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Loewner Theory for Quasiconformal Extensions: Old and New

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This survey article gives an account of quasiconformal extensions of univalent functions with its motivational background from Teichmüller theory and classical and modern approaches based on Loewner theory.

KEYWORDS: Loewner theory, quasiconformal mapping, evolution family, universal Teichmüller space, univalent function

1. Introduction

The Loewner differential equation is an ordinary/partial differential equation introduced by Löwner in 1923 for the study of conformal mappings in geometric function theory. Originally, a slit mapping, namely, a conformal map on the open unit disk onto the complex plane minus a Jordan arc whose one end point is ∞, is dealt with. His work was later developed by Kufarev and Pommerenke to a conformal map whose image is a simply-connected domain on the complex plane.

Löwner's theory has been successfully used to various problems in geometric function theory. In particular the Bieberbach conjecture, an extremal problem of the Taylor coefficients of the normalized univalent functions on the unit disk, was solved with the help of the Loewner's method. Recently, the Schramm–Loewner Evolution (SLE), a stochastic variant of the Loewner differential equation discovered by Oded Schramm in 2000, has attracted substantial attention in probability theory and conformal field theory.

In this survey article we discuss one of applications of the Loewner differential equation, quasiconformal extensions of univalent functions. It is structured as follows: In Sect. 2, a motivational background behind the quasiconformal extension problem is explained. The key is the construction of the equivalent models of the universal Teichmüller spaces. Before entering the main topic of this article, in Sect. 3 we summarize classical results on the theory of univalent functions and quasiconformal extensions. In Sect. 4, we present a short introduction of the classical theory of the Loewner differential equation and its connection to quasiconformal extensions. In Sect. 5, a modern approach to Loewner theory introduced and developed in the last decade is reviewed. It gives a unified treatment of the known types of the classical Loewner equations. The paper is closed with a brief consideration of quasiconformal extension problems in the modern framework of the Loewner theory.

2. Universal Teichmüller Spaces

The notion of the universal Teichmüller spaces was illuminated in the theory of quasiconformal mappings as an embedding of the Teichmüller spaces of compact Riemann surfaces of finite genus. Several equivalent models of universal Teichmüller spaces are known (see e.g., [Sug07]). In this article we will focus on the connection with a space of the Schwarzian derivatives of conformal extensions of quasiconformal mappings defined on the upper half-plane $\mathbb{H}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

2.1 Quasiconformal mappings

A homeomorphism f of a domain $G \subset \mathbb{C}$ is called ***k*-quasiconformal** if f_z and $f_{\bar{z}}$, the partial derivatives in z and $\bar{z}$ in the distributional sense, are locally integrable on G and satisfy

$$|f_{\bar{z}}(z)| \leq k|f_z(z)| \tag{2.1}$$

almost everywhere in G, where $k \in [0, 1)$. The above definition implies that a quasiconformal map f is sense-preserving, namely, the Jacobian $J_f := |f_z|^2 - |f_{\bar{z}}|^2$ is always positive.

In order to observe the geometric interpretation of the inequality (2.1), assume for a while $f \in C^1(G)$. Then the

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differential is

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

We shall consider the $\mathbb{R}$-linear transformation $T(z) := f_z z + f_{\bar{z}}\bar{z}$. Denote $\alpha := \arg f_z$ and $\beta := \arg f_{\bar{z}}$. Then we have

$$\begin{aligned} T(re^{i\theta}) &= |f_z|e^{i\alpha}re^{i\theta} + |f_{\bar{z}}|e^{i\beta}re^{-i\theta} \\ &= re^{i\psi}(|f_z|e^{i(\theta-\phi)} + |f_{\bar{z}}|e^{-i(\theta-\phi)}) \end{aligned}$$

where $\phi := (\beta - \alpha)/2$ and $\psi := (\beta + \alpha)/2$. Consequently, f maps each infinitesimal circle in G onto an infinitesimal ellipse with axis ratio bounded by $(1+k)/(1-k)$ (Fig. 1).

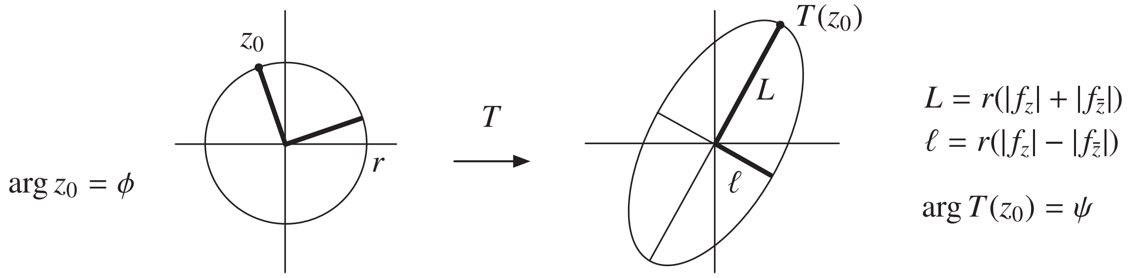

Fig. 1. An infinitesimal circle is mapped to an infinitesimal ellipse.

Suppose that f is conformal (i.e., holomorphic injective) on G. Recall that $f_z = (f_x - if_y)/2$ and $f_{\bar{z}} = (f_x + if_y)/2$. Thus $f_{\bar{z}}(z) = 0$ for all $z \in G$ which is exactly the Cauchy-Riemann equations. Hence f is 0-quasiconformal. In this case $L = \ell$ in Fig. 1. Conversely, if f is 0-quasiconformal, then by (2.1) $f_{\bar{z}}(z) = 0$ for almost all $z \in G$. By virtue of the following Weyl's lemma, we conclude that f is conformal on G.

Lemma 2.1 (Weyl's lemma (see e.g., [IT92, p. 84]). *Let f be a continuous function on G whose distributional derivative $f_{\bar{z}}$ is locally integrable on G. If $f_{\bar{z}} = 0$ in the sense of distributions on G, then f is holomorphic on G.*

Let $B(G)$ be the open unit ball $\{\mu \in L^\infty(G) : \|\mu\|_\infty < 1\}$ of $L^\infty(G)$, where $L^\infty(G)$ is the complex Banach space of all bounded measurable functions on G, and $\|\mu\|_\infty := \operatorname{ess\,sup}_{z\in G} |\mu(z)|$ for a $\mu \in L^\infty(G)$. An element $\mu \in B(G)$ is called the **Beltrami coefficient**. If f is a k-quasiconformal mapping on G, then it is verified that $f_z(z) \neq 0$ for almost all $z \in G$ (e.g., [LV73, Theorem IV-1.4 in p. 166]). Hence $\mu_f := f_{\bar{z}}/f_z$ defines a function belongs to $B(G)$. μ_f is called the **complex dilatation** of f, and the quantity $k := k(f) := \|\mu_f\|_\infty$ is called the **maximal dilatation** of f. Conversely, the following fundamental existence and uniqueness theorem is known.

Theorem 2.2 (The measurable Riemann mapping theorem). *For a given measurable function $\mu \in B(\mathbb{C})$, there exists a unique solution f of the equation*

$$f_{\bar{z}} = \mu f_z \tag{2.2}$$

for which $f : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping fixing the points 0 and 1.

The Eq. (2.2) is called the **Beltrami equation**.

Here we give some fundamental properties of quasiconformal mappings we will use later. For the general theory of quasiconformal mappings in the plane, the reader is referred to [Ahl06], [LV73], [AIM09], [Hub06] and [IT92].

f is 0-quasiconformal if and only if f is conformal, as discussed above. If f is k-quasiconformal, then so is its inverse f^{-1} as well. A composition of a k_1- and k_2-quasiconformal map is $(k_1+k_2)/(1+k_1k_2)$-quasiconformal. The composition property of the complex dilatation is the following; Let f and g be quasiconformal maps on G. Then the complex dilatation $\mu_{g\circ f^{-1}}$ of the map $g \circ f^{-1}$ is given by

$$\mu_{g\circ f^{-1}}(f) = \frac{f_z}{\overline{f_z}} \cdot \frac{\mu_g - \mu_f}{1 - \mu_g\overline{\mu_f}}. \tag{2.3}$$

Since a 0-quasiconformal map is conformal, the above formula concludes that if $\mu_f = \mu_g$ almost everywhere in G then $g \circ f^{-1}$ is conformal on $f(G)$.

As the case of conformal mappings, isolated boundary points of a domain G are removable singularities of every quasiconformal mapping of G. It follows from this property that quasiconformal and conformal mappings divide simply connected domains into the same equivalence classes.

2.2 Schwarzian derivatives

Let f be a non-constant meromorphic function with $f' \neq 0$. Then we define the **Schwarzian derivative** by means of

$$S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

It is known that f is a Möbius transformation if and only if $S_f \equiv 0$. Further, a direct calculation shows that

$$S_{f \circ g} = (S_f \circ g)g'^2 + S_g.$$

Hence it follows the invariance property of S_f that if f is a Möbius transformation then $S_{f \circ g} = S_g$. One can interpret that the Schwarzian derivative measures the deviation of f from Möbius transformations. In order to describe it precisely, we introduce the norm of the Schwarzian derivative $\|S_f\|_G$ of a function f on G by

$$\|S_f\|_G := \sup_{z \in G} |S_f(z)| \eta_G(z)^{-2},$$

where η_G is a Poincaré density of G. One of the important properties of $\|S_f\|$ is the following; Let f be meromorphic on G and g and h Möbius transformations, then $\|S_f\|_G = \|S_{h \circ f \circ g}\|_{g^{-1}(G)}$. It shows that $\|S_f\|$ is completely invariant under compositions of Möbius transformations. We note that if $G = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then $\|S_f\|_{\mathbb{D}} = \sup_{z \in \mathbb{D}} (1 - |z|)^2 |S_f(z)|$. For later use, we denote $\|S_f\|_{\mathbb{D}}$ by simply $\|S_f\|$.

2.3 Bers embedding of Teichmüller spaces

Let us consider the family $\mathcal{F}$ of all quasiconformal automorphisms of the upper half-plane $\mathbb{H}^+$. Since all mappings in $\mathcal{F}$ can be extended to homeomorphic self-mappings of the closure of $\mathbb{H}^+$, all elements of $\mathcal{F}$ are recognized as self-homeomorphisms of $\overline{\mathbb{H}^+}$. We define an equivalence relation $\sim$ on $\mathcal{F}$ according to which $f \sim g$ for $f, g \in \mathcal{F}$ if and only if there exists a holomorphic automorphism M of $\mathbb{H}^+$, a Möbius transformation having the form $M(z) = (az + b)/(cz + d), a, b, c, d \in \mathbb{R}$, such that $f \circ M = g$ on $\mathbb{H}^+$. The equivalence relation on $\mathcal{F}$ induces the quotient space $\mathcal{F}/\sim$, which is called the **universal Teichmüller space** and denoted by $\mathcal{T}$. Theorem 2.2 with (2.3) tells us that there is a one-to-one correspondence between $\mathcal{T}$ and $B(\mathbb{H}^+)$. If $f \sim g$, then the corresponding complex dilatations μ_f and ν_g are also said to be **equivalent**.

Another equivalent class of $\mathcal{T}$ is given by the following profound observation due to Bers [Ber60]. Let $\mu \in B(\mathbb{H}^+)$. We extend μ to the lower half-plane $\mathbb{H}^- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ by putting 0 everywhere. By Theorem 2.2, there exists a quasiconformal mapping f^μ fixing $0, 1, \infty$ associated with such an extended μ. Then $f^\mu|_{\mathbb{H}^-}$ is conformal.

