vertex set by

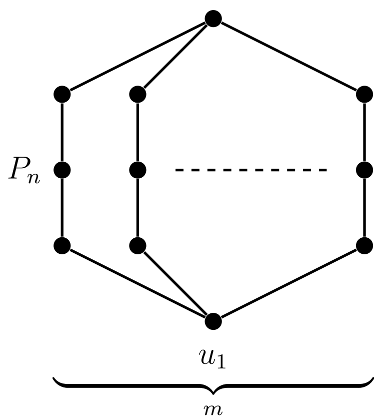


Figure 4.4: The graph $P_{n,m}$

$$V(P_{n,m}) = \{u_0, u_1\} \cup \{w_j^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq n\},$$

with $w_1^{(i)} = u_0$ and $w_n^{(i)} = u_1$ for $1 \leq i \leq m$. Define the edge set by

$$E(P_{n,m}) = \{w_j^{(i)} w_{j+1}^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq n-1\}.$$

Theorem 4.2.4. *The graph $P_{n,m}$ is Grover-periodic for $n, m \in \mathbb{N}$, and the period is $2(n-1)$.*

Proof. Let $G = P_{n,m}$. Here, we will find $|V(G)| = m(n-2)+2$ linearly independent eigenfunctions of $T(G)$. We define functions on $\ell^2(V(G))$ by $\Psi_1 = \delta_{u_0}$, $\Psi_n = \delta_{u_1}$ and

$$\Psi_j = \frac{1}{m} \sum_{i=1}^{m} \delta_{w_j^{(i)}}, 2 \leq j \leq n-1.$$

Let $\mathcal{A}$ be the subset of $\ell^2(V(G))$ spanned by $\Psi_1, \Psi_2, \ldots, \Psi_n$. Then it follows that

$$\mathcal{A}^\perp = \left\{ f \in \ell^2(V(G)) \mid \sum_{i=1}^{n} f(w_j^{(i)}) = 0, 1 \leq j \leq n \right\}.$$

It is easily checked that

$$T(G)\Psi_1 = \Psi_2,$$
$$T(G)\Psi_n = \Psi_{n-1},$$

and

$$T(G)\Psi_j = \frac{1}{2}(\Psi_{j-1} + \Psi_{j+1}), \quad 2 \le j \le n-1.$$

Then $T(G)$ is invariant on $\mathcal{A}$ and we have

$$T(G)|_{\mathcal{A}} = T(P_n).$$

Hence, $\sigma(T(P_n)) \subset \sigma(T(G)))$ and we find n linearly independent functions according to the Theorem 3.5.1.

The remaining eigenvalue of $T(G)$ is

$$\lambda_k = \cos \frac{\pi}{2(n-1)} k$$

for an even k with $0 < k < 2(n-1)$. For an even k with $0 < k < 2(n-1)$ and $1 \le i \le m$, let us define $g_k^{(i)} \in \ell^2(V(G))$ by

$$g_k^{(i)}(v) = \begin{cases} U_{j-2}(\lambda_k) & \text{if } v = w_j^{(1)} \ (2 \le j \le n) \\ -U_{j-2}(\lambda_k) & \text{if } v = w_j^{(i)} \ (2 \le j \le n) \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

for $2 \le j \le n$. Then $g_k^{(i)} \in \mathcal{A}^\perp$. It is easily checked that $T(G)g_k^{(i)}(x) = \lambda_k g_k^{(i)}(x)$ for $x \in V(G) \backslash u_1$. Remark that $U_{n-2}(\lambda_k) = 0$ for an even k. Thus, we have

$$\begin{aligned} T(G)g_k^{(i)}(u_1) &= \frac{1}{m}(g_k^{(i)}(w_{n-1}^{(1)}) + g_k^{(i)}(w_{n-1}^{(i)})) \\ &= \frac{1}{m}(U_{n-3}(\lambda_k) - U_{n-3}(\lambda_k)) \\ &= 0 \\ &= \lambda_k U_{n-2}(\lambda_k) \\ &= \lambda_k g_k^{(i)}(u_1). \end{aligned}$$

Hence, $g_k^{(i)}$ becomes an eigenfunction of $T(G)$ associated to λ_k. We have thus found $(n-2)(m-1)$ linearly independent eigenfunctions of $T(G)$. Therefore, the number of the linearly independent eigenfunctions that we have obtained is

$$n + (n-2)(m-1) = m(n-2) + 2,$$

which coincides with $|V(G)|$. Furthermore, the eigenvalues is

$$\lambda = \cos \frac{\pi}{n-1} k, \quad 0 \le k \le n-1.$$

Therefore, the graph $P_{n,m}$ is Grover-periodic by Proposition 3.2.3 and the period is $2(n-1)$. $\square$

4.3 Grover-odd-periodic graphs

In this Section, we treat the Grover-odd-periodic graphs. We give a simple necessary condition for a graph to be Grover-odd-periodic.

Theorem 4.3.1. *If a graph G is Grover-odd-periodic, then it holds that $b_1(G) = 1$ and $\mathbf{1_B}(G) = 0$.*

Proof. Let G be a Grover-odd-periodic graph. Then it holds that $-1 \notin \sigma(U)$ by Proposition 3.2.1, which implies that $b_1(G) - 1 + \mathbf{1_B}(G) = 0$. Then it follows that $b_1(G) = 1, \mathbf{1_B}(G) = 0$ or that $b_1(G) = 0, \mathbf{1_B}(G) = 1$. If $b_1(G) = 0, \mathbf{1_B}(G) = 1$, then G is a tree. Then G is a bipartite graph and it holds that $-1 \in \sigma(T)$. Hence, it follows that $-1 \in \{e^{\pm i \cos^{-1}(\sigma(T))}\} \subset \sigma(U)$. Therefore, it holds that $b_1(G) = 1$ and $\mathbf{1_B}(G) = 0$. □