Theorem 2.3 (see e.g., [Leh87, Theorem III-1.2]). *The complex dilatations μ and ν are equivalent if and only if $f^\mu|_{\mathbb{H}^-} \equiv f^\nu|_{\mathbb{H}^-}$.*

By the above theorem, the universal Teichmüller space $\mathcal{T}$ can be understood as the set of the normalized conformal mappings $f^\mu|_{\mathbb{H}^-}$ which can be extended quasiconformally to the upper half-plane $\mathbb{H}^+$. Recall that for a Möbius transformation f we have $S_{f \circ g} = S_g$. Therefore, it is natural to consider the mapping

$$\mathcal{T} \ni [f] \mapsto S_{f^\mu|_{\mathbb{H}^-}} \in \mathcal{Q}, \tag{2.4}$$

between $\mathcal{T}$ and $\mathcal{Q}$, where $\mathcal{Q}$ is the space of functions ϕ holomorphic in $\mathbb{H}^-$ for which the hyperbolic sup norm $\|\phi\|_{\mathbb{H}^-} = \sup_{z \in \mathbb{H}^-} 4(\operatorname{Im} z)^2 |\phi(z)|$ is finite.

In order to investigate a detailed property of the mapping (2.4), we define a metric on $\mathcal{T}$ by

$$d_t(p, q) := \frac{1}{2} \inf_{p,q \in \mathcal{T}} \{\log K(g \circ f^{-1}) : f \in p, g \in q\},$$

where $K(f) := (1 + k(f))/(1 - k(f))$, and on $\mathcal{Q}$ by

$$d_q(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_\infty.$$

d_t is called the **Teichmüller distance**. As a consequence of the fact that d_t and d_q are topologically equivalent, we obtain the following theorem which provides a new model of the universal Teichmüller space.

Theorem 2.4. *The mapping* (2.4) *is a homeomorphism of the universal Teichmüller space $\mathcal{T}$ onto its image in $\mathcal{Q}$.*

The mapping (2.4) is called the **Bers embedding** of Teichmüller space. We denote the image of $\mathcal{T}$ under (2.4) by $\mathcal{T}_1$. It is known that $\mathcal{T}_1$ is a bounded, connected and open subset of $\mathcal{Q}$ ([Ahl63]).

From the viewpoint of the theory of univalent functions, $\mathcal{T}_1$ is characterized as follows. Let $\mathcal{A}$ be the family of functions f holomorphic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$ and $\mathcal{S}$ be the subfamily of $\mathcal{A}$ whose components are univalent on $\mathbb{D}$. We define $\mathcal{S}(k)$ and $\mathcal{S}^*(k)$ as the families of functions in $\mathcal{S}$ which can be extended to k-quasiconformal mappings of $\mathbb{C}$ and $\widehat{\mathbb{C}}$. Set $\mathcal{S}(1) := \cup_{k \in [0,1)} \mathcal{S}(k)$. Then $\mathcal{T}_1$ is written by

$$\mathcal{T}_1 = \{S_f : f \in \mathcal{S}(1)\}.$$

We give a short account of the relation to the Teichmüller spaces. Let S_1 and S_2 be Riemann surfaces and G_1 and G_2 the covering groups of $\mathbb{H}$ over S_1 and S_2, respectively. For the Riemann surfaces S_1 and S_2, the Teichmüller spaces $\mathcal{T}_{S_1}$ and $\mathcal{T}_{S_2}$ are defined. If G_1 is a subgroup of G_2, then the relation $\mathcal{T}_{S_2} \subset \mathcal{T}_{S_1}$ holds. In particular, if G_1 is trivial, then T_{S_1}

is the universal Teichmüller space which includes all the other Teichmüller spaces as subspaces. For this reason the name "universal" is used to $\mathcal{T}_1$.

3. Quasiconformal Extensions of Univalent Functions

In Sect. 1 we have introduced $\mathcal{S}(k)$ to characterize the universal Teichmüller space $\mathcal{T}_1$. Before entering the main part concerning with Loewner theory, we present some results of the general study of univalent functions and quasiconformal extensions.

3.1 Univalent functions

First of all, we review some results for the class $\mathcal{S}$. A number of properties for this class have been investigated by elementary methods.

Σ, the family of univalent holomorphic maps $g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}$ mapping $\mathbb{D}^* := \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ into $\widehat{\mathbb{C}} \backslash \{0\}$, also plays a key role in the theory of univalent functions. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, define

$$g(\zeta) := \frac{1}{f(1/\zeta)} = \zeta - a_2 + \frac{a_2^2 - a_3}{\zeta} + \cdots \quad (\zeta \in \mathbb{D}^*).$$

Then $g \in \Sigma$. On the other hand it is not always true that for a given $g \in \Sigma$, $f(z) := 1/g(1/z) \in \mathcal{S}$ because g may take 0. Hence $\Sigma_0 := \{g \in \Sigma : g(\zeta) \neq 0 \text{ on } \zeta \in \mathbb{D}^*\}$ and $\mathcal{S}$ have a one-to-one correspondence.

Theorem 3.1 (Gronwall's area theorem). *For a $g \in \Sigma$, we have*

$$m(\mathbb{C} - g(\mathbb{D}^*)) = \pi \left(1 - \sum_{n=1}^{\infty} n |b_n|^2 \right),$$

where m stands for the Lebesgue measure.

In particular $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. In particular $|b_1| \leq 1$. Here the equality $|b_1| = 1$ holds if and only if $g(\zeta) = \zeta + b_0 + e^{i\theta}/\zeta$. Since $(f(z^n))^{1/n} \in \mathcal{S}$ for all $f \in \mathcal{S}$ and all $n \in \mathbb{N}$, we have

$$\frac{1}{\sqrt{f(1/\zeta^2)}} = \zeta - \frac{1}{2} a_2 \cdot \frac{1}{\zeta} + \cdots \in \Sigma \quad (\zeta \in \mathbb{D}^*).$$

Hence by the estimate for $|b_1|$, we obtain the following.

Theorem 3.2 ([Bie16]). *If $f \in \mathcal{S}$, then $|a_2| \leq 2$. Equality holds if and only if $f(z)$ is a rotation of the Koebe function defined by*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4} \left(\left(\frac{1+z}{1-z} \right)^2 - 1 \right) = z + \sum_{n=2}^{\infty} n z_n. \tag{3.1}$$

Then, in a footnote of the paper [Bie16] Bieberbach wrote "*Vielleicht ist überhaupt* $k_n = n$" (where $k_n := \max_{f \in \mathcal{S}} |a_n|$) which means that *probably* $k_n = n$ *in general*. This statement is called the **Bieberbach conjecture**. In 1923 Löwner [Löw23] proved $|a_3| \leq 3$, in 1955 Garabedian and Schiffer [GS55b] proved $|a_4| \leq 4$, in 1969 Ozawa [Oza69a, Oza69b] and in 1968 Pederson [Ped68] proved $|a_6| \leq 6$ independently and in 1972 Pederson and Schiffer [PS72] proved $|a_5| \leq 5$. Finally, in 1985 de Branges [dB85] proved $|a_n| \leq n$ for all n. For the historical development of the conjecture, see e.g., [Zor86] and [Koe07]. The coefficient problem for Σ appears to be even more difficult than for $\mathcal{S}$. One reason is that there can be no single extremal function for all coefficients as the Koebe function. In 1914, Gronwall [Gro15] proved $|b_1| \leq 1$ as above, in 1938 Schiffer [Sch38] proved $|b_2| \leq 2/3$ and in 1955 Garabedian and Schiffer [GS55a] proved $|b_3| \leq 1/2 + \exp(-6)$. We do not even have a general coefficient conjecture for Σ. For this problem, see e.g., [NN57], [SW84] and [CJ92].

We get back to the main story. The next is an important application of Theorem 3.2.

Theorem 3.3 (The Koebe 1/4-theorem). *If $f \in \mathcal{S}$, then $f(\mathbb{D})$ contains the disk centered at the origin with radius 1/4.*

Since the class $\mathcal{S}$ is closed with respect to the Koebe transform

$$f_K(z) := \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2) f'(\zeta)} = z + \left(\frac{1}{2}(1-|\zeta|^2) \frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta} \right) z^2 + \cdots, \tag{3.2}$$

applying Theorem 3.2 to (3.2) we have the inequality

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right|\leq 4.$$

It derives the distortion theorems for the class $\mathcal{S}$.

Theorem 3.4. *If $f\in\mathcal{S}$, then*

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2|z|^2}{1-|z|^2}\right|\leq\frac{4|z|}{1-|z|^2},$$

$$\frac{1-|z|}{(1+|z|)^3}\leq|f'(z)|\leq\frac{1+|z|}{(1-|z|)^3},$$

$$\frac{|z|}{(1+|z|)^2}\leq|f(z)|\leq\frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{1+|z|}\leq\left|\frac{zf'(z)}{f(z)}\right|\leq\frac{1+|z|}{1-|z|},$$

for all $z\in\mathbb{D}$. In each case, equality holds if and only if f is a rotation of the Koebe function (3.1).

In particular, the third estimate implies that f is locally uniformly bounded. Hence $\mathcal{S}$ forms a normal family. Further, Hurwitz's theorem states that if a sequence of univalent functions on $\mathbb{D}$ converges to a holomorphic function locally uniformly on $\mathbb{D}$, then the limit function is univalent or constant. Since constant functions do not belong to $\mathcal{S}$ by the normalization, we conclude that $\mathcal{S}$ is compact in the topology of locally uniform convergence.

3.2 Examples of quasiconformal extensions

For a given conformal mapping f of a domain D, we say that f has a **quasiconformal extension** to $\mathbb{C}$ if there exists a k-quasiconformal mapping F such that its restriction $F|_D$ is equal to f. For some fundamental conformal mappings, we can construct quasiconformal extensions explicitly. Below we summarize such examples which are sometimes useful. Some more examples can be found in [IT92, p. 78]. We remark that (2.1) is written by the polar coordinates as

$$\left|\frac{ir\partial_r f(re^{i\theta})-\partial_\theta f(re^{i\theta})}{ir\partial_r f(re^{i\theta})+\partial_\theta f(re^{i\theta})}\right|\leq k,$$

where $\partial_r:=\partial/\partial r$ and $\partial_\theta:=\partial/\partial\theta$.

Example 3.5. A very simple but important example is

$$f(z)=\begin{cases} z+\dfrac{k}{z}, & |z|>1,\\ z+k\bar{z}, & |z|\leq 1,\end{cases}$$

where $k\in[0,1)$. Then $|f_{\bar{z}}/f_z|=k$ on $|z|\leq 1$. The case $k=1$ reflects the Joukowsky transform in $|z|>1$, though in this case f is not a quasiconformal mapping any more.

Example 3.6. An identity mapping of $\mathbb{D}$ has trivially a quasiconformal extension. In fact, the following extension,

$$f(z)=\begin{cases} re^{i\theta}, & r<1,\\ \phi(r)e^{i\theta}, & r\geq 1,\end{cases}$$

is given, where $\phi:[1,\infty)\to[1,\infty)$ is bi-Lipschitz continuous and injective with $\phi(1)=1$ and $\phi(\infty)=\infty$. The maximal dilatation is given by

$$|\mu_f|=\left|\frac{\phi(r)-r\phi'(r)}{\phi(r)+r\phi'(r)}\right|.$$

Let $M>1$ be a Lipschitz constant. Then $1/M\leq\phi'(r)\leq M$ and $1/M\leq\phi(r)/r\leq M$ and therefore $1\leq r\phi'(r)/\phi(r)\leq M^2$. We conclude that the extension is $|\mu_f|\leq|M^2-1|/|M^2+1|$-quasiconformal.