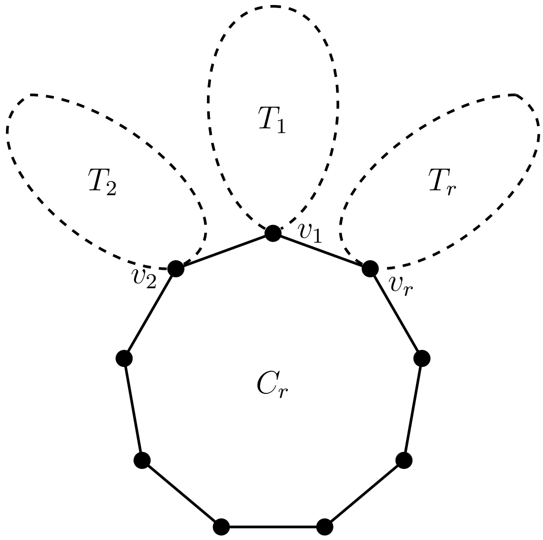


Figure 4.5: An odd-unicycle graph

The conditions $b_1(G) = 1$ and $\mathbf{1_B}(G) = 0$ imply that G is an odd-unicycle graph. Then the graph G has exactly one essential cycle with odd length. Let C_r be the essential cycle with $V(C_r) = \{v_1, v_2, \ldots, v_r\}$, where r is an odd integer. Let T_i be a tree connecting to v_i for $1 \leq i \leq r$(see Figure 4.3). If $T_i = \phi$ for $1 \leq i \leq r$, then $G = C_r$, which is a Grover-periodic graph. Now, we suppose that

$$1 \leq {}^{\exists} i \leq r \quad \text{such that} \quad T_i \neq \phi. \tag{4.31}$$

Theorem 4.3.2. *Let G be an odd-unicycle graph fulfilling (4.31). If G is Grover-periodic, there exists a vertex $v \in V(C_r)$ such that* $\deg(v) = 4$ *and* $\deg(u) = 2$ *for* $u \in V(C_r) \backslash \{v\}$.

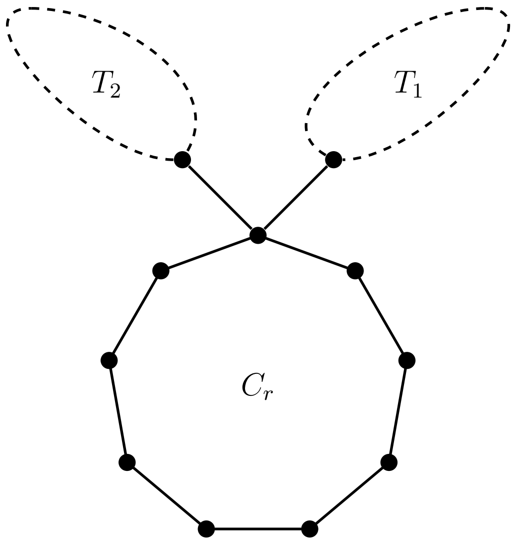


Figure 4.6: A shape of a Grover-odd-periodic graph

Proof. Let $|V(G)| = n$ and ρ_j be defined as in (4.1) for $0 \leq j \leq n$. We consider ρ_{n-r}. Then the permutations which only contribute to ρ_{n-r} are

$$\sigma = \begin{pmatrix} v_1 & v_2 & \dots & v_r \\ v_2 & v_3 & \dots & v_1 \end{pmatrix}, \begin{pmatrix} v_1 & v_2 & \dots & v_r \\ v_r & v_1 & \dots & v_{r-1} \end{pmatrix}$$

and $E_r = \{[(v_1, v_2, \dots, v_r)], [(v_r, v_{r-1}, \dots, v_1)]\}$ since C_r is the unique essential cycle on G and r is odd. Thus, we have

$$\begin{aligned} \rho_{n-r} &= (-1) \cdot (M_r([(v_1, v_2, \dots, v_r)] + M_r([(v_r, v_{r-1}, \dots, v_1)]) \\ &= -2 \prod_{i=1}^{r} \frac{1}{\deg(v_i)} \end{aligned}$$

by Proposition 4.1.3. Then it follows that

$$2^r \rho_{n-r} = -2^{r+1} \prod_{i=1}^{r} \frac{1}{\deg(v_i)} \in \mathbb{Z}$$

by Theorem 4.1.2. Since $\deg(v_i) \geq 2$ for $1 \leq i \leq r$, there exists a vertex $v \in V(C_r)$ such that $\deg(v) = 4$ and $\deg(u) = 2$ for $u \in V(C_r) \backslash \{v\}$. $\square$

Without loss of generality we set $\deg(v_1) = 4$. Thus, we have $T_i = \phi$ for $2 \leq i \leq r$ and a shape of a Grover-odd-periodic graph is seen in Figure 4.3. Along the above argument we come to the following conjecture.

Conjecture 4.3.3. *For an odd $k \in \mathbb{N}$ with $k \geq 3$, a graph G is Grover-k-periodic if and only if $G = C_k$.*

Chapter 5

Graph-transformations preserving periodicity

In this Chapter, we discuss graph-transformations which preserve the periodicity of quantum walks. In particular, we focus on multiplex graphs, subdivision graphs and line graphs induced by a Hoffman graph. In the last part, we study another quantum walk called a *staggered walk*.

5.1 Multiplex graph

Theorem 5.1.1. *Let G be a simple graph. If G is Grover-k-periodic, so is $MP_n(G)$ for $n \in \mathbb{N}$.*

Proof. For $uv \in E(MP_n(G))$, let m_{uv} be the multiplicity of $uv \in E(G)$. Put $M_u = \sum_{v \sim u} m_{uv}$. By definition we have $m_{uv} = n$ for $uv \in E(MP_n(G))$ and $M_u = n \deg_G(u)$. Then it follows by (2.1) that

$$T(MP_n(G)) = T(G).$$

This completes the proof by Lemma 3.2.2. $\square$

5.2 Subdivision

Spectral mappings of several operators, e.g., the adjacency operator and the Laplacian operator by subdivision are referred to in [24], [60]. Below we show the spectral mapping of the transition operator by subdivision and apply it to periodicity of the Grover walk.