Example 3.7. Let $K(z):=(1+z)/(1-z)$ be the Cayley map and $P_\beta(z):=z^\beta$. For a fixed $\beta\in(0,2)$, the function

$$f(z):=(P_\beta\circ K)(z)$$

maps $\mathbb{D}$ onto the sector domain $\Delta(-\beta,\beta):=\{z:-\pi\beta/2<\arg z<\pi\beta/2\}$. We shall construct a quasiconformal extension of f. The function $g(z):=(-P_{2-\beta}\circ -K)(z)$ maps $\mathbb{C}\backslash\overline{\mathbb{D}}$ onto $\Delta(\beta,4-\beta)$. But in this case $f(e^{i\theta})\neq g(e^{i\theta})$ for each $\theta\in(0,2\pi)$. In order to sew these two functions on their boundaries, define $h(re^{i\theta}):=r^{\beta/(2-\beta)}e^{i\theta}$. Then $(-P_{2-\beta}\circ h\circ -K)(z)$ takes the same value as f on $\partial\mathbb{D}$. Hence it gives a quasiconformal extension of f. A calculation shows that its maximal dilatation is $|1-\beta|$.

Example 3.8. For a given $\lambda \in (-\pi/2, \pi/2)$, a function defined by

$$f(re^{i\theta}) = e^{i\theta} \exp(e^{i\lambda} \log r)$$

is a $\tan(\lambda/2)$-quasiconformal mapping of $\mathbb{C}$ onto $\mathbb{C}$. On the other hand, since the above f maps a radial segment $[0, \infty)$ to a logarithmic spiral, it is not differentiable at the origin. By calculation we have $|f| = \exp(\cos\lambda \log r)$ and $\arg f = \theta + \sin\lambda \log r$. Therefore f with a proper rotation gives a $\tan(\lambda/2)$-quasiconformal extension for a function $f(z) = cz$ on $\mathbb{D}$ or $\mathbb{D}^* := \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$, where c is some constant.

Example 3.9. The functions $K(z) = z/(1-z)^2$ and $f(z) = z - z^2/2$ are typical examples in $\mathcal{S}$ which do not have any quasiconformal extensions. The first one is the Koebe function (3.1) which maps $\mathbb{D}$ onto $\mathbb{C} \backslash (-\infty, -1/4]$. There does not exist a homeomorphism which maps $\mathbb{D}^*$ onto $(-\infty, -1/4]$. As for the second function, $\partial\mathbb{D}$ is mapped to a cardioid which has a cusp at $z = 1$.

3.3 Extremal problems on $\mathcal{S}(k)$

In order to investigate the structure of the family of functions, the extremal problems sometimes provide us quite beneficial information. The Bieberbach conjecture is one of the most known such problems. A similar problem for $\mathcal{S}(k)$ and $\Sigma(k)$, a subclass of Σ such that all elements have k-quasiconformal extensions to $\widehat{\mathbb{C}}$, were proposed, and many mathematicians have worked on this problem. We note that in spite of such a circumstance, there are many open problems in this field including the coefficient problem.

Our argument is built on the following fact.

Theorem 3.10. *$\mathcal{S}(k)$, $\mathcal{S}^*(k)$ and $\Sigma(k)$ are compact families.*

Kühnau gave a fundamental contribution to the coefficient problem with the variational method.

Theorem 3.11 ([Küh69]). *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(k)$ and $g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n z^{-n} \in \Sigma(k)$. Then the followings hold; $|b_0| \leq 2k$, $|b_1| \leq k$ and $|a_3 - a_2^2| \leq k$, in particular $|a_2| \leq 2k$.*

We note that in the case when $k = 1$ we obtain estimates for the classes $\mathcal{S}$ and Σ.

As more general approach to this problem, the distortion theorem for bounded functional was studied. We basically follow the description of the survey paper by Krushkal [Kru05b, Chapter 3]. The reader is also referred to [KK83].

Let $E \subset \widehat{\mathbb{C}}$ be a measurable set whose complement $E^* := \widehat{\mathbb{C}} \backslash E$ has positive measure, and set

$$B^*(E) := \{\mu \in B(\widehat{\mathbb{C}}) : \mu|_{E^*} = 0\}.$$

Denote by $Q(E)$ the family of normalized quasiconformal mappings $f_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ where $\mu \in B(E)$, and $Q_k(E) := \{f \in Q(E) : \|\mu_f\| \leq k\}$ for a $k \in [0, 1)$. Now let $F : Q(E) \to \mathbb{C}$ be a non-trivial holomorphic functional, where holomorphic means that it is complex Gateaux differentiable. Lastly, set $\|F\|_1 := \sup_{f \in Q(E)} |F(f)|$ and $\|F\|_k := \max_{f \in Q_k(E)} |F(f)|$.

Theorem 3.12. *Let $F : Q(E) \to \mathbb{C}$ be bounded. Then we have $\|F\|_k \leq k\|F\|_1$.*

Some applications of the theorem are demonstrated in [Kru05b, Chapter 3.4]. One of them is the distortion theorem for the class $S(k)$ (see also [Gut73, Corollary 7]);

$$\left(\frac{1-|z|}{1+|z|}\right)^k \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \left(\frac{1+|z|}{1-|z|}\right)^k.$$

For more results and proofs, see [Sch75], [Kru05b], [Kru05a].

The estimate of $|a_2|$ for the class $\mathcal{S}^*(k)$ is obtained by Schiffer and Schober.

Theorem 3.13 ([SS76]). *For all $f \in \mathcal{S}^*(k)$, we have the sharp estimate*

$$|a_2| \leq 2 - 4\left(\frac{\arccos k}{\pi}\right)^2.$$

For the sharp function, see [SS76, Eq. (4.2)]

Since the class $\mathcal{S}^*(k)$ is closed with respect to the Koebe transform (3.2), we have the fundamental estimate for $\mathcal{S}^*(k)$

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \left(2 - 4\left(\frac{\arccos k}{\pi}\right)^2\right)\frac{2|z|}{1-|z|^2}.$$

Following the standard argument for the class $\mathcal{S}$ (see Sect. 2.1, or [Pom75, pp. 21–22]), we have distortions of f and f' for $\mathcal{S}^*(k)$. We note that the same method as this is not valid for the class $\mathcal{S}(k)$ because the Koebe transform (3.2) does not fix ∞ except the case $\zeta = 0$.

As is written before, while the coefficient problem has been completely solved in the class $\mathcal{S}$, the question remains open for the class $\mathcal{S}(k)$. However, if we restrict ourselves to that k is sufficiently small, then the sharp result is established by Krushkal.

Theorem 3.14 ([Kru88, Kru95]). *For a function $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}(k)$, we have the sharp estimate*

$$|a_n| \leq \frac{2k}{n-1} \tag{3.3}$$

for $k \leq 1/(n^2+1)$.

The extremal function of the estimate (3.3) is given by

$$f_2(z) := \frac{z}{(1-kz)^2} \quad (k \in [0,1)),$$

$$f_n(z) := (f_2(z^{n-1}))^{1/(n-1)} = z + \frac{2k}{n-1}z^n + \cdots \quad n = 3, 4, \ldots.$$

To see $f_n \in \mathcal{S}(k)$, calculate $zf_n'(z)/f_n(z)$ and apply the quasiconformal extension criterion for starlike functions in Sect. 3.4.

3.4 Sufficient conditions for $\mathcal{S}(k)$

Since Bers introduced a new model of the universal Teichmüller space, numerous sufficient conditions for the class $\mathcal{S}(k)$ have been obtained. In this subsection we introduce only a few remarkable results.

In 1962, the first sufficient condition for $\mathcal{S}(k)$ was provided by Ahlfors and Weill.

Theorem 3.15 ([AW62]). *Let f be a non-constant meromorphic function defined on $\mathbb{D}$ and $k \in [0,1)$ be a constant. If f satisfies $\|S_f\| \leq 2k$, then f can be extended to a quasiconformal mapping F to $\widehat{\mathbb{C}}$. In this case the dilatation μ_F is given by*

$$\mu_F(z) := \begin{cases} -\dfrac{1}{2}(|z|^2-1)^2 S_F\left(\dfrac{1}{\bar{z}}\right)\dfrac{1}{\bar{z}^4}, & |z| > 1 \\ 0, & |z| < 1. \end{cases}$$

1972, Becker gave a sufficient condition in connection with the pre-Schwarzian derivative. Later it was generalized by Ahlfors.

Theorem 3.16 ([Ahl74]). *Let $f \in \mathcal{A}$. If there exists a $k \in [0,1)$ such that for a constant $c \in \mathbb{C}$ the inequality*

$$\left| c|z|^2 + (1-|z|^2)\frac{f''(z)}{f'(z)} \right| \leq k \tag{3.4}$$

holds for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

The case when $c = 0$ is due to Becker [Bec72]. Remark that the condition $|c| \leq k$ which was stated in the original form is embedded in the inequality (3.4) (see [Hot10]).

It is known that many univalence criteria are refined to quasiconformal extension criteria. For instance, Fait, Krzyż and Zygmunt proved the following theorem which is the refinement of the definition of strongly starlike functions (for the definition, see Sect. 3.3).

Theorem 3.17 ([FKZ76]). *Every strongly starlike functions of order α has a $\sin(\pi\alpha/2)$-quasiconformal extension to $\mathbb{C}$.*

This is generalized to strongly spiral-like functions [Sug12]. Some more results are obtained in [Bro84, Hot09] with explicit quasiconformal extensions which correspond to each subclass of $\mathcal{S}$. In particular, in [Hot09] the research relies on the (classical) Loewner theory, which will be mentioned in the next section.

Sugawa approached this problem by means of the holomorphic motions with extended λ-Lemma ([MSS83], [Slo91]).

Theorem 3.18 ([Sug99]). *Let $k \in [0,1)$ be a constant. For a given $f \in \mathcal{A}$, let p denote one of the quantities $zf'(z)/f(z), 1 + zf''(z)/f'(z)$ and $f'(z)$. If*

$$\left| \frac{1-p(z)}{1+p(z)} \right| \leq k$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

We note that in most of the sufficient conditions of quasiconformal extensions including the above theorems the case $k = 1$ reflects univalence criteria.

4. Classical Loewner Theory

The idea of the parametric representation method of conformal maps was introduced by Löwner [Löw23], and later developed by Kufarev [Kuf43] and Pommerenke [Pom65]. It describes a time-parametrized conformal map on $\mathbb{D}$ whose image is a continuously increasing simply connected domain. The key point is that such a family can be represented by a partial differential equation. Loewner's approach also made a significant contribution to quasiconformal extensions of univalent functions. This method was discovered by Becker.

Since our focus in this note is on univalent functions with quasiconformal extensions, we will deal with Loewner chains in the sense of Pommerenke (see [Pom75]). For one-slit maps as Löwner originally considered, see e.g., [DMG16] which also contains a list of references. For the classical theory, the reader is also referred to [Gol69, Chapter III-2], [Tsu75, Chapter IX-9], [Dur83, Chapter 3], [Hen86, Chapter 19], [RR94, Chapter 7-8], [Hay94, Chapter 7-8], [Con95, Chapter 17], [GK03, Chapter 3].