Theorem 5.2.1. *Let $k, l \in \mathbb{N}$. If a graph G is Grover-periodic, so is $S_l(G)$.*

Proof. Let G be a Grover-k-periodic graph with $|V(G)| = n$ and $|E(G)| = m$. Put $G' = S_l(G)$. We decompose $V(G')$ into the following disjoint subsets: $V_0 := V(G) = \{v_1, \cdots, v_n\}, V_1 := V(G') \backslash V(G) = \{w_{i,j}^{(r)} \mid 0 \leq r \leq l, v_i v_j \in E(G)\}$, where setting of $w_{i,j}^{(r)}$ is seen in subsection 2.1.2. Namely, V_0 is the vertex set of the original graph G and V_1 is the set of the additional vertices. For the proof, it is sufficient to find $|V(G')| = n + (l-1)m$ linearly independent eigenfunctions of $T(G')$ and verify that all the eigenvalues satisfy the condition in Proposition 3.2.3. Let $f \in \ell^2(V(G'))$ be an eigenfunction of $T(G')$ associated to μ. For $v_i \in V_0$ and $v_j \in V_0$ with $v_iv_j \in E(G)$, it holds that

$$\mu f(v_i) = \frac{1}{\deg(v_i)} \sum_{v_iv_j \in E(G)} f(w_{i,j}^{(1)}), \tag{5.1}$$

$$\mu f(w_{i,j}^{(r)}) = \frac{1}{2}(f(w_{i,j}^{(r-1)}) + f(w_{i,j}^{(r+1)})), \qquad 1 \le r \le l-1. \tag{5.2}$$

Applying Lemma 2.2.1 to (5.2), we have

$$f(w_{i,j}^{(r)}) = U_{r-1}(\mu) f(w_{i,j}^{(1)}) - U_{r-2}(\mu) f(v_i), \quad 1 \le r \le l. \tag{5.3}$$

When $r = l$, we obtain

$$f(w_{i,j}^{(l)}) = U_{l-1}(\mu) f(w_{i,j}^{(1)}) - U_{l-2}(\mu) f(v_i).$$

First, we suppose that $U_{l-1} \neq 0$. Then we have

$$f(w_{i,j}^{(1)}) = \frac{U_{l-2} f(v_i) + f(v_j)}{U_{l-1}}. \tag{5.4}$$

Inserting the above equation to (5.1), we have

$$\begin{aligned}
\mu f(v_i) &= \frac{1}{\deg(v_i)} \sum_{v_iv_j \in E(G)} \frac{U_{l-2} f(v_i) + f(v_j)}{U_{l-1}}, \\
\mu U_{l-1} f(v_i) &= U_{l-2} f(v_i) + \frac{1}{\deg(v_i)} \sum_{v_iv_j \in E(G)} f(v_j), \\
(\mu U_{l-1} - U_{l-2}) f(v_i) &= \frac{1}{\deg(v_i)} \sum_{v_iv_j \in E(G)} f(v_j), \\
T_l f(v_i) &= \frac{1}{\deg(v_i)} \sum_{v_iv_j \in E(G)} f(v_j),
\end{aligned}$$

where we applied (i) of Lemma 2.2.2 to the third equation. Now, we have

$$T_l(\mu) f|_{V_0} = T(G) f|_{V_0}$$

and $T_l(\mu)$ is an eigenvalue of $T(G)$ and $f|_{V_0}$ is the associated eigenfunction. Here, we note that $f|_{V_0} = \mathbf{0}_{V(G)}$ implies $f = \mathbf{0}_{V(G')}$ by (5.3) and (5.4). Then we have $f|_{V_0} \neq \mathbf{0}_{V(G)}$. Thus, there is an eigenvalue $\lambda \in \sigma(T(G))$ such that $T_l(\mu) = \lambda$. Put $\lambda = \cos\theta$ and $\mu = \cos\varphi$ for $0 < \theta, \varphi < \pi$. Then we have

$$\cos l\varphi = \cos\theta,$$

that is,

$$-2\sin\frac{l\varphi+\theta}{2}\sin\frac{l\varphi-\theta}{2}=0.$$

Hence, φ is expressed as

$$\varphi=\frac{\theta+2\pi s}{l},\quad \frac{-\theta+2\pi s'}{l},$$

for

$$0\le s\le\left\lfloor\frac{l-1}{2}\right\rfloor, 1\le s'\le\left\lfloor\frac{l}{2}\right\rfloor.$$

We have thus found l linearly independent eigenfunctions of $T(G')$ for fixed θ with $\theta\in\cos^{-1}(\sigma(T(G))\backslash\{\pm1\})$. If G is a bipartite graph (resp. non-bipartite graph), there are $l(n-2)$ (resp. $l(n-1)$) linearly independent eigenfunctions of $T(G')$.

The remaining eigenvalues are zeros of U_{l-1}, that is,

$$\mu_t=\cos\frac{\pi}{l}t \tag{5.5}$$

for $0<t<l$. For the above μ_t we have $U_{l-2}(\mu_t)=(-1)^t$ and $T_{l-1}(\mu_t)=(-1)^t\mu_t$. First, we suppose that G is a bipartite graph. Let C be an essential cycle on G with $V(C)=\{v_{\nu_1},v_{\nu_2},\cdots,v_{\nu_d}\}$. Note that the lengths of all the essential cycles are even. For an essential cycle C and μ_t in (5.5), let us define a function $g_t^{(C)}\in\ell^2(V(G'))$ by

$$g_t^{(C)}(v)=\begin{cases}(-1)^{tj}U_{r-1}(\mu_t), & \text{if } v=w^{(r)}_{\nu_j,\nu_{j+1}},\\ 0, & \text{otherwise}\end{cases} \tag{5.6}$$

for $1\le j\le d$, $1\le r\le l-1$ and $0<t<l$, where we set $v_{\nu_{d+1}}=v_{\nu_1}$. By $U_{l-2}(\mu_t)=(-1)^t$, it is checked that $g_t^{(C)}$ is an eigenfunction of $T(G')$ for μ_t. Therefore, there are $l-1$ linearly independent eigenfunctions of $T(G')$ for a fixed essential cycle C. Then the number of linearly independent eigenfunctions for (5.6) is $(l-1)(m+n-1)$. Since G is a bipartite graph, V_0 can be decomposed into two disjoint subsets. Name the subsets X and Y. For $0<t<l$, let us define a function $\hat{g}_t\in\ell^2(V(G'))$ by

$$\hat{g}_t(v)=\begin{cases}1, & \text{if } v\in X,\\ (-1)^t, & \text{if } v\in Y,\\ T_r(\mu_t), & \text{if } v=w^{(r)}_{v_i,v_j},\end{cases}$$

for $1\le r\le l-1, 0<t<l$ and every $v_iv_j\in E(G)$ with $v_i\in X, v_j\in Y$. By $T_{l-1}(\mu_t)=(-1)^t\mu_t$, it is easily checked that $\hat{g}\in\ell^2(V(G'))$ is an eigenfunction of $T(G')$ associated to μ_t. We have thus found $l-1$ linearly independent eigenfunctions associated to μ_t. The remaining eigenvalues are ±1 since G' is a bipartite graph as G is a bipartite graph. Then the number of the linearly independent eigenfunctions of $T(G')$ that we have found so far is

$$l(n-2)+(l-1)(m+n-1)+(l-1)+2=n+(l-1)m,$$

which coincides with $|V(G')|$. Therefore, we have exhausted the linearly independent eigenfunctions of $T(G')$.

Next, we suppose that G is a non-bipartite graph. Note that there exist at least one essential cycles with odd length. For the eigenvalues in (5.5) with even t, $g_t^{(C)}$ becomes an eigenfunction of $T(G')$ for μ_t for every essential cycle C. In addition, $\hat{g}^{(t)}$ is an eigenfunctions of $T(G')$ associated to μ_t when t is even. There are $\left\lfloor \frac{l-1}{2} \right\rfloor (m-n+1) + \left\lfloor \frac{l-1}{2} \right\rfloor = \left\lfloor \frac{l-1}{2} \right\rfloor (m-n+2)$ linearly independent eigenfunctions. For an eigenvalue μ_t with odd t, $g_t^{(C)}$ becomes an eigenfunction of $T(G')$ associated to μ_t if the length of C is even. Let $C_1, C_2, \ldots, C_q$ be the distinct essentials cycles with odd length in G. From the connectivity of G, there exits a path on G connecting a vertex in C_1 and a vertex in C_i for $2 \le i \le q$. Let $V(C_1) = \{v_{\nu_1}, v_{\nu_2} \ldots, v_{\nu_d}\}$ and $V(C_i) = \{v_{\eta_1}, v_{\eta_2}, \ldots, v_{\eta_e}\}$. We find a path $P = \{v_{\rho_1}, v_{\rho_2}, \cdots, v_{\rho_f}\}$ with $v_{\rho_1} = v_{\nu_1}, v_{\rho_f} = v_{\eta_1}$. For $2 \le i \le q$ and odd t with $0 < t < l$, let us define a function $h_t^{(i)} \in \ell^2(V(G'))$ by

$$h_t^{(i)}(v) = \begin{cases} (-1)^a U_{r-1}(\mu_t), & \text{if } v = w^{(r)}_{\nu_a,\nu_{a+1}}, \\ 2 \cdot (-1)^{b+1} U_{r'-1}(\mu_t), & \text{if } v = w^{(r)}_{\rho_b,\rho_{b+1}}, \\ (-1)^{b+c} U_{r''-1}(\mu_t), & \text{if } x = w^{(r)}_{\eta_c,\eta_{c+1}}, \\ 0, & \text{otherwise} \end{cases}$$

for $1 \le a \le d, 1 \le b \le e, 1 \le c \le f$ and $1 \le r \le l-1$, where we set $\nu_{d+1} = \nu_1, \eta_{e+1} = \eta_1, \rho_{f+1} = \rho_1$. By $U_{l-2}(\mu_t) = (-1)^t$, it is similarly checked that $h_t^{(i)}$ is an eigenfunction of $T(G')$ for μ_t with odd t. We have thus found $\left\lceil \frac{l-1}{2} \right\rceil (m-n)$ linearly independent eigenfunctions of $T(G')$. Furthermore, if l even, G' becomes a bipartite graph and we have $-1 \in \sigma(T(G'))$. The remaining eigenvalue is 1. Hence, the number of the linearly independent eigenfunctions that we have found is

$$l(n-1) + \left\lfloor \frac{l-1}{2} \right\rfloor (m-n+2) + \left\lceil \frac{l-1}{2} \right\rceil (m-n) + 1 + \mathbf{1}_{\mathbf{B}}(G') = n + (l-1)m,$$

which coincides with $|V(G')|$. Therefore, we have exhausted all the linearly independent eigenfunctions of $T(G')$.

After all, we obtain

$$\begin{aligned} \sigma(T(G')) = \bigcup_{\lambda \in \sigma(T(G)) \setminus \{\pm 1\}} & \{\mu \in \mathbb{R} \mid T_l(\mu) = \lambda, \} \\ & \cup \{\mu \in \mathbb{R} \mid U_{l-1}(\mu) = 0, U_{l-2}(\mu) = -1\}^{b_1(G)+1} \\ & \cup \{\mu \in \mathbb{R} \mid U_{l-1}(\mu) = 0, U_{l-2}(\mu) = 1\}^{b_1(G)-1+2\cdot\mathbf{1}_{\mathbf{B}}(G)} \\ & \cup \{-1\}^{\mathbf{1}_{\mathbf{B}}(G')} \cup \{1\} \end{aligned}$$

and the eigenvalues of $T(G')$ are given by

$$\mu = \cos \frac{\pm\theta + 2\pi s}{l}, \quad \cos \frac{\pi}{l} t, \quad \pm 1$$

for $\theta \in \cos^{-1}(\sigma(T(G))\backslash\{\pm 1\})$,where the ranges of s, s' and t are as mentioned above. Now we recall that G is Grover-k-periodic. Then we have $\theta \in \frac{2\pi}{k}\mathbb{Z}$. If k is even, the eigenvalues

$$\cos \frac{\pm\theta + 2\pi s}{l}$$

are the real parts of the kl-th root of unity. Therefore, G' is Grover-periodic and its period is $\mathrm{lcm}(kl, 2l, 2, 1) = kl$ by Proposition 3.2.3. If k is odd, we have $b_1(G) = 1, \mathbf{1_B}(G) = 0$ by Theorem 4.3.1. Then the eigenvalues satisfying $U_{l-1}(\mu) = 0, U_{l-2}(\mu) = 1$ will vanish. Hence G' is also Grover-periodic by Proposition 3.2.3 and the period is $\mathrm{lcm}(kl, l, 2, 1) = kl$ if l is even, $\mathrm{lcm}(kl, l, 1) = kl$ otherwise. In any case, the period is kl and we complete the proof.