4.1 Classical Loewner chains

Let $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ be a function defined on $\mathbb{D} \times [0, \infty)$. f_t is said to be a (**classical**) **Loewner chain** if f_t satisfies the conditions (Fig. 2);

1. f_t is holomorphic and univalent in $\mathbb{D}$ for each $t \in [0, \infty)$;
2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$.

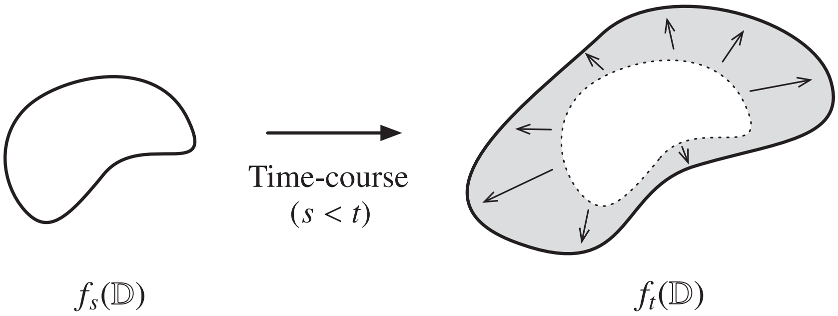

Fig. 2. The image of the unit disk $\mathbb{D}$ under f_t expands continuously as t increases.

One can also characterize it in the geometrical sense. Let $\{D_t\}_{t \geq 0}$ be a family of simply connected domains having the following properties;

1′. $0 \in D_0$;
2′. $D_s \subsetneq D_t$ for all $0 \leq s < t < \infty$;
3′. $D_{t_n} \to D_t$ if $t_n \to t < \infty$ and $D_{t_n} \to \mathbb{C}$ if $t_n \to \infty$ $(n \to \infty)$, in the sense of the kernel convergence.

Then by the Riemann mapping theorem there exists a family of conformal mappings $\{f_t\}_{t \geq 0}$ such that $f_t(0) = 0$ and $f_t'(0) > 0$ for all $t \geq 0$. We note that f_t is continuous on $t \in [0, \infty)$, and $f_s'(0) < f_t'(0)$ for all $s < t$ (for otherwise by the Schwarz Lemma $f_t^{-1} \circ f_s$ is an identity, which contradicts $D_s \subsetneq D_t$). So after rescaling as $f_0 \in \mathcal{S}$ and reparametrizing as $f_t'(0) = e^t$, we obtain a Loewner chain.

f_t has a time derivative almost everywhere on $[0, \infty)$ for each fixed $z \in \mathbb{D}$. In fact, applying the distortion theorem for $\mathcal{S}$ (Theorem 3.4), the next estimate follows.

Lemma 4.1. *For each fixed $z \in \mathbb{D}$, a Loewner chain f_t satisfies*

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4} |e^t - e^s|$$

for all $0 \leq s \leq t < \infty$.

Hence f_t is absolutely continuous on $t \in [0, \infty)$ for all fixed $z \in \mathbb{D}$.

A necessary and sufficient condition for a Loewner chain is shown by Pommerenke.

Theorem 4.2 ([Pom65, Pom75]). *Let $0 < r_0 \leq 1$. Let $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ be a function defined on $\mathbb{D} \times [0, \infty)$. Then f_t is a Loewner chain if and only if the following two conditions are satisfied;*

(i) f_t is holomorphic in $z \in \mathbb{D}_{r_0}$ for each $t \in [0, \infty)$, absolutely continuous in $t \in [0, \infty)$ for each $z \in \mathbb{D}_{r_0}$ and satisfies

$$|f_t| \leq K_0 e^t \quad (z \in \mathbb{D}_{r_0}, t \in [0, \infty)) \tag{4.1}$$

for some positive constant K_0.

(ii) There exists a function $p(z, t)$ analytic in $z \in \mathbb{D}$ for each $t \in [0, \infty)$ and measurable in $t \in [0, \infty)$ for each $z \in \mathbb{D}$ satisfying

$$\operatorname{Re} p(z, t) > 0 \quad (z \in \mathbb{D}, t \in [0, \infty))$$

such that

$$\dot{f}_t(z) = zf_t'(z)p(z,t) \quad (z \in \mathbb{D}_{r_0}, \text{a.e. } t \in [0,\infty)) \tag{4.2}$$

where $\dot{f} = \partial f/\partial t$ and $f' = \partial f/\partial z$.

The partial differential equation (4.2) is called the **Loewner–Kufarev PDE**, and the function p in (4.2) is called a **Herglotz function**.

Remark 4.3. Inequality (4.1) and the following classical result due to Dieudonné [Die31] (for the proof, see e.g., [Tsu75, p. 259]) ensure the existence of the uniform radius of univalence of $\{f_t\}_{t\geq 0}$; *Let f be holomorphic on $\mathbb{D}$ satisfying $f(0) = 0$, $f'(0) = a > 0$ and $|f(z)| < M$ for all $z \in \mathbb{D}$. Then f is univalent on the disk $\{|z| < \rho < 1\}$, where*

$$\rho := \frac{a}{M + \sqrt{M^2 - a^2}}.$$

Hence, although it is not written on the sufficient conditions of Theorem 4.2, f_t is implicitly assumed to be univalent on a certain disk whose radius is determined independently from $t \in [0,\infty)$.

Remark 4.4. (4.2) describes an expanding flow of the image domain $f_t(\mathbb{D})$ of a Loewner chain. Indeed, (4.2) can be written as

$$|\arg \dot{f}_t(z) - \arg zf_t'(z)| = |\arg p(z,t)| < \frac{\pi}{2}.$$

It implies that the velocity vector $\dot{f}_t$ at a boundary point of the domain $f_t(\mathbb{D}_r)$ points out of this set and therefore all points on $\partial f_t(\mathbb{D}_r)$ moves to outside of $\overline{f_t(\mathbb{D}_r)}$ when t increases (Fig. 3).

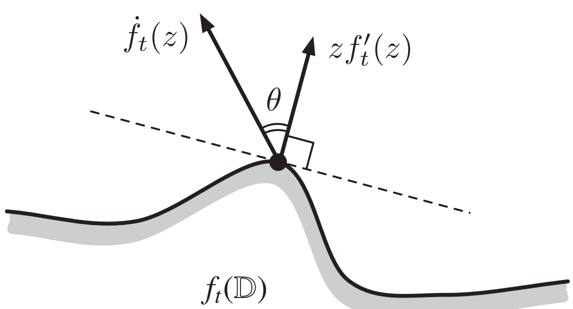

Fig. 3. The angle θ between the normal vector zf_t' of the tangent line and the velocity vector $\dot{f}_t$ satisfies $|\theta| < \pi/2$.

The next property is also important.

Theorem 4.5. *For any $f \in \mathcal{S}$, there exists a Loewner chain f_t such that $f_0 = f$.*

4.2 Evolution families

In Loewner theory, a two-parameter family of holomorphic self-maps of the unit disk $(\varphi_{s,t})$, $0 \leq s \leq t < \infty$, called an **evolution family**, plays a key role. To be precise, $(\varphi_{s,t})$ satisfies the followings;

1. $\varphi_{s,s}(z) = z$;
2. $\varphi_{s,t}(0) = 0$ and $\varphi_{s,t}'(0) = e^{s-t}$;
3. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < \infty$.

We note that $\varphi_{s,t}$ is not assumed to be univalent on $\mathbb{D}$. By means of the same idea as Lemma 4.1, we have the estimate for $0 \leq s \leq u \leq t < \infty$,

$$|\varphi_{s,t}(z) - \varphi_{u,t}(z)| \leq \frac{2|z|}{(1-|z|)^2}(1 - e^{s-u}),$$
$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq 2|z|\frac{1+|z|}{1-|z|}(1 - e^{u-t}),$$

for all $z \in \mathbb{D}$.

For a Loewner chain f_t, the function $\varphi_{s,t}(z) := (f_t^{-1} \circ f_s)(z)$ defines an evolution family. Since $f_t(\varphi_{s,t}(z)) = f_s$, differentiating both sides of the equation with respect to t we have $\dot{f}_t(\varphi_{s,t}) + f_t'(\varphi_{s,t})\dot{\varphi}_{s,t} = 0$. Hence one can obtain by (4.2)

$$\dot{\varphi}_{s,t}(z) = -\varphi_{s,t}(z)p(\varphi_{s,t}(z), t). \tag{4.3}$$

This is called the **Loewner–Kufarev ODE**. The following is the basic result on existence and uniqueness of a solution of the ODE.

Theorem 4.6. *Suppose that a function $p(z,t)$ is holomorphic in $z \in \mathbb{D}$ and measurable in $t \in [0,\infty)$ satisfying $\operatorname{Re} p(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \in [0,\infty)$. Then, for each fixed $z \in \mathbb{D}$ and $s \in [0,\infty)$, the initial value problem*

$$\frac{dw}{dt} = -wp(w,t)$$

for almost all $t \in [s,\infty)$ has a unique absolutely continuous solution $w(t)$ with the initial condition $w(s) = z$. If we write $\varphi_{s,t}(z) := w(t)$, then $\varphi_{s,t}$ is an evolution family and univalent on $\mathbb{D}$. Further, the function $f_s(z)$ defined by

$$f_s(z) := \lim_{t\to\infty} e^t \varphi_{s,t}(z) \tag{4.4}$$

exists locally uniformly in $z \in \mathbb{D}$ and is a Loewner chain.

Conversely, if f_t is a Loewner chain and $\varphi_{s,t}$ is an evolution family associated with f_t by $\varphi_{s,t} := f_t^{-1} \circ f_s$. Then for almost all $t \in [s,\infty)$, $\varphi_{s,t}$ satisfies

$$\frac{d\varphi_{s,t}}{dt} = -\varphi_{s,t} p(\varphi_{s,t}, t)$$

for all $z \in \mathbb{D}$, and (4.4) is satisfied.

In the first assertion of Theorem 4.6, it may happen that two different Herglotz functions p_1 and p_2 generate the same evolution family $\varphi_{s,t}$. Then $p_1(z,t) = p_2(z,t)$ for almost all $t \geq 0$. Hence Theorem 4.6 says that there is a one-to-one correspondence between an evolution family and a Herglotz function in such a sense.

4.3 Loewner chains and quasiconformal extensions

An interesting method connecting Loewner theory and quasiconformal extensions was obtained by Becker.

Theorem 4.7 ([Bec72], [Bec80]). *Suppose that f_t is a Loewner chain for which $p(z,t)$ in (4.2) satisfying the condition*

$$p(z,t) \in U(k) := \left\{ w \in \mathbb{C} : \left| \frac{1-w}{1+w} \right| \leq k \right\} \tag{4.5}$$

i.e., $p(z,t)$ lies in the closed hyperbolic disk $U(k)$ in the right half-plane centered at 1 with radius $\operatorname{arctanh} k$*, for all $z \in \mathbb{D}$ and almost all $t \geq 0$. Then f_t admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the map F defined by*

$$F(re^{i\theta}) = \begin{cases} f_0(re^{i\theta}), & \text{if } r < 1, \\ f_{\log r}(e^{i\theta}), & \text{if } r \geq 1, \end{cases} \tag{4.6}$$

is a k-quasiconformal extension of f_0 to $\mathbb{C}$.