□

5.3 Line graphs induced by Hoffman graphs

We need the notion of a *tessellation* of a graph. A graph G is said to be 2-*tessellable* if there exists two clique partitions $C = \{C_1, C_2, \dots, C_k\}$ and $D = \{D_1, D_2, \dots, D_l\}$ such that

$$E(G) = \bigsqcup_{i=1}^{k} E(G[C_i]) \cup \bigsqcup_{j=1}^{l} E(G[D_j])$$

and we call $C \sqcup D$ a *tessellation cover*. Here, we treat the *staggered walk* defined in a 2-tessellable graph. Ambainis *et al.* [7] suggested a staggered walk on the two-dimensional lattice and also formulated an efficient searching algorithm driven by a staggered walk. Staggered walks on graphs have been analyzed along with problems such as searching problems in [55] and [56]. In this Section, we define a staggered walk on a generalized line graph induced by a Hoffman graph and analyze the periodicity. In particular, we provide a class of Hoffman graphs which induce periodic staggered walks.

5.3.1 Hoffman graphs

The *Hoffman graphs* were studied by Woo and Neumaier [65] as a generalization of line graphs. A Hoffman graph consists of two disjoint sets of vertices called *fat* and *slim*. A Hoffman graph $\mathfrak{h}$ is a pair (H, μ), where $H = (V, E)$ is a graph and μ is a labeling map from V to $\{f, s\}$ satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) the vertices with label f are pairwise non-adjacent.

A vertex labeled by s is called *slim*, and one labeled by f is *fat*. Figure 5.1 shows examples of Hoffman graphs, where a fat vertex is denoted by a big dot and a slim one by a small dot.

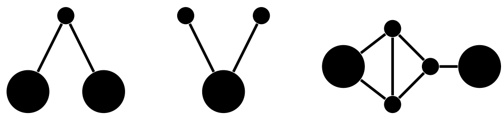
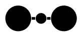


Figure 5.1: Examples of Hoffman graphs

From now on, we always consider Hoffman graphs with two fat vertices. For such a Hoffman graph $\mathfrak{h} = (H, \mu)$, we set

$$F(\mathfrak{h}) := \{v \in V(H) \mid \mu(v) = f\},$$
$$S(\mathfrak{h}) := \{v \in V(H) \mid \mu(v) = s\}.$$

Let $G = (V, E)$ be a simple graph. It is noticeable that the line graph $L(G)$ is obtained by using a Hoffman graph $\mathfrak{h} =$  as follows:

(i) Embed the Hoffman graph $\mathfrak{h}$ to each edge of the original graph G.

(ii) Connect two distinct vertices if they have a common fat neighbor.

(iii) Remove all fat vertices.

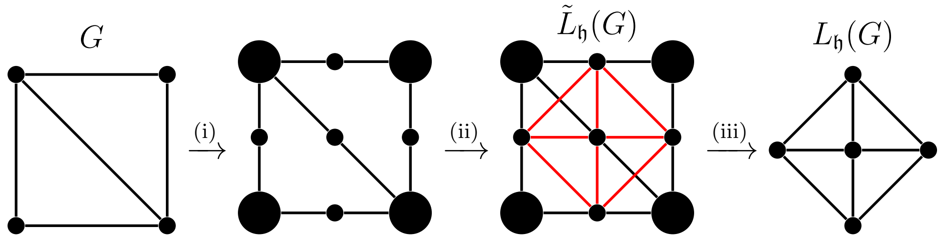


Figure 5.2: An example of construction of an ordinary line graph by Hoffman graph

Figure 5.2 illustrates the procedure. The concept of a line graph is generalized by replacing

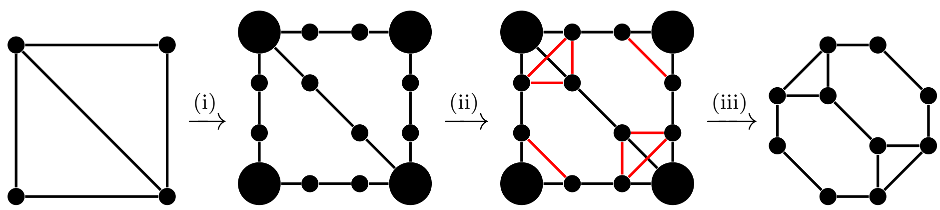
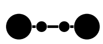


Figure 5.3: An example of generalized line graph by $\mathfrak{h}'$

the Hoffman graph $\mathfrak{h}$ with a general one, for example, $\mathfrak{h}' =$  . (See Figure 5.3)