The idea of the theorem is the following. By Koebe's 1/4-Theorem (Theorem 3.3), $f_t(\mathbb{D})$ must contain the disk whose center is 0 with radius $e^t/4$. Thus $f_t(\mathbb{D})$ tends to $\mathbb{C}$ as $t \to \infty$. This fact implies that the boundary $\partial f_t(\mathbb{D})$ runs throughout on $\mathbb{C}\backslash f_0(\mathbb{D})$. Therefore the mapping $F : \mathbb{D}^* \to \mathbb{C}\backslash f_0(\mathbb{D})$ is constructed by (4.6) which gives a correspondence between the circle $\{|z| = e^t\}$ and the boundary $\partial f_t(\mathbb{D})$. Its quasiconformality follows from the condition (4.5).

Betker generalized Theorem 4.7 by introducing an inverse version of Loewner chains. Let $\omega_t(z) = \sum_{n=1}^{\infty} b_n(t) z^n$, $b_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0,\infty)$, where $b_1(t)$ is a complex-valued, locally absolutely continuous function on $[0,\infty)$. Then ω_t is said to be an **inverse Loewner chain** if;

1. ω_t is univalent in $\mathbb{D}$ for each $t \geq 0$;
2. $|b_1(t)|$ decreases strictly monotonically as t increases, and $\lim_{t\to\infty} |b_1(t)| \to 0$;
3. $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$ for $0 \leq s < t < \infty$;
4. $\omega_0(z) = z$ and $\omega_s(0) = \omega_t(0)$ for $0 \leq s \leq t < \infty$.

ω_t also satisfies the partial differential equation

$$\dot{\omega}_t(z) = -z\omega_t'(z) q(z,t) \quad (z \in \mathbb{D}, \text{a.e. } t \geq 0), \tag{4.7}$$

where q is a Herglotz function. Conversely, we can construct an inverse Loewner chain by means of (4.7) according to the following lemma:

Lemma 4.8. *Let $q(z,t)$ be a Herglotz function. Suppose that $q(0,t)$ be locally integrable in $[0,\infty)$ with $\int_0^\infty \operatorname{Re} q(0,t)dt = \infty$. Then there exists an inverse Loewner chain w_t with (4.7).*

By applying the notion of an inverse Loewner chain, we obtain a generalization of Becker's result.

Theorem 4.9 ([Bet92]). *Let $k \in [0,1)$. Let f_t be a Loewner chain for which $p(z,t)$ in (4.2) satisfying the condition*

$$\left|\frac{p(z,t)-\overline{q(z,t)}}{p(z,t)+q(z,t)}\right| \leq k \quad (z \in \mathbb{D}, \text{a.e. } t \geq 0),$$

where $q(z,t)$ is a Herglotz function. Let ω_t be the inverse Loewner chain which is generated with q by Lemma 4.8. Then f_t and ω_t are continuous and injective on $\overline{\mathbb{D}}$ for each $t \geq 0$, and f_0 has a k-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$ which is defined by

$$F\left(\frac{1}{\overline{\omega_t(e^{i\theta})}}\right) = f_t(e^{i\theta}) \quad (\theta \in [0,2\pi), t \geq 0).$$

We obtain Becker's result for $q(z,t) = 1$. In this case an inverse Loewner chain is given by $\omega_t(z) = e^{-t}z$. Further, choosing ω as $p = q$, we have the following corollary:

Corollary 4.10 ([Bet92]). *Let $\alpha \in [0,1)$. Suppose that f_t is a Loewner chain for which $p(z,t)$ in* (4.2) *satisfies*

$$p(z,t) \in \Delta(-\alpha,\alpha) = \left\{z : -\frac{\alpha\pi}{2} \leq \arg z \leq \frac{\alpha\pi}{2}\right\}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0,\infty)$. Then f_t admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and f_0 has a $\sin\alpha\pi/2$-quasiconformal extension to $\mathbb{C}$.

Corollary 4.10 does not include Theorem 4.7 in view of the dilatation of the extended quasiconformal map. In fact, the following relation holds;

$$U(k) \subset \Delta(-k_0,k_0) \quad \text{where} \quad k_0 := \frac{2}{\pi}\arcsin\left(\frac{2k}{1+k^2}\right) \geq k.$$

Remark that $k_0 = k$ if and only if $k = 0$.

In contrast to Becker's quasiconformal extension theorem, the theorem due to Betker does not always provide an explicit quasiconformal extension. The reason is based on the fact that it is difficult to express an inverse Loewner chain ω_t which has the same Herglotz function as a given Loewner chain f_t in an explicit form. For details, see [HW17, Sect. 5].

4.4 Applications to the theory of univalent functions

Here we will see some applications of Theorem 4.2 and Theorem 4.7. In order to find out explicit Loewner chains which corresponds to the typical subclasses of $\mathcal{S}$, we need to observe their geometric features. Some Loewner chains are not normalized as $f'(0) = e^t$. In [Hot11], it is discussed that Theorem 4.2 and Theorem 4.7 work well without such a normalization. In fact, a Loewner chain is generalized for a function $f_t(z) = \sum_{n=1}^{\infty} a_t(z)z^t$ where $a_1(t) \neq 0$ is a complex-valued, locally absolutely continuous function on $t \in [0,\infty)$ with $\lim_{n\to\infty} |a_1(t)| = \infty$. Further, either the condition that $|a_1(t)|$ is strictly increasing with respect to $t \in [0,\infty)$, or $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$ should be assumed.

I Convex functions

A function $f \in \mathcal{S}$ is said to be **convex** and belongs to $\mathcal{K}$ if $f(\mathbb{D})$ is a convex domain. It is known that $f \in \mathcal{K}$ if and only if

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0$$

for all $z \in \mathbb{D}$. A flow of the expansion for a convex function is considered as following. If a boundary point $\zeta \in \partial f(\mathbb{D})$ moves to the direction of their normal vector $\zeta f'(\zeta)$ according to the parameter t increases, then ζ always runs on the complement of $f(\mathbb{D})$ and their trajectories do not cross each other. In view of this, it is natural to set a Loewner chain as

$$f_t(z) = f(z) + t \cdot zf'(z) \tag{4.8}$$

Then we have $1/p(z,t) = 1 + t \cdot [1 + (zf''(z)/f'(z))]$ and hence f_t is a Loewner chain if $f \in \mathcal{K}$.

II Starlike functions

Next, consider a **starlike function** (with respect to 0), i.e., a function $f \in \mathcal{S}$ such that for every $z \in \mathbb{D}$ the segment connecting $f(z)$ and 0 lies in $f(\mathbb{D})$. Denote by $\mathcal{S}^*$ the family of starlike functions. An analytic characterization for starlike functions is

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$$

for all $z \in \mathbb{D}$. It follows from the definition that for a boundary point $\zeta \in \partial f(\mathbb{D})$, the ray $\{t\zeta : t \geq 1\}$ always lies in the exterior of $f(\mathbb{D})$. Hence the possible chain for $\mathcal{S}^*$ is

$$f_t(z) := e^t f(z). \tag{4.9}$$

A simple calculation shows that $1/p(z,t) = zf'(z)/f(z)$ and therefore f_t is a Loewner chain if $f \in \mathcal{S}^*$. In the case of **spiral-like functions**, i.e., functions $f \in \mathcal{S}$ defined by the condition

$$\mathrm{Re}\left[e^{-i\lambda}\frac{zf'(z)}{f(z)}\right] > 0$$

for some $\lambda \in (-\pi/2, \pi/2)$, a Loewner chain is given by

$$f_t(z) := e^{ct} f(z) \tag{4.10}$$

with $c := e^{i\lambda}$ whose trajectories draw logarithmic spirals. The case $\lambda = 0$ corresponds to starlike functions.

III Close-to-convex functions

For a given $f \in \mathcal{S}$, if there exists a $g \in \mathcal{S}^*$ such that

$$\mathrm{Re}\left[e^{-i\lambda}\frac{zf'(z)}{g(z)}\right] > 0$$

for some $\lambda \in (-\pi/2, \pi/2)$ and all $z \in \mathbb{D}$, then f is said to be **close-to-convex** and we denote by $f \in \mathcal{C}$. The image $f(\mathbb{D})$ by a close-to-convex function is known to be a **linearly accessible domain**, namely, $\mathbb{C}\backslash f(\mathbb{D})$ is a union of closed half-lines which are mutually disjoint except their end points. f is said to be **linearly accessible** if $f(\mathbb{D})$ is a linearly accessible domain.

A Loewner chain corresponding to the class $\mathcal{C}$ is given by

$$f_t(z) := f(z) + t \cdot e^{i\lambda} g(z). \tag{4.11}$$

Then $1/p(z,t) = e^{-i\lambda}(zf'(z)/g(z)) + t(zg'(z)/g(z))$ and hence $\mathrm{Re}\, p(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \geq 0$. The validity of the chain (4.11) is given by the following consideration.

Below we consider the case $\lambda = 0$. Take a fixed $\rho \in (0,1)$ and set $f_\rho(z) := f(\rho z)/\rho$ and $g_\rho(z) := g(\rho z)/\rho$. Then $f_t^\rho := f_\rho + tg_\rho$ is well-defined on $\overline{\mathbb{D}}$. For each boundary point $\zeta_0 \in \partial\mathbb{D}$,

$$\gamma_{\zeta_0} := \{f_t^\rho(\zeta_0) : t \in [0,\infty)\}$$

defines a half-line with an inclination of $\arg g_\rho(\zeta_0)$. Let $\zeta_1 \in \partial\mathbb{D}$ be another boundary point with $\zeta_1 \neq \zeta_0$. Since f_t^ρ is a Loewner chain, γ_{ζ_0} and γ_{ζ_1} do not have any intersection. Further, by the property $f_t^\rho(\mathbb{D}) \to \mathbb{C}$ as $t \to \infty$, γ_ζ runs throughout $\mathbb{C}\backslash f_\rho(\mathbb{D})$ if $\arg\zeta$ is taken from 0 to 2π. Therefore $\bigcup_{\zeta\in\partial\mathbb{D}} \gamma_\zeta = \mathbb{C}\backslash f_\rho(\mathbb{D})$ which proves that every $f_\rho \in \mathcal{C}$ is linearly accessible. It is known that the family of linearly accessible functions $f \in \mathcal{S}$ is compact in the topology of locally uniform convergence ([Bie36]). Hence we conclude that $f = \lim_{\rho\to1} f_0^\rho \in \mathcal{C}$ is linearly accessible.

One can prove it without compactness of the family of linearly accessible functions. Let $p_\zeta[f]$ be the prime end defined on a domain $f(\mathbb{D})$ corresponding to a boundary point $\zeta \in \partial\mathbb{D}$ and $I_\zeta[f]$ be the impression of the prime end $p_\zeta[f]$. It is known that there is a one-to-one correspondence among ζ, $p_\zeta[f]$ and $I_\zeta[f]$ (see [Pom92, Chapter 2]). Since g is starlike, for all $w_g \in I_{\zeta_0}[g]\backslash\{\infty\}$, $\arg w_g$ reflects one real value. Then redefine γ_{ζ_0} as a family of rays (may consist of only one ray) by

$$\gamma_{\zeta_0} := \{w_f + t\exp(i\arg w_g) : w_f \in I_{\zeta_0}[f]\backslash\{\infty\}, w_g \in I_{\zeta_0}[g]\backslash\{\infty\}, t \in [0,\infty)\}.$$

Then $\bigcup_{\zeta\in\partial\mathbb{D}} \gamma_\zeta = \mathbb{C}\backslash f(\mathbb{D})$, for otherwise there exists a point $z \in \mathbb{C}\backslash f(\mathbb{D})$ such that $z \notin \gamma_\zeta$ for any $\zeta \in \partial\mathbb{D}$ which contradicts the fact that f_t is a Loewner chain. By choosing proper components of $\bigcup_{\zeta\in\partial\mathbb{D}} \gamma_\zeta$, a union of closed half-lines for that $f(\mathbb{D})$ is a linearly accessible domain is given.