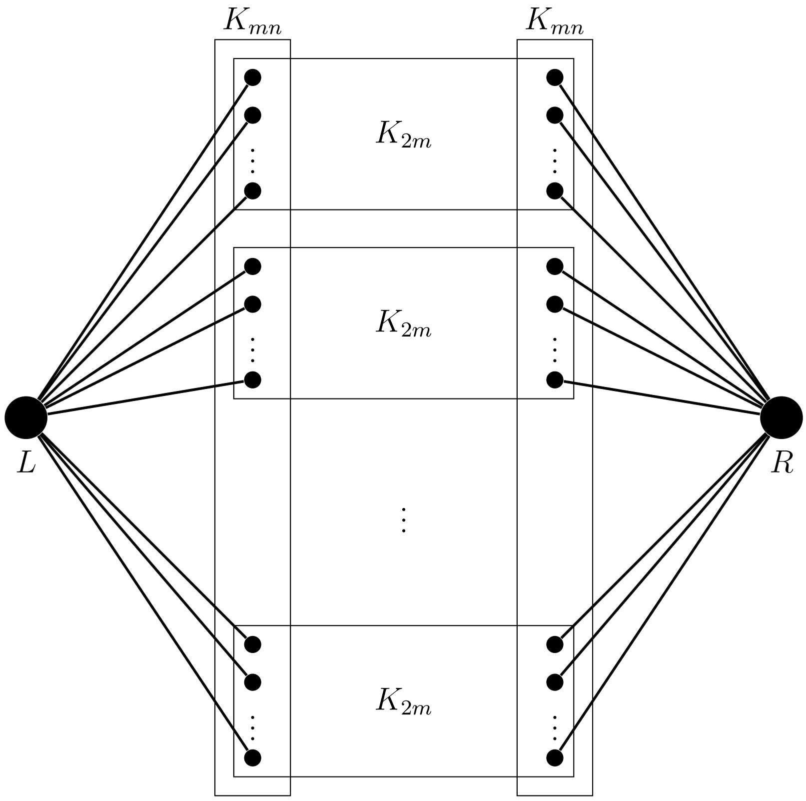


Figure 5.4: The Hoffman graph $\mathfrak{h}_{m,n}$

For a graph G and a Hoffman graph $\mathfrak{h}$ the graph obtained by the above procedure (i)–(iii) is called a *generalized line graph* and is denoted by $L_{\mathfrak{h}}(G)$. Furthermore, we denote the graph obtained by the process (i)–(ii) by $\tilde{L}_{\mathfrak{h}}(G)$. By definition of the generalized line graph, there is a correspondence between the vertices of G and the fat vertices of $\tilde{L}_{\mathfrak{h}}(G)$. In addition, there is a correspondence between the edges of G and the Hoffman graphs embedded in $\tilde{L}_{\mathfrak{h}}(G)$. For $v_i \in V(G)$ and $e_j \in E(G)$, let us denote by $\nu_i \in V(\tilde{L}_{\mathfrak{h}}(G))$ and $\mathfrak{e}_j$ the corresponding fat vertex and embedded Hoffman graph, respectively. Let $C = \{C_1, C_2, \ldots, C_k\}, D = \{D_1, D_2, \ldots, D_l\}$ be two partitions of the vertices of $L_{\mathfrak{h}}(G)$ defined by

$$
\begin{aligned}
C_i &:= N(\nu_i) \\
D_j &:= S(\mathfrak{e}_j),
\end{aligned}
$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. Then C and D constitute a 2-tessellation of $L_{\mathfrak{h}}(G)$ and $L_{\mathfrak{h}}(G)$ is a 2-tessellable graph. We call $C \sqcup D$ *the natural tesellation.* There are natural correspondences between C and $V(G)$, D and $E(G)$.

For positive integers m and n, we define the Hoffman graph $\mathfrak{h}_{m,n}$ as follows: The vertex set of $\mathfrak{h}_{m,n}$ is

$$\{(a, i, j) \mid 1 \leq a \leq 2, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{L, R\}.$$

Two vertices (a,i,j) and (b,i',j') are adjacent if and only if $a=b$ or $i=i'$. Moreover, L is adjacent to $(1,i,j)$ for $1\le i\le m$ and $1\le j\le n$. Similarly, R is adjacent to every $(2,i,j)$ for $1\le i\le m$ and $1\le j\le n$. The labeling map μ is

$$\mu(v)=\begin{cases} f & \text{if } v\in\{L,R\},\\ s & \text{otherwise.}\end{cases}$$

The shape of $\mathfrak{h}_{m,n}$ is illustrated in Figure 5.4.

For a 2-tessellable graph H with tessellations $C=\{C_1,\dots,C_k\}$ and $D=\{D_1,\dots,D_l\}$, the *intersection graph* is a multigraph defined by

- $V(\mathcal{I}(H)) := C\sqcup D$.
- $E(\mathcal{I}(H)) := \{C_{\tau(x)}D_{\epsilon(x)}(m_x)\mid x\in V(H)\}$,

where the multiplicity is given by $m_x=|C_{\tau(x)}\cap D_{\epsilon(x)}|$. Note that $V(H)\simeq E(\mathcal{I}(H))$. Let us denote two arcs on $\mathcal{I}(H)$ associated with $x\in V(H)$ by $(C_{\tau(x)},D_{\epsilon(x)};x)$ and $(D_{\epsilon(x)},C_{\tau(x)};x)$. Then

$$\mathcal{A}(\mathcal{I}(H)) := \{(C_{\tau(x)},D_{\epsilon(x)};x),(D_{\epsilon(x)},C_{\tau(x)};x)\mid x\in V(H)\}.$$

We give an example of an intersection graph in Figure 5.5.

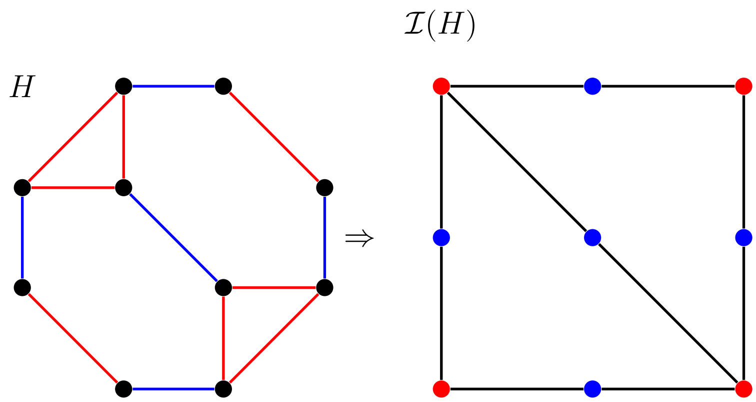


Figure 5.5: An example of an intersection graph

5.3.2 Definition of staggered walks

The staggered walk is a quantum walk defined on a 2-tessellable graph. Let $H=(V,E)$ be a 2-tessellable graph with two clique partitions $C=\{C_1,\dots,C_k\}$ and $D=\{D_1,\dots,D_l\}$. Then for $x\in V(H)$ there uniquely exist $1\le i\le k$ and $1\le j\le l$ such that $x\in C_i\cap D_j$. Let us denote the indices i and j by $\tau(x)$ and $\epsilon(x)$, respectively. Define a unitary operator $\hat{C}$ on $\ell^2(V(H))$ by

$$\hat{C}f(x)=\sum_{y\in C_{\tau(x)}}\left(\frac{2}{|C_{\tau(x)}|}-\delta_{x,y}\right)f(y),\quad f\in\ell^2(V(H)).$$

Similarly, we define a unitary operator $\hat{D}$ on $\ell^2(V(H))$ by

$$\hat{D}f(x) = \sum_{y \in D_{\epsilon(x)}} \left(\frac{2}{|D_{\epsilon(x)}|} - \delta_{x,y} \right) f(y), \quad f \in \ell^2(V(H)).$$

We set $\hat{U} = \hat{D}\hat{C}$. The quantum walk on H defined by $\hat{U} = \hat{U}(H)$ is called the *staggered walk* with respect to $C = \{C_1, \ldots, C_k\}$ and $D = \{D_1, \ldots, D_l\}$. If there exits an integer $k \in \mathbb{N}$ such that $\hat{U}^k = \hat{U}^k(H) = I_{V(H)}$ we say that H is a *staggered-periodic graph.*

5.3.3 Periodicity of staggered walks

Let A and B be operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then A and B are *unitarily equivalent* if there exists a unitary operator $\chi : \mathcal{H} \to \mathcal{K}$ such that $A = \chi^{-1} B \chi$.

Let G be a simple graph. Here, $H = L_{\mathfrak{h}_{m,n}}(G)$ is the 2-tessellable graph with respect to the natural tessellation $C = \{C_1, \cdots, C_t\}, D = \{D_1, \cdots, D_k\}$. It follows immediately that $\mathcal{I}(H) \simeq MP_m(S_2(MP_n(G)))$. Since $V(G) \simeq C$, there exits a bijective map $\phi : C \to V(G)$ which maps a clique C_i to the corresponding vertex of G. Then we have $|C_i| = mn \deg(\phi(C_i))$ for $1 \leq i \leq t$. In addition, we define the following subspace $\mathcal{B} \subset \mathcal{A}(\mathcal{I}(H))$:

$$\mathcal{B} := \{(D_{\epsilon(x)}, C_{\tau(x)}; x) \mid x \in V(H)\}.$$

Then $\mathcal{B} \simeq V(H)$. To prove our main result in this Section, we show the unitary equivalence between the staggered walk on a generalized line graph induced from $\mathfrak{h}_{m,n}$ and the Grover walk on the original graph G by following [42].

Theorem 5.3.1. *For a graph G, let H and H' be $L_{\mathfrak{h}_{m,n}}(G)$ and its intersection graph, respectively. Then $\hat{U}(H)$ and $U^2(H')|_{\ell^2(\mathcal{B})}$ are unitarily equivalent.*

Proof. We fix $x \in V(H)$. Let $e_x = (D_{\epsilon(x)}, C_{\tau(x)}; x)$. Using the correspondence between $V(H)$ and $\mathcal{B}$, we define a unitary operator $\chi : \ell^2(V(H)) \to \ell^2(\mathcal{B})$ by

$$\chi g(e_x) = g(x), \quad g \in \ell^2(V(H)).$$

Note that

$$\chi^{-1}\varphi(x) = \varphi((e_x)), \quad \varphi \in \ell^2(\ell^2(\mathcal{B})).$$

For $g \in \ell^2(V(H))$ we have

$$\begin{aligned}(\hat{U}g)(x) &= (\hat{D}\hat{C}g)(x) \\ &= \sum_{y \in D_{\epsilon(x)}} \left(\frac{2}{|D_{\epsilon(x)}|} - \delta_{x,y} \right) \hat{C}g(y) \\ &= \sum_{y \in D_{\epsilon(x)}} \left(\frac{2}{|D_{\epsilon(x)}|} - \delta_{x,y} \right) \sum_{z \in C_{\tau(y)}} \left(\frac{2}{|C_{\tau(y)}|} - \delta_{y,z} \right) g(z).\end{aligned}$$

Since $|C_{\tau(y)}| = mn \deg_G \phi(C_{\tau(y)})$ and $|D_{\epsilon(x)}| = 2mn$, we have

$$(\hat{U}g)(x) = \sum_{y \in D_{\epsilon(x)}} \left(\frac{1}{mn} - \delta_{x,y} \right) \sum_{z \in C_{\tau(y)}} \left(\frac{2}{mn \deg_G \phi(C_{\tau(y)})} - \delta_{y,z} \right) g(z). \tag{5.7}$$

Next, we consider the action of $\chi^{-1}U^2|_{\ell^2(\mathcal{B})}(H')\chi$. Let $\varphi = \chi g \in \ell^2(\mathcal{B})$. Then we have

$$\begin{aligned}
(\chi^{-1}U^2|_{\ell^2(\mathcal{B})}\chi g)(x) &= (\chi^{-1}UU\varphi)(x) \\
&= (UU\varphi)(e_x) \\
&= \sum_{\substack{f \in \mathcal{A}(H') \\ t(f)=o(e_x)}} \left(\frac{2}{\deg_{H'}(t(f))} - \delta_{e_x, f^{-1}} \right) (U\varphi)(f) \\
&= \sum_{\substack{f \in \mathcal{A}(H') \\ t(f)=o(e_x)}} \left(\frac{2}{\deg_{H'}(t(f))} - \delta_{e_x, f^{-1}} \right) \sum_{\substack{h \in \mathcal{A}(H') \\ t(h)=o(f)}} \left(\frac{2}{\deg_{H'}(t(h))} - \delta_{f, h^{-1}} \right) \varphi(h).
\end{aligned} \tag{5.8}$$

Since $o(e_x) = D_{\epsilon(x)}$, an arc $f \in \mathcal{A}(H')$ with $t(f) = o(e_x)$ is written as $(C_{\tau(y)}, D_{\epsilon(x)}; y)$ for $y \in C_{\tau(x)}$. Note that $(C_{\tau(y)}, D_{\epsilon(x)}; y) = (C_{\tau(y)}, D_{\epsilon(y)}; y)$ since $y \in D_{\epsilon(x)}$. Then each arc $f \in \mathcal{A}(H')$ with $t(f) = o(e_x)$ corresponds to a vertex $y \in D_{\epsilon(x)}$, where the inverse arc $e_x^{-1} = (C_{\tau(x)}, D_{\epsilon(x)}; x)$ corresponds to x itself. For the above f, an arc $h \in \mathcal{A}(H')$ with $t(h) = o(f)$ is written as $(D_{\epsilon(z)}, C_{\tau(y)}; z)$ for $z \in C_{\tau(y)}$ with $y \in D_{\epsilon(x)}$. Note that $(D_{\epsilon(z)}, C_{\tau(y)}; z) = (D_{\epsilon(z)}, C_{\tau(z)}; z)$ since $y \in D_{\epsilon(x)}$. Then each arc $h \in \mathcal{A}(H')$ with $t(h) = o(f)$ corresponds to a vertex $z \in C_{\tau(y)}$, where the inverse arc $f^{-1} = (D_{\epsilon(y)}, C_{\tau(y)}; y)$ corresponds to y itself. Then it follows that

$$\begin{aligned}
\varphi(h) &= (\chi g)(h) \\
&= (\chi g)((D_{\epsilon(z)}, C_{\tau(z)}; z)) \\
&= (\chi g)(e_z) \\
&= g(z).
\end{aligned}$$