The Noshiro–Warschawski class is known as the special case of close-to-convex functions. Noshiro [Nos35] and Warschawski [War35] independently proved that if a function $f \in \mathcal{A}$ satisfies

$$\mathrm{Re}\, f'(z) > 0$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}$ (see e.g., [HW17]). We denote the family of such functions by $\mathcal{R}$. Choosing $g(z) = z$ and $\lambda = 0$ in (4.11), we have the chain

$$f_t(z) := f(z) + tz \tag{4.12}$$

which proves $\mathcal{R} \subset \mathcal{C} \subset \mathcal{S}$. By this consideration, the following property is derived.

Proposition 4.11. *For a function $f \in \mathcal{R}$, if the boundary of $f(\mathbb{D})$ is locally connected, then $e^{i\theta} \mapsto f(e^{i\theta}) \in \mathbb{C}$ is one-to-one.*

Further, we can make use of (4.12) to observe the shape of $f(\mathbb{D})$ for an $f \in \mathcal{R}$. We assume that the boundary of $f(\mathbb{D})$ is locally connected. Then the half-line $\gamma_{e^{i\theta}} := \{f(e^{i\theta}) + te^{i\theta} : t \in [0,\infty)\}$ is well-defined. Since the inclination of $\gamma_{e^{i\theta}}$ is exactly θ, we obtain the following property for $\mathcal{R}$;

Proposition 4.12. *Let $f \in \mathcal{S}$. If $f(\mathbb{D})$ contains some sector domain in $\mathbb{C}$, then f does not belong to $\mathcal{R}$.*

For example, $f(z) = ((1+z)/(1-z) - 1)/2$ maps $\mathbb{D}$ onto the half-plane. Hence we immediately conclude that $f \notin \mathcal{R}$ (of course in this case it is easy to see that f does not satisfy $\mathrm{Re}\, f' > 0$ by calculation).

IV Bazilevič functions

For real constants $\alpha > 0$ and $\beta \in \mathbb{R}$, set $\gamma = \alpha + i\beta$. In 1955, Bazilevič [Baz55] showed that the function defined by

$$f(z) = \left[(\alpha + i\beta) \int_0^z h(u) g(u)^\alpha u^{i\beta - 1} du\right]^{1/(\alpha + i\beta)}$$

where g is a starlike univalent function and h is an analytic function with $h(0) = 1$ satisfying $\mathrm{Re}(e^{i\lambda} h) > 0$ in $\mathbb{D}$ for some $\lambda \in \mathbb{R}$ belongs to the class $\mathcal{S}$. It is called a **Bazilevič function of type** (α, β) and we denote by $\mathcal{B}(\alpha, \beta)$ the family of Bazilevič functions of type (α, β). A simple observation shows that $f \in \mathcal{B}(\alpha, \beta)$ if and only if

$$\mathrm{Re}\left\{ e^{i\lambda} \frac{z f'(z)}{f(z)} \left(\frac{f(z)}{g(z)}\right)^\alpha \left(\frac{f(z)}{z}\right)^{i\beta} \right\} > 0 \quad (z \in \mathbb{D})$$

for some $g \in \mathcal{S}^*$. A Loewner chain for the class $\mathcal{B}(\alpha, \beta)$ is known ([Pom65, p. 166]) as

$$f_t(z) = (f(z)^\gamma + t \cdot \gamma g(z)^\alpha z^{i\beta})^{1/\gamma}. \tag{4.13}$$

By using the previous argument for close-to-convex functions, we can derive some geometric features for the class $\mathcal{B}(\alpha, \beta)$. We consider the simple case that the boundaries of $f(\mathbb{D})$ and $g(\mathbb{D})$ are locally connected. Then for each point $\zeta_0 \in \partial\mathbb{D}$, the curve $\{\delta_{\zeta_0}(t) := (f(\zeta_0)^\gamma + t \cdot \gamma g(\zeta_0)^\alpha {\zeta_0}^{i\beta})^{1/\gamma} : t \in [0, \infty)\}$ is defined. Hence $f(\mathbb{D})$ is described as the complement of a union of such curves.

Observe the behavior of the curve. If $\beta > 0$ (or $\beta < 0$), then it draws an asymptotically similar curve as a logarithmic spiral which evolves counterclockwise (or clockwise). On the other hand, in the case when $\beta = 0$, firstly it draws a spiral, then tends to a straight line as t gets large. In both cases, the curvature $d_t \arg \delta'_{\zeta_0}(t) = \mathrm{Im}[\delta''_{\zeta_0}(t)/\delta'_{\zeta_0}(t)]$ is always positive or negative. From this fact one can construct functions which do not belong to any $\mathcal{B}(\alpha, \beta)$ easily. Consider a slit domain $\mathbb{C}\backslash\gamma$. If the curvature of the slit γ takes both positive and negative values (ex. $\gamma = \{x + iy : y = \sin x \text{ and } x > 0\}$), or γ is not smooth (ex. $\gamma = \{x \geq 0\} \cup \{iy : y \in (0, 1)\}$), then such slit domains cannot be images of $\mathbb{D}$ under any $f \in \mathcal{B}(\alpha, \beta)$.

4.5 Applications to quasiconformal extensions

Applying Theorem 4.7 to the chains (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) we obtain quasiconformal extension criteria for each subclass of $\mathcal{S}$ with explicit extensions. In this case the chains (4.8), (4.11), (4.12) and (4.13) should be reparametrized by $e^t - 1$. The theorems can be found in [Hot09, Hot11, HW11, Hot13]. Further, by Theorem 4.10 with the chains (4.9) and (4.10) we obtain quasiconformal extension criteria given by [FKZ76] and [Sug12]. For an explicit extension of these cases, see [HW17].

The other typical example is Theorem 3.16, Ahlfors's quasiconformal extension criterion. It can be obtained by Theorem 4.7 with the chain

$$f_t(z) := f(e^{-t}z) + \frac{1}{1+c}(e^t - e^{-t}) z f'(e^{-t}z),$$

for then

$$\frac{1 - p(z,t)}{1 + p(z,t)} = \frac{z f_t'(z) - \dot{f}_t(z)}{z f_t'(z) + \dot{f}_t(z)} = c\,\frac{1}{e^{2t}} + \left(1 - \frac{1}{e^{2t}}\right) \frac{e^{-t} z f'(e^{-t}z)}{f''(e^{-t}z)}.$$

5. Modern Loewner Theory

Recently a new approach to treat evolution families and Loewner chains in a general framework has been suggested by Bracci, Contreras, Díaz-Madrigal and Gumenyuk ([BCDM12], [BCDM09], [CDMG10b]). It enables us to describe a variety of the dynamics of one-parameter family of conformal mappings. In this section we outline the theory of generalized evolution families and Loewner chains. The key fact is that there is an (essentially) one-to-one correspondence among evolution families and Herglotz vector fields. We also present some results about generalized Loewner chains with quasiconformal extensions.

5.1 Semigroups of holomorphic mappings

Let D be a simply connected domain in the complex plane $\mathbb{C}$. We denote the family of all holomorphic functions on D by $\mathrm{Hol}(D, \mathbb{C})$. If $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is a self-mapping of $\mathbb{D}$, then we will denote the family of such functions by $\mathrm{Hol}(\mathbb{D})$.

An easy consequence of the well-known Schwarz-Pick Lemma, $f \in \mathrm{Hol}(\mathbb{D})\backslash\{\mathrm{id}\}$ may have at most one fixed point in $\mathbb{D}$. If such a point exists, then it is called the **Denjoy–Wolff point** of f. On the other hand, if f does not have a fixed point in $\mathbb{D}$, then the Denjoy–Wolff theorem (see e.g., [ES10]) claims that there exists a unique boundary fixed point $\angle\lim_{z\to\tau} f(z) = \tau \in \partial\mathbb{D}$ such that the sequence of iterates $\{f^n\}_{n\in\mathbb{N}}$ converges to τ locally uniformly, where $\angle\lim$ denotes an angular (or non-tangential) limit, and f^n an n-th iterate of f, namely, $f^1 := f$ and $f^n := f \circ f^{n-1}$. In this case the boundary point τ is also called the **Denjoy–Wolff point**. Remark that a boundary fixed point is not always the Denjoy–Wolff point. A simple example is observed with a holomorphic automorphism of $\mathbb{D}$, $f(z) = (z+a)/(1+\bar{a}z)$ with $a \in \mathbb{D}\backslash\{0\}$. f has two boundary fixed points $\pm a/|a|$, but only one $a/|a|$ can be the Denjoy–Wolff point.

A family $(\phi_t)_{t\geq 0}$ of holomorphic self-mappings of $\mathbb{D}$ is called a **one-parameter semigroup** if;

1. $\phi_0 = id_{\mathbb{D}}$;
2. $\phi_t \circ \phi_s = \phi_{s+t}$ for all $s, t \in [0, \infty)$;
3. $\lim_{t\to 0^+} \phi_t(z) = z$ locally uniformly on $\mathbb{D}$;

In the definition, only right continuity at 0 is required.

The following theorem is fundamental in the theory of one-parameter semigroups.

Theorem 5.1. *Let $(\phi_t)_{t\geq 0}$ be a one-parameter semigroup of holomorphic self-mappings of $\mathbb{D}$. Then for each $z \in \mathbb{D}$ there exists the limit*

$$\lim_{t\to 0^+} \frac{\phi_t(z) - z}{t} =: G(z) \tag{5.1}$$

such that $G \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$. The convergence in (5.1) *is uniform on each compact subset of $\mathbb{D}$. Moreover, the semigroup $(\phi_t)_{t\geq 0}$ can be defined as a unique solution of the Cauchy problem*

$$\frac{d\phi_t(z)}{dt} = G(\phi_t(z)) \quad (t \geq 0)$$

with the initial condition $\phi_0(z) = z$.

The above function $G \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is called the **infinitesimal generator** of the semigroup. Various criteria which guarantee that a homeomorphic function $G \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is the infinitesimal generator are known. As one of them, in 1978 Berkson and Porta gave the following fundamental characterization.

Theorem 5.2 ([BP78]). *A holomorphic function $G \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator if and only if there exists a $\tau \in \overline{\mathbb{D}}$ and a function $p \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with $\mathrm{Re}\, p(z) \geq 0$ for all $z \in \mathbb{D}$ such that*

$$G(z) = (\tau - z)(1 - \bar{\tau}z)p(z) \tag{5.2}$$

for all $z \in \mathbb{D}$.

The Eq. (5.2) is called the **Berkson–Porta representation**. In fact, the point τ in (5.2) is the Denjoy–Wolff point of the one-parameter semigroup generated with G.

5.2 Generalized evolution families in the unit disk

We have discussed in Sect. 3.1 that a Loewner chain f_t (in the classical sense) defines a function $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}$ which is called an evolution family. Recently, this notion and one-parameter semigroups are unified and generalized as following.

Definition 5.3 ([BCDM12, Definition 3.1]). A family of holomorphic self-maps of the unit disk $(\varphi_{s,t})$, $0 \leq s \leq t < \infty$, is an **evolution family** if;

- EF1. $\varphi_{s,s}(z) = z$;
- EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < \infty$;
- EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative locally integrable function $k_{z,T} : [0, T] \to \mathbb{R}_{\geq 0}$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi$$

for all $0 \leq s \leq u \leq t \leq T$.

We denote the family of evolution families by EF.

Remark 5.4. If $(\phi_t) \subset \mathrm{Hol}(\mathbb{D})$ is a one-parameter semigroup, then $(\varphi_{s,t})_{0\leq s\leq t<\infty} := (\phi_{t-s})_{0\leq s\leq t<\infty}$ forms an evolution family.

Remark 5.5. In [BCDM12] and [CDMG10b], the definitions of evolution families and some other relevant notions contain an integrability order $d \in [1, +\infty]$. Since this parameter is not important for the discussions in this article, we assume that $d = 1$ which is the most general case of the order.

Some fundamental properties of EF are derived as follows.

Theorem 5.6 ([BCDM12, Proposition 3.7, Corollary 6.3]). *Let $(\varphi_{s,t}) \in$ EF.*

(i) $\varphi_{s,t}$ is univalent in $\mathbb{D}$ for all $0 \le s \le t < \infty$.

(ii) For each $z_0 \in \mathbb{D}$ and $s_0 \in [0, \infty)$, $\varphi_{s_0,t}(z_0)$ is locally absolutely continuous on $t \in [s_0, \infty)$.

(iii) For each $z_0 \in \mathbb{D}$ and $t_0 \in (0, \infty)$, $\varphi_{s,t_0}(z_0)$ is absolutely continuous on $s \in [0, t_0]$.

Next, we extend the notion of infinitesimal generators to the same structure as evolution families.

Definition 5.7 ([BCDM12, Definition 4.1, Definition 4.3]). **A Herglotz vector field** on the unit disk $\mathbb{D}$ is a function $G : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties;

HV1. for all $z \in \mathbb{D}$, $G(z, \cdot)$ is measurable on $[0, \infty)$;

HV2. for any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative locally integrable function $k_{K,T} : [0, T] \to \mathbb{R}_{\ge 0}$ such that

$$|G(z, t)| \le k_{K,T}(t)$$

for all $z \in K$ and for almost every $t \in [0, T]$;

HV3. for almost all $t \in [0, \infty)$, $G(\cdot t)$ is an infinitesimal generator.

We denote by HV the family of all Herglotz vector fields.

The following theorem states the relation between $(\varphi_{s,t}) \in$ EF and $G \in$ HV.

Theorem 5.8 ([BCDM12, Theorem 5.2, Theorem 6.2]). *For any $(\varphi_{s,t}) \in$ EF, there exists an essentially unique $G \in$ HV such that*

$$\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z), t) \tag{5.3}$$

for all $z \in \mathbb{D}$, all $s \in [0, \infty)$ and almost all $t \in [s, \infty)$. Conversely, for any $G \in$ HV, a family of unique solutions of (5.3) *with the initial condition $\varphi_{s,s}(z) = z$ generates an evolution family.*

Here, **essentially unique** means that if $G^*(z, t)$ is another Herglotz vector field which satisfies (5.3), then $G(\cdot, t) = G^*(\cdot, t)$ for almost every $t \ge 0$.

The similar mutual characterization holds between a Herglotz vector field and a pair of the generalized Denjoy–Wolff point τ and a generalized Herglotz function.

Definition 5.9 ([BCDM12, Definition 4.5]). **A Herglotz function** on the unit disk $\mathbb{D}$ is a function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ with the following properties;

HF1. for all $z \in \mathbb{D}$, $p(z, \cdot)$ is locally integrable on $[0, \infty)$;

HF2. for almost all $t \in [0, \infty)$, $p(\cdot, t)$ is holomorphic on $\mathbb{D}$;

HF3. $\operatorname{Re} p(z, t) \ge 0$ for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$.

We denote HF the family of all Herglotz functions.

Theorem 5.10 ([BCDM12, Theorem 4.8]). *Let $G \in$ HV. Then there exists an essentially unique measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ and $p \in$ HF such that*

$$G(z, t) = (\tau(t) - z)(1 - \overline{\tau(t)}z)p(z, t) \tag{5.4}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Conversely, for a given measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ and $p \in$ HF, the Eq. (5.4) *forms a Herglotz vector field.*

For convenience, we call the above measurable function $\tau : [0, \infty) \to \overline{\mathbb{D}}$ the **Denjoy–Wolff function** and denote by $\tau \in$ DW. A pair (p, τ) of $p \in$ HV and $\tau \in$ DW is called the **Berkson–Porta data** for $G \in$ HV. We denote the set of all Berkson–Porta data by BP. If two $(p, \tau), (\tilde{p}, \tilde{\tau}) \in$ BP generate the same $G \in$ HV up to a set of measure zero, then $p = \tilde{p}$ for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$ and $\tau = \tilde{\tau}$ for almost all $[t, \infty)$ such that $G(\cdot, t) \not\equiv 0$.

Hence, there is a one-to-one correspondence among $(\varphi_{s,t}) \in$ EF, $G \in$ HV and $(p, \tau) \in$ BP. In particular, the relation of $\varphi_{s,t}$ and (p, τ) is described by the ordinary differential equation

$$\dot{\varphi}_{s,t}(z) = (\tau(t) - \varphi_{s,t}(z))(1 - \overline{\tau(t)}\varphi_{s,t}(z))p(\varphi_{s,t}(z), t) \tag{5.5}$$

which incorporates the Loewner–Kufarev ODE (4.3) and the Berkson–Porta representation (5.2) as special cases.

5.3 Generalized Loewner chains

According to the notion of evolution families, Loewner chains are also generalized as follows.

Definition 5.11 ([CDMG10b, Definition 1.2]). A family of holomorphic functions $(f_t)_{t \ge 0}$ on the unit disk $\mathbb{D}$ is called

a **Loewner chain** if;
LC1. $f_t : \mathbb{D} \to \mathbb{C}$ is univalent for each $t \in [0, \infty)$;
LC2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \le s < t < \infty$;
LC3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$, there exists a non-negative function $k_{K,T} : [0, T] \to \mathbb{R}_{\ge 0}$ such that

$$|f_s(z) - f_t(z)| \le \int_s^t k_{K,T}(\xi) d\xi$$

for all $z \in K$ and all $0 \le s \le t \le T$.
Further, a Loewner chain will be said to be **normalized** if $f_0 \in \mathcal{S}$.

We denote a family of Loewner chains by LC. Remark that in Definition 5.11, any assumption is not required to $f_t(0)$ and $f_t'(0)$. It implies that a subordination property that $f_s(\mathbb{D}_r) \subset f_t(\mathbb{D}_r)$ for all $r \in (0, 1)$ and $0 \le s < t < \infty$ does not hold any longer in general. Further, we even do not know whether the **Loewner range**

$$\Omega[(f_t)] := \bigcup_{t \ge 0} f_t(\mathbb{D})$$

is the whole complex plane or not.
The next theorem gives a relation between Loewner chains and evolution families.

Theorem 5.12 ([CDMG10b, Theorem 1.3]). *For any $(f_t) \in$ LC, if we define*

$$\varphi_{s,t}(z) := (f_t^{-1} \circ f_s)(z) \quad (z \in \mathbb{D}, 0 \le s \le t < \infty)$$

then $(\varphi_{s,t}) \in$ EF. Conversely, for any $(\varphi_{s,t}) \in$ EF, there exists an $(f_t) \in$ LC such that the following equality holds

$$(f_t \circ \varphi_{s,t})(z) = f_s(z) \quad (z \in \mathbb{D}, 0 \le s \le t < \infty). \tag{5.6}$$

Differentiate both sides of (5.6) with respect to t then $f_t'(\varphi_{s,t}) \cdot \dot{\varphi}_{s,t} + \dot{f}_t(\varphi_{s,t}) = 0$ and therefore combining to (5.5) we have the following generalized Loewner–Kufarev PDE

$$\dot{f}_t(z) = (z - \tau(t))(1 - \overline{\tau(t)}z) f_t'(z) p(z, t). \tag{5.7}$$

We shall observe (5.7). Since the term $\dot{f}_t(z)$ gives a velocity vector at the point $f_t(z)$, the right-hand side of the Eq. (5.7) defines a vector field on $f_t(\mathbb{D})$. Assume that p is not identically equal to zero. Then $\dot{f}_t(z) = 0$ if $z = \tau(t)$. It implies that the point $f_t(\tau(t))$ plays a role of an "eye" of the flow described by $f_t(z)$. Since the Denjoy–Wolff function τ is assumed to be only measurable w.r.t. t, the origin $f_t(\tau(t))$ of the vector field moves measurably. This observation indicates that Loewner chain describes various flows of expanding simply connected domains. The classical radial Loewner–Kufarev PDE is given as the special case of (5.7) with $\tau \equiv 0$.
In general, for a given evolution family $(\varphi_{s,t})$, the Eq. (5.6) does not define a unique Loewner chain. That is, there is no guarantee that $\mathcal{L}[(\varphi_{s,t})]$, the family of normalized Loewner chains associated with $(\varphi_{s,t}) \in$ EF, consists of one function. However, $\mathcal{L}[(\varphi_{s,t})]$ always includes one special Loewner chain (in [CDMG10b], such a chain is called **standard**) and in this sense (f_t) is determined uniquely. Further, it is sometimes the only member of $\mathcal{L}[(\varphi_{s,t})]$. The following theorem states such properties of the uniqueness for Loewner chains.

Theorem 5.13 ([CDMG10b, Theorem 1.6 and Theorem 1.7]). *Let $(\varphi_{s,t}) \in$ EF. Then there exists a unique normalized $(f_t) \in$ LC such that $\Omega[(f_t)]$ is either $\mathbb{C}$ or an Euclidean disk in $\mathbb{C}$ whose center is the origin. Furthermore;*
- *The following 4 statements are equivalent;*
  - *(i) $\Omega[(f_t)] = \mathbb{C}$;*
  - *(ii) $\mathcal{L}[(\varphi_{s,t})]$ consists of only one function;*
  - *(iii) $\beta(z) = 0$ for all $z \in \mathbb{D}$, where*

$$\beta(z) := \lim_{t \to +\infty} \frac{|\varphi_{0,t}'(z)|}{1 - |\varphi_{0,t}(z)|^2};$$

  - *(iv) there exists at least one point $z_0 \in \mathbb{D}$ such that $\beta(z_0) = 0$.*
- *On the other hand, if $\Omega[(f_t)] \ne \mathbb{C}$, then it is written by*

$$\Omega[(f_t)] = \left\{ w : |w| < \frac{1}{\beta(0)} \right\},$$

and for the other normalized Loewner chain g_t associated with $(\varphi_{s,t})$, there exists $h \in \mathcal{S}$ such that

$$g_t(z) = \frac{h(\beta(0) f_t(z))}{\beta(0)}.$$

Here we demonstrate how to construct a normalized Loewner chain $(f_t) \in$ LC from a given evolution family

$(\varphi_{s,t}) \in \mathtt{EF}$. Firstly, define $(\psi_{s,t})_{0\leq s\leq t<\infty}$ by

$$\psi_{s,t} := h_t^{-1} \circ \varphi_{s,t} \circ h_s,$$

where h_t is a Möbius transformation given by

$$h_t(z) := \frac{b(t)z + a(t)}{1 + \overline{a(t)}b(t)z}, \quad a(t) := \varphi_{0,t}(0), \quad b(t) := \frac{\varphi'_{0,t}(0)}{|\varphi'_{0,t}(0)|}.$$

Then $(\psi_{s,t}) \in \mathtt{EF}$ ([CDMG10b, Proposition 2.9]). Further it is easy to see that $\psi_{s,t}(0) = 0$ and $\psi'_{s,t}(0) > 0$ for all $0 \leq s \leq t < \infty$. By the $(\psi_{s,t})$, define $(g_s)_{s\geq 0}$ as

$$g_s(z) := \lim_{t\to\infty} \frac{\psi_{s,t}(z)}{\psi'_{0,t}(0)}. \tag{5.8}$$

Remark that the limit in (5.8) is attained locally uniformly on $\mathbb{D}$. One can show that $(g_t) \in \mathtt{LC}$ associated with $(\psi_{s,t}) \in \mathtt{EF}$ and $g_0 \in \mathcal{S}$ ([CDMG10b, Theorem 3.3]). Finally, set

$$f_t := g_t \circ h_t^{-1}.$$

We conclude that $(f_t)_{t\geq 0} \in \mathtt{LC}$ associated with $(\varphi_{s,t}) \in \mathtt{EF}$ and $f_0 \in \mathcal{S}$. In the classical radial case, (5.8) corresponds to (4.4).