Since $\deg_{H'}(t(f)) = 2mn$ and $\deg_{H'}(t(h)) = mn \deg(\phi(C_{\tau(y)}))$, (5.8) is rewritten as

$$\sum_{y \in D_{\epsilon(x)}} \left(\frac{1}{mn} - \delta_{x,y} \right) \sum_{z \in C_{\tau(y)}} \left(\frac{2}{mn \deg_G \phi(C_{\tau(y)})} - \delta_{y,z} \right) g(z),$$

which equals (5.7). We complete the proof. □

Theorem 5.3.2. *If a graph G is Grover-periodic, $L_{\mathfrak{h}_{m,n}}(G)$ is staggered-periodic for $m, n \in \mathbb{N}$.*

Proof. Let $H = L_{\mathfrak{h}_{m,n}}(G)$ and $H' = \mathcal{I}(H)$. Note that $H' \simeq MP_m(S_2(MP_n(G)))$. If G is a Grover-periodic graph, so is $MP_n(G)$ by Theorem 5.1.1. Note also that $S_2(MP_n(G))$

is obtained by replacing each edge of G with $P_{3,n}$. By a similar way as in the proof of Theorem 4.2.4, there exists a subset $\mathcal{A} \subset \ell^2(V(S_2(MP_n(G))))$ such that

$$T(S_2(MP_n(G)))|_{\mathcal{A}} = T(S_2(G)).$$

Hence, we have

$$\sigma(T(S_2(G)) \subset \sigma(T(S_2(MP_n(G))))$$

and the remaining eigenvalues of $T(S_2(MP_n(G)))$ are

$$\lambda_k = \cos\frac{\pi}{4}k$$

for $k \in \{0, 2, 4\}$ with associated eigenfunction as in (4.30). Therefore, $S_2(MP_n(G))$ is Grover-periodic by Theorem 5.2.1. Moreover, $MP_m(S_2(MP(G)))$ is Grover-periodic by Theorem 5.1.1. Then there exists an integer k such that

$$(U^2(H')|_{\ell^2(\mathcal{B})})^k = I_{\ell^2(\mathcal{B})}.$$

By Theorem 5.3.1, we have

$$\hat{U}^k(H) = \chi^{-1}(U(H')|_{\ell^2(\mathcal{B})})^k\chi = I_{V(H)}.$$

Therefore, H is staggered-periodic, which completes the proof.

□

Chapter 6

Summary and discussions

In this paper, we are mainly concerned with characterization of graphs which admit periodic Grover walks (named Grover-periodic graph). Periodicity treated in this paper is regarded as a discrete version of perfect state transfer [18] and [19]. As an extension of [28], characterization of concrete classes of Grover-periodic graphs, e.g., cycle graphs, path graphs, Hamming graphs, Johnson graphs and generalized Bethe trees are obtained by spectral analysis. The results are summarized as follows:

- Cycle graphs.

  For $n \in \mathbb{N}$, the cycle graph C_n is Grover-periodic and its period is n.

- Path graphs.

  For $n \in \mathbb{N}$, the path graph P_n is Grover-periodic and its period is $2(n-1)$.

- Hamming graphs.

  The Grover-periodic Hamming graphs are only

  $$H(1,2), H(1,3), H(2,2), H(3,3), H(4,2)$$

  and their periods are $2, 3, 4, 12$ and 12, respectively.

- Johnson graph.

  The Grover-periodic Johnson graphs are only

  $$J(2,1), J(3,1), J(4,2)$$

  and their periods are $2, 3,$ and 12, respectively.

- Generalized Bethe trees.

  The Grover-periodic generalized Bethe trees are only

  $$S_k(B(1,2,3)), S_k(B(s,3))$$

  for $k, s \in \mathbb{N}$ and both of their periods are 12. (The graphs $B(1,2,3)$ and $B(s,3)$ are illustrated in Figures 3.2.)

Through the research, we aim to see how graph structure affects an induced quantum walk. Our concern is to see restriction of graphs when we suppose that the quantum walk induced by the graph is periodic. By analysis of the transition operator, we obtain the following condition. Let T be the transition operator of a simple graph G on n vertices and

$$\det(xI_n - T) = \sum_{i=0}^{n} \rho_i x^i, \quad 0 \le i \le n,$$

where I_n is the $n \times n$-identity matrix. Then it holds necessarily that

$$2^j \rho_{n-j} \in \mathbb{Z}, \quad 0 \le i \le n.$$

By using the condition, we constructed a new class of Grover-periodic graphs obtained by the join, e.g., a cycle and a cycle, a cycle and a claw. Moreover, thanks of the condition, we approached our conjecture for the Grover-odd-periodic graph, that is, the Grover-odd-periodic graph is only the cycle graph with odd length. Furthermore, we introduced some graph-transformations, e.g., multiplex graphs, subdivision graphs and line graphs induced by a Hoffman graph preserving the periodicity. If a simple graph G is Grover-periodic, these graphs $MP_k(G), S_l(G)$ and $L_{\mathfrak{h}_{m,n}}(G)$ are also Grover-periodic for $k, l, m, n \in \mathbb{N}$, where a Hoffman graph $\mathfrak{h}_{m,n}$ is seen in Figure 5.4.

In the next stage, we aim to enlarge graphs while maintaining periodicity of quantum walks. We believe that these results give great contribution to the purpose. In particular, the condition (6.1) will be a useful tool for our future directions. Characterization of Grover-k-periodic graphs is still an open problem for every $k \in \mathbb{N}$. Furthermore, we may consider more extended setting. Here, we will treat graphs satisfying

$$U^k(G) = I_{\mathcal{A}(G)}$$

for $k \in \mathbb{N}$. This condition can be extended to

$$U^k(G) = P,$$

where P is a permutation operator. Then there exist $e, f \in \mathcal{A}(G)$ such that

$$U^k(G)\delta_e = \delta_f.$$

In addition, our condition tacitly requires that $U^k(G)\varphi_0 = \varphi_0$ for any $\varphi_0 \in \ell^2(\mathcal{A}(G))$. It can be loosened to the condition that there exists φ such that $U^k(G)\varphi = \varphi$. We may call such a graph *partial periodic*. To see graph structure under such relaxed conditions is also our future work. We ultimately aim to reveal the relation between quantum walks and graph structure.

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