5.4 Quasiconformal extensions for Loewner chains of radial type

In view of Theorem 4.7, a natural question is proposed that whether the same assumption for $p \in \mathtt{HF}$ that $p \in U(k)$ deduces quasiconformal extensibility of the corresponding $(f_t) \in \mathtt{LC}$ or not. We give a positive answer to this problem under the special situation that $\tau \in \mathtt{DW}$ is constant. According to the case that $\tau \in \mathbb{D}$ or $\tau \in \partial\mathbb{D}$, the corresponding setting is called the **radial case** or **chordal case**. In the classical Loewner theory, the first is the original case introduced by Löwner, and the second is investigated firstly by Kufarev and his students [KSS68].

We employ the following definition due to [CDMG10a].

Definition 5.14 ([CDMG10a, Definition 1.2]). Let $(\varphi_{s,t}) \in \mathtt{EF}$. Suppose that all non-identical elements of $(\varphi_{s,t})$ share the same point $\tau_0 \in \overline{\mathbb{D}}$ such that $\varphi_{s,t}(\tau_0) = \tau_0$ and $|\varphi'_{s,t}(\tau_0)| \leq 1$ for all $s \geq 0$ and $t \geq s$, where $\varphi_{s,t}(\tau_0)$ and $\varphi'_{s,t}(\tau_0)$ are to be understood as the corresponding angular limit if $\tau_0 \in \partial\mathbb{D}$. Then $\varphi_{s,t}$ is said to be a **radial evolution family** if $\tau_0 \in \mathbb{D}$, or a **chordal evolution family** if $\tau_0 \in \partial\mathbb{D}$.

Then the radial and chordal version of Loewner chains are defined.

Definition 5.15 ([CDMG10a, Definition 1.5]). Let $(f_t) \in \mathtt{LC}$. If $(\varphi_{s,t})_{0\leq s\leq t<\infty} := (f_t^{-1} \circ f_s)_{0\leq s\leq t<\infty}$ is a radial (or chordal) evolution family, then we call (f_t) a **Loewner chain of radial** (or **chordal**) **type**.

Now we prove the following quasiconformal extension criterion for a Loewner chain of radial type.

Theorem 5.16. *Let $k \in [0, 1)$ be a constant. Suppose that (f_t) is a Loewner chain of radial type for which $p \in \mathtt{HF}$ associated with (f_t) by* (5.7)*, satisfies*

$$p(z, t) \in U(k)$$

for all $z \in \mathbb{D}$ and almost all $t \geq 0$ and $\tau \in \mathtt{DW}$ is equal to 0. Then the following assertions hold;

(i) f_t admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$;
(ii) F defined in (4.6) *gives a k-quasiconformal extension of f_0 to $\mathbb{C}$;*
(iii) $\Omega[(f_t)] = \mathbb{C}$.

Proof. With no loss of generality, we may assume $(f_t) \in \mathtt{LC}$ is normalized, i.e. $f_0 \in \mathcal{S}$. Let $\rho \in (c, 1)$ with some constant $c \in (0, 1)$ and define $f_t^\rho(z) := f_t(\rho z)/\rho$. Then accordingly F_ρ is defined. Since $f_t^\rho(z)$ satisfies $\partial_t f_t^\rho(z) := z\partial_z f_t^\rho(z)p(\rho z, t)$, f_t^ρ satisfies all the assumptions of our theorem. Further, f_t^ρ is well-defined on $\overline{\mathbb{D}}$ for all $t \geq 0$.

Take two distinct points $z_1, z_2 \in \mathbb{C}$. If either z_1 or z_2 is in $\mathbb{D}$, then it is clear that $F_\rho(z_1) \neq F_\rho(z_2)$. Suppose $z_1 := r_1e^{i\theta_1}, z_2 := r_2e^{i\theta_2} \in \mathbb{C}\backslash\mathbb{D}$ such that $F_\rho(z_1) = F_\rho(z_2)$, namely $f_{\log r_1}(\rho e^{i\theta_1}) = f_{\log r_2}(\rho e^{i\theta_2})$. Denote $t_1 := \log r_1$ and $t_2 := \log r_2$. Since $f_t^\rho(\partial\mathbb{D})$ is a Jordan curve, it follows that $t_1 \neq t_2$. By the equality condition of the Schwarz lemma we have $\varphi_{t_1,t_2}(z) := f_{t_2}^{-1} \circ f_{t_1}(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Hence $p(\mathbb{D}, t)$ lies on the imaginary axis for all $t \in [t_1, t_2]$ which contradicts our assumption. We conclude that F_ρ is a homeomorphism on $\mathbb{C}$.

A simple calculation shows that

$$\left|\frac{\partial_{\bar{z}}F_\rho(z)}{\partial_z F_\rho(z)}\right| = \left|\frac{\partial_t f_t^\rho(z) - z\partial_z f_t^\rho(z)}{\partial_t f_t^\rho(z) + z\partial_z f_t^\rho(z)}\right| \leq k$$

Hence F_ρ is k-quasiconformal on $\mathbb{C}$. Since the k does not depend on $\rho \in (c, 1)$, $(F_\rho)_{\rho\in(c,1)}$ forms a family of k-quasiconformal mappings on $\mathbb{C}$ and it is normal. Therefore the limit $F(z) = \lim_{\rho\to 1} F_\rho(z)$ exists which gives a

k-quasiconformal extension of f_0. In particular, f_t is defined on $\partial\mathbb{D}$ for all $t \geq 0$. It also follows from quasiconformality of F that $F(\mathbb{C}) = \Omega[(f_t)] = \mathbb{C}$. $\square$

If (f_t) is a Loewner chain of radial type and $(p, \tau) \in \mathtt{BP}$ associated with (f_t) where $p \in U(k)$ and $\tau \in \mathbb{D}\backslash\{0\}$, (f_t) satisfies $\dot{f}_t(z) = (z - \tau)(1 - \bar{\tau}z)f_t'(z)p(z, t)$ for all $z \in \mathbb{D}$ and almost all $t \geq 0$. Let M be a Möbius transformation defined by

$$M(z) := \frac{z + \tau}{1 + \bar{\tau}z}.$$

Then $(g_t)_{t\geq 0} := (f_t \circ M)_{t\geq 0}$ is a family of univalent maps satisfying $\dot{g}_t(z) = zg_t'(z)p(M(z), t)$ for all $z \in \mathbb{D}$ and almost all $t \geq 0$. By [CDMG10b, Theorem 4.1], (g_t) is a Loewner chain whose Berkson–Porta data is $(p, 0) \in \mathtt{BP}$. Applying Theorem 5.16, g_0, and hence f_0, has a k-quasiconformal extension to $\mathbb{C}$.

5.5 Quasiconformal extensions for Loewner chains of chordal type

A Loewner chain of chordal type (see Definition 5.15) with a quasiconformal extension is discussed by Gumenyuk and the author [GH17]. In the chordal case, $\tau \in \mathtt{DW}$ is a boundary fixed point of $\mathbb{D}$. By some rotation we may assume that $\tau = 1$.

It is sometimes convenient to discuss the chordal case on the not $\mathbb{D}$ but rather the half-plane. In fact, by means of the conjugation with a Cayley map $K(z) = (1 + z)/(1 - z)$, everything can be transferred from the unit disk to the right half-plane. For instance, a family $(\Phi_{s,t})_{0\leq s\leq t<\infty}$ of holomorphic self-maps of the right half-plane $\mathbb{H}$ is an **evolution family** if $(K^{-1} \circ \Phi_{s,t} \circ K)_{0\leq s\leq t<\infty}$ is an evolution family on the unit disk $\mathbb{D}$. Then the generalized chordal Loewner–Kufarev PDE and ODE are written by

$$\dot{\Phi}_{s,t}(\zeta) = p_{\mathbb{H}}(\Phi_{s,t}(\zeta), t) \quad \text{and} \quad \dot{f}_t(\zeta) = -f_t'(\zeta)p_{\mathbb{H}}(\zeta, t) \quad (\zeta \in \mathbb{H}), \tag{5.9}$$

where $p_{\mathbb{H}}(\zeta, t) := 2p(K^{-1}(\zeta), t)$ stands for the right half-plane version of the Herglotz function. A special case of (5.9) that $p_{\mathbb{H}}(\zeta) = 1/(\zeta + i\lambda(t))$ has attracted great attention since the work by Schramm [Sch00] was provided, where $\lambda : [0, \infty) \to \mathbb{R}$ is a measurable function.

The next theorem states the chordal variant of Becker's theorem.

Theorem 5.17 ([GH17]). *Suppose that a family of holomorphic functions $(f_t)_{t\geq 0}$ on the right half-plane $\mathbb{H}$ is a Loewner chain of chordal type. If there exists a uniform constant $k \in [0, 1)$ such that $p_{\mathbb{H}}$, a Herglotz function associated with (f_t), satisfies*

$$p_{\mathbb{H}}(\zeta, t) \in U(k) \tag{5.10}$$

for all $\zeta \in \mathbb{H}$ and almost all $t \geq 0$, then

(i) f_t admits a continuous extension to $\mathbb{H} \cup i\mathbb{R}$;

(ii) f_t has a k-quasiconformal extension to $\mathbb{C}$ for each $t \geq 0$. In this case the extension F is explicitly given by

$$F(\zeta) := \begin{cases} f_0(\zeta), & \zeta \in \mathbb{H}, \\ f_{-\operatorname{Re}\zeta}(i \operatorname{Im}\zeta), & \zeta \in \mathbb{C}\backslash\overline{\mathbb{H}}; \end{cases}$$

(iii) $\Omega[(f_t)] = \mathbb{C}$.

If $\tau \in \mathtt{DW}$ is a boundary point on $\partial\mathbb{D}\backslash\{1\}$, then composing a proper rotation we obtain the same result as Theorem 5.17. In fact, by setting $g_t(z) := f_t(\bar{\tau}z)$ we have $g_t(z) = (z - \tau)(1 - \bar{\tau}z)g'(z)p(\bar{\tau}z, t)$. After transferring g_t to the right half-plane, Theorem 5.17 with the same k as f_t is applied.

